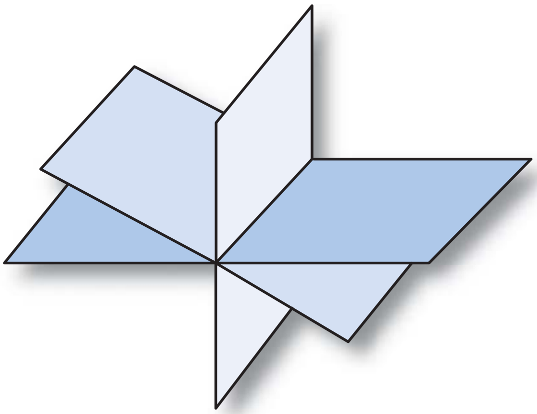

CHAPTER

1

Linear Equations

1.1 Introduction to Linear Systems

Traditionally, algebra was the art of solving equations and systems of equations. The word *algebra* comes from the Arabic *al-jabr* (الجبر), which means *restoration* (of broken parts).[1] The term was first used in a mathematical sense by Mohammed al-Khowarizmi (c. 780–850), who worked at the House of Wisdom, an academy established by Caliph al-Ma'mun in Baghdad. Linear algebra, then, is the art of solving systems of linear equations.

The need to solve systems of linear equations frequently arises in mathematics, statistics, physics, astronomy, engineering, computer science, and economics.

Solving systems of linear equations is not conceptually difficult. For small systems, ad hoc methods certainly suffice. Larger systems, however, require more systematic methods. The approach generally used today was beautifully explained 2,000 years ago in a Chinese text, the *Nine Chapters on the Mathematical Art* (Jiuzhang Suanshu, 九章算術).[2] Chapter 8 of that text, called *Method of Rectangular Arrays* (Fang Cheng, 方程), contains the following problem:

> The yield of one bundle of inferior rice, two bundles of medium-grade rice, and three bundles of superior rice is 39 *dou* of grain.[3] The yield of one bundle of inferior rice, three bundles of medium-grade rice, and two bundles of superior rice is 34 *dou*. The yield of three bundles of inferior rice, two bundles of medium-grade rice, and one bundle of superior rice is 26 *dou*. What is the yield of one bundle of each grade of rice?

In this problem the unknown quantities are the yields of one bundle of inferior, one bundle of medium-grade, and one bundle of superior rice. Let us denote these quantities by x, y, and z, respectively. The problem can then be represented by the

[1] At one time, it was not unusual to see the sign *Algebrista y Sangrador* (bone setter and blood letter) at the entrance of a Spanish barber's shop.

[2] Shen Kangshen et al. (ed.), *The Nine Chapters on the Mathematical Art*, Companion and Commentary, Oxford University Press, 1999.

[3] The *dou* is a measure of volume, corresponding to about 2 liters at that time.

following system of linear equations:

$$\left|\begin{array}{r} x + 2y + 3z = 39 \\ x + 3y + 2z = 34 \\ 3x + 2y + z = 26 \end{array}\right|.$$

To solve for x, y, and z, we need to transform this system from the form

$$\left|\begin{array}{r} x + 2y + 3z = 39 \\ x + 3y + 2z = 34 \\ 3x + 2y + z = 26 \end{array}\right| \quad \text{into the form} \quad \left|\begin{array}{l} x \quad = \ldots \\ \quad y \quad = \ldots \\ \quad\quad z = \ldots \end{array}\right|.$$

In other words, we need to eliminate the terms that are off the diagonal, those circled in the following equations, and make the coefficients of the variables along the diagonal equal to 1:

$$\begin{array}{rcrcrcl} x & + & \textcircled{2y} & + & \textcircled{3z} & = & 39 \\ \textcircled{x} & + & 3y & + & \textcircled{2z} & = & 34 \\ \textcircled{3x} & + & \textcircled{2y} & + & z & = & 26. \end{array}$$

We can accomplish these goals step by step, one variable at a time. In the past, you may have simplified systems of equations by adding equations to one another or subtracting them. In this system, we can eliminate the variable x from the second equation by subtracting the first equation from the second:

$$\left|\begin{array}{r} x + 2y + 3z = 39 \\ x + 3y + 2z = 34 \\ 3x + 2y + z = 26 \end{array}\right| \quad \begin{array}{c} \longrightarrow \\ -1\text{st equation} \\ \end{array} \quad \left|\begin{array}{r} x + 2y + 3z = 39 \\ y - z = -5 \\ 3x + 2y + z = 26 \end{array}\right|.$$

To eliminate the variable x from the third equation, we subtract the first equation from the third equation three times. We multiply the first equation by 3 to get

$$3x + 6y + 9z = 117 \qquad (3 \times 1\text{st equation})$$

and then subtract this result from the third equation:

$$\left|\begin{array}{r} x + 2y + 3z = 39 \\ y - z = -5 \\ 3x + 2y + z = 26 \end{array}\right| \quad \begin{array}{c} \longrightarrow \\ \\ -3 \times 1\text{st equation} \end{array} \quad \left|\begin{array}{r} x + 2y + 3z = 39 \\ y - z = -5 \\ -4y - 8z = -91 \end{array}\right|.$$

