MA 351: Introduction to Linear Algebra and Its Applications
Fall 2021, Test One

Instructor: Yip

- This test booklet has FIVE QUESTIONS, totaling 100 points for the whole test. You have 75 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is **closed book, with no electronic device**.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.

Name: Answer Key (Major:)

Question	Score
1.(20 pts)	
2.(20 pts)	
3.(20 pts)	
4.(20 pts)	
5.(20 pts)	
Total (100 pts)	

1. Let $A = \begin{pmatrix} 2 & 3 & 1 & -19 & 1 \\ -1 & 0 & -2 & 8 & 7 \\ 2 & -1 & 5 & -15 & -19 \end{pmatrix}$.

(a) Find an explicit, *easily checkable* condition on $B = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ such that $AX = B$ is solvable.

Find also an explicit *numerical* example of B such that $AX = B$ is *not* solvable.

(b) Find a basis for Col(A) by selecting appropriate columns of A. Write also the columns of A that are not used as as a linear combination of the basis vectors.

(c) Find a basis for Null(A).

(a) $\left(\begin{array}{ccccc|c} 2 & 3 & 1 & -19 & 1 & a \\ -1 & 0 & -2 & 8 & 7 & b \\ 2 & -1 & 5 & -15 & -19 & c \end{array}\right) \to \left(\begin{array}{ccccc|c} 1 & 0 & 2 & -8 & -7 & -b \\ 2 & 3 & 1 & -19 & 1 & a \\ 2 & -1 & 5 & -15 & -19 & c \end{array}\right)$

$\to \left(\begin{array}{ccccc|c} 1 & 0 & 2 & -8 & -7 & -b \\ 0 & 3 & -3 & -3 & 15 & a+2b \\ 0 & -1 & 1 & 1 & -5 & c+2b \end{array}\right)$

Take e.g $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\to \left(\begin{array}{ccccc|c} 1 & 0 & 2 & -8 & -7 & -b \\ 0 & 1 & -1 & -1 & 5 & \frac{a+2b}{3} \\ 0 & -1 & 1 & 1 & -5 & c+2b \end{array}\right)$

so that $a+8b+3c \neq 0$

$\to \left(\begin{array}{ccccc|c} 1 & 0 & 2 & -8 & -7 & -b \\ 0 & 1 & -1 & -1 & 5 & (a+2b)/3 \\ 0 & 0 & 0 & 0 & 0 & c+2b+\frac{a+2b}{3} \end{array}\right)$

So we need $c+2b+\frac{a+2b}{3} = 0$, i.e. $\boxed{a+8b+3c=0}$

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(b) Consider dependency relation for the cols. of A:

$$c_1 u_1 + c_2 u_2 + c_3 u_3 + c_4 u_4 + c_5 u_5 = 0$$

$$\Longrightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 2 & -8 & -7 & 0 \\ 0 & 1 & -1 & -1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

pivots free

Hence throw away v_3, v_4, v_5 of A and keep u_1, u_2.

$$\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \right\}$$

basis of Col(A)

$c_3 = \alpha$, $c_4 = \beta$, $c_5 = \gamma$, $c_1 = -2\alpha + 8\beta + 7\gamma$

$c_2 = \alpha + \beta - 5\gamma$.

so that

$$(-2\alpha + 8\beta + 7\gamma)u_1 + (\alpha + \beta - 5\gamma)u_2 + \alpha u_3 + \beta u_4 + \gamma u_5 = 0$$

$\alpha = 1, \beta = 0, \gamma = 0 \Longrightarrow u_3 = 2u_1 - u_2$

$\alpha = 0, \beta = 1, \gamma = 0 \Longrightarrow u_4 = -8u_1 - u_2$

$\alpha = 0, \beta = 0, \gamma = 1 \Longrightarrow u_5 = -7u_1 + 5u_2$.

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(c) $\text{Null}(A) = \{X : AX = 0\}$

$$\rightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 2 & -8 & -7 & 0 \\ 0 & 1 & -1 & -1 & 5 & 0 \\ 0 & 0 & \underbrace{0 \quad\quad 0 \quad\quad 0}_{\text{free}} & & & 0 \end{array}\right)$$

pivots

$x_3 = \alpha,\ x