MA 351: Introduction to Linear Algebra and Its Applications
Fall 2026, Midterm Two

Instructor: Yip

- This test booklet has **five questions**, totaling 100 points for the whole test. You have 75 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is **closed book, closed note, with no electronic devices**.
- In order to get full credits, you need to give **correct**, **simplified**, and **complete** answers and explain in a **comprehensible way** how you arrive at them.
- **As a rule of thumb, you should give explicit and useful answers.** No points will be given for just writing down some generically true statements. In other words, your solutions and answers should be relevant to the information given in the question.
- **As a rule of thumb, you should only use those methods that have been covered in class.** If you use some other methods "for the sake of convenience", at our discretion, we might not give you any credit. You have the right to contest. In that event, **you will be asked to explain your answer using only what has been covered in class up to the point of time of this exam.**

Read the above instructions!

Name: Answer Key (Major:)

Question	Score
1.(20 pts)	
2.(20 pts)	
3.(20 pts)	
4.(20 pts)	
5.(20 pts)	
Total (100 pts)	

1. Find a basis for the following subspace:

$$U = \left\{ \begin{bmatrix} a+3b+c & -2b+2c & 2a+8c \\ -3a+2b-14c & a-3b+7c & 5a+3b+17c \end{bmatrix} : a,b,c \in R \right\}$$

$$= a\begin{bmatrix} 1 & 0 & 2 \\ -3 & 1 & 5 \end{bmatrix} + b\begin{bmatrix} 3 & -2 & 0 \\ 2 & -3 & 3 \end{bmatrix} + c\begin{bmatrix} 1 & 2 & 8 \\ -14 & 7 & 17 \end{bmatrix}$$

M_1 M_2 M_3

$U = \text{Span}\{M_1, M_2, M_3\}$

check for lin ind for M_1, M_2, M_3

$aM_1 + bM_2 + cM_3 = 0$

$$\Leftrightarrow \begin{cases} a+3b+c = 0 \\ -2b+2c = 0 \\ 2a+8c = 0 \\ -3a+2b-14c = 0 \\ a-3b+7c = 0 \\ 5a+3b+17c = 0 \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 2 & 0 & 8 & 0 \\ -3 & 2 & -14 & 0 \\ 1 & -3 & 7 & 0 \\ 5 & 3 & 17 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -6 & 6 & 0 \\ 0 & 11 & -11 & 0 \\ 0 & -6 & 6 & 0 \\ 0 & -12 & 12 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$U = \text{Span}\{M_1, M_2\}$

$(M_3 = 4M_1 - M_2)$

2. Consider the following matrix:

$$A = \begin{pmatrix} 1 & 1 & 3 & 2 \\ 2 & 2 & 6 & 4 \\ 10 & 2 & 14 & 20 \\ 2\sqrt{2} & -\sqrt{2} & 0 & 4\sqrt{2} \\ \sqrt{2} & \sqrt{3} & \sqrt{2}+2\sqrt{3} & 2\sqrt{2} \\ 2 & 2 & 6 & 4 \\ 1 & 1 & 3 & 2 \end{pmatrix}$$

(a) Find the rank and nullity of A.

(b) Find a basis for the Col and Null and Row spaces of A.

(c) For any b, can you find an X such that $AX = b$? Explain.

(d) Does the solution $AX = b$ has a unique solution (if a solution exists)? Explain.

(a)

$$\left(\begin{array}{cccc|c} 1 & 1 & 3 & 2 & 0 \\ 2 & 2 & 6 & 4 & 0 \\ 10 & 2 & 14 & 20 & 0 \\ 2\sqrt{2} & -\sqrt{2} & 0 & 4\sqrt{2} & 0 \\ \sqrt{2} & \sqrt{3} & \sqrt{2}+2\sqrt{3} & 2\sqrt{2} & 0 \\ 2 & 2 & 6 & 4 & 0 \\ 1 & 1 & 3 & 2 & \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -8 & -16 & 0 & 0 \\ 0 & -3\sqrt{2} & -6\sqrt{2} & 0 & 0 \\ 0 & \sqrt{3}-\sqrt{2} & -2\sqrt{2}+2\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

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$$\to \left(\begin{array}{cccc|c} 1 & 1 & 3 & 2 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

(a) Rank $= 2$, Nullity $= 2$

(b) Basis for $Col(A) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 10 \\ 2\sqrt{2} \\ \sqrt{2} \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \\ -\sqrt{2} \\ \sqrt{3} \\ 2 \\ 1 \end{pmatrix} \right\}$

Basis for $Null(A)$: $x_3 = \alpha$, $x_4 = \beta$

$x_1 = -\alpha - 2\beta$

$x_2 = -2\alpha$

Basis

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -\alpha - 2\beta \\ -2\alpha \\ \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

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Basis for Row(A) = $\{(1\ 0\ 1\ 2), (0\ 1\ 2\ 0)\}$

(c) $AX = b$ does not have a solution for any b

(∵ Rank = $2 < 7$)

(d) $AX = b$ can have infinitely many solution

(∵ Nullity = $2 > 0$, there are 2 free vars.)

3. Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$. Find a basis and the dimension for the following subspace of 2×2 matrices:

$$\mathcal{M} = \{B : AB = BA\}.$$

$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{pmatrix} a+2b & -a \\ c+2d & -c \end{pmatrix} = \begin{pmatrix} a-c & b-d \\ 2a & 2b \end{pmatrix}$$

$a+2b = a-c \iff c = -2b$ ①

$-a = b-d \iff a+b = d$ ②

$c+2d = 2a \iff 2a = c+2d$ ③

$-c = 2b \iff c = -2b$ ④

$2a = -2b+2d$

use b & d as free vars. $a = -b+d$ same as ②

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -b+d & b \\ -2b & d \end{pmatrix} = b\begin{pmatrix} -1 & 1 \\ -2 & 0 \end{pmatrix} + d\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

M_1 M_2

Basis vector

Alternatively,

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$$\Rightarrow \begin{cases} 2b+c = 0 \\ a+b-d = 0 \\ 2a-c-2d = 0 \\ 2b+c = 0 \end{cases} \Rightarrow \left(\begin{array}{cccc|c} 0 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 2 & 0 & -1 & -2 & 0 \\ 0 & 2 & 1 & 0 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & -1 & -2 & 0 \\ 0 & 2 & 1 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & -2 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \begin{array}{l} b = -\frac{c}{2} \\ a = -b+d \\ a = +\frac{c}{2}+d \end{array}$$

c, d are free

basis

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{c}{2}+d & -\frac{c}{2} \\ c & d \end{pmatrix} = c\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

4. Let $\mathcal{S} = \{A^{2\times 2} : A = A^t\}$ and $\mathcal{K} = \{B^{2\times 2} : B = -B^t\}$. (The superscript ^{t} means taking the transpose of a matrix. The set $\mathcal{S}$ and $\mathcal{K}$ are called the space of symmetric and skew-symmetric matrices.)

   (a) Find a basis and the dimension for $\mathcal{S}$.

   (b) Find a basis and the dimension for $\mathcal{K}$.

   (c) If you collect the basis vectors for $\mathcal{S}$ and $\mathcal{K}$ together, do they form a basis for the space of all 2×2 matrices? If so, write a general 2×2 matrix as a linear combination of these basis vectors.

(a) $A = A^T$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$\Rightarrow \quad a = a, \; b = c, \; c = b, \; d = d$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ b & d \end{pmatrix} = a\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{S_1} + b\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{S_2} + d\underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{S_3}$$

basis for $\mathcal{S}$ dim = 3

(b) $B = -B^T$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$\Rightarrow \quad \underbrace{a = -a}_{a=0}, \; b = -c, \; c = -b, \; \underbrace{d = -d}_{d=0}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = b\underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{K_1}$$

basis, dim = 1

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(c) $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c_4 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{aligned} a &= c_1, \\ b &= c_2 + c_4 \\ c &= c_2 - c_4 \\ d &= c_3 \end{aligned} \Rightarrow \begin{aligned} c_1 &= a \\ c_2 &= \frac{b+c}{2} \\ c_4 &= \frac{b-c}{2} \\ c_3 &= d \end{aligned}$$

Yes. They do form a basis. (They are lin. ind. (why?))

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{b+c}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{b-c}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

5. Let T be a linear transformation from R^2 to R^3 such that

$$T\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

(a) Find the matrix representation of T, i.e. find the matrix A such that $T(X) = AX$.

(b) Is T onto/surjective, i.e. for any Y, can you find (its pre-image) X such that $T(X) = Y$? Either prove it or find a vector Y such that it *does not* have a pre-image.

(c) Is T one-to-one/injective, i.e. does $T(X_1) = T(X_2)$ imply $X_1 = X_2$? Either prove it or find *two* vectors X_1 and X_2 such that $T(X_1) = T(X_2)$ but $X_1 \neq X_2$.

(a) $\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$2a+3b=1$ $\quad 2c+3d=0$ $\quad 2e+3f=-1$

$a+2b=0$ $\quad c+2d=1$ $\quad e+2f=1$

$\Rightarrow$ $b=-1$ $\quad d=2$ $\quad f=3$

$a=2$ $\quad c=-3$ $\quad e=-5$

$$A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \\ -5 & 3 \end{pmatrix}$$

[Check $\begin{pmatrix} 2 & -1 \\ -3 & 2 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 2 & -1 \\ -3 & 2 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$]

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(b) $\left(\begin{array}{cc|c} 2 & -1 & a \\ -3 & 2 & b \\ -5 & 3 & c \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -\frac{1}{2} & \frac{a}{2} \\ -3 & 2 & b \\ -5 & 3 & c \end{array}\right)$

$\to \left(\begin{array}{cc|c} 1 & -1/2 & a/2 \\ 0 & 1/2 & \frac{3a}{2}+b \\ 0 & \frac{1}{2} & \frac{5a}{2}+c \end{array}\right)$

$\to \left(\begin{array}{cc|c} 1 & -1/2 & a/2 \\ 0 & 1/2 & \frac{3a}{2}+b \\ 0 & 0 & a+c-b \end{array}\right)$

We need $a+c-b=0$. Choose for example $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

No pre-image

T is not onto

(c) T is one-to-one as there is no free variable.