Similarly, we eliminate the variable y above and below the diagonal:

$$\left|\begin{array}{r} x + 2y + 3z = 39 \\ y - z = -5 \\ -4y - 8z = -91 \end{array}\right| \quad \begin{array}{c} -2 \times 2\text{nd equation} \\ \longrightarrow \\ +4 \times 2\text{nd equation} \end{array} \quad \left|\begin{array}{r} x \quad + 5z = 49 \\ y - z = -5 \\ -12z = -111 \end{array}\right|.$$

Before we eliminate the variable z above the diagonal, we make the coefficient of z on the diagonal equal to 1, by dividing the last equation by -12:

$$\left|\begin{array}{r} x \quad + 5z = 49 \\ y - z = -5 \\ -12z = -111 \end{array}\right| \quad \begin{array}{c} \\ \longrightarrow \\ \div(-12) \end{array} \quad \left|\begin{array}{r} x \quad + 5z = 49 \\ y - z = -5 \\ z = 9.25 \end{array}\right|.$$

Finally, we eliminate the variable z above the diagonal:

$$\left|\begin{array}{r} x \quad + 5z = 49 \\ y - z = -5 \\ z = 9.25 \end{array}\right| \quad \begin{array}{c} -5 \times \text{third equation} \\ + \text{third equation} \\ \longrightarrow \end{array} \quad \left|\begin{array}{r} x \quad\quad = 2.75 \\ y \quad = 4.25 \\ z = 9.25 \end{array}\right|.$$

The yields of inferior, medium-grade, and superior rice are 2.75, 4.25, and 9.25 *dou* per bundle, respectively.

By substituting these values, we can check that $x = 2.75$, $y = 4.25$, $z = 9.25$ is indeed the solution of the system:

$$\begin{aligned} 2.75 + 2 \times 4.25 + 3 \times 9.25 &= 39 \\ 2.75 + 3 \times 4.25 + 2 \times 9.25 &= 34 \\ 3 \times 2.75 + 2 \times 4.25 + \quad 9.25 &= 26. \end{aligned}$$

Happily, in linear algebra, you are almost always able to check your solutions. It will help you if you get into the habit of checking now.

Geometric Interpretation

How can we interpret this result geometrically? Each of the three equations of the system defines a plane in x–y–z-space. The solution set of the system consists of those points (x, y, z) that lie in all three planes (i.e., the intersection of the three planes). Algebraically speaking, the solution set consists of those ordered triples of numbers (x, y, z) that satisfy all three equations simultaneously. Our computations show that the system has only one solution, $(x, y, z) = (2.75, 4.25, 9.25)$. This means that the planes defined by the three equations intersect at the point $(x, y, z) = (2.75, 4.25, 9.25)$, as shown in Figure 1.

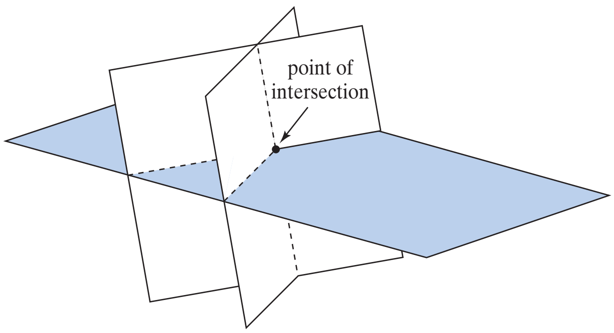

Figure 1 Three planes in space, intersecting at a point.

While three different planes in space usually intersect at a point, they may have a line in common (see Figure 2a) or may not have a common intersection at all, as shown in Figure 2b. Therefore, a system of three equations with three unknowns may have a unique solution, infinitely many solutions, or no solutions at all.

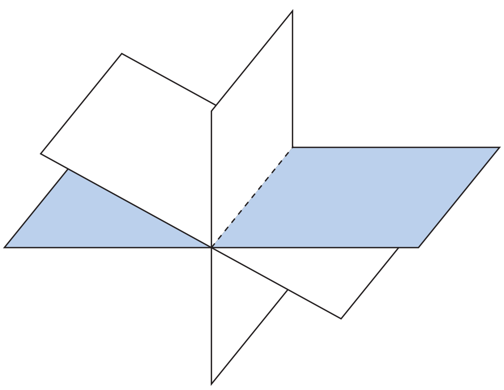

Figure 2(a) Three planes having a line in common.

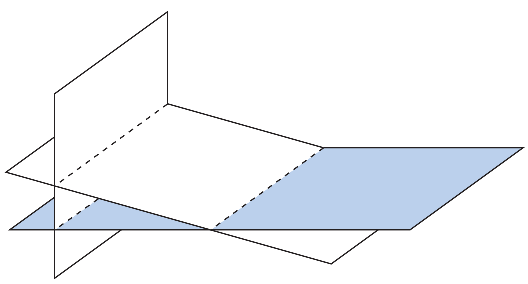

Figure 2(b) Three planes with no common intersection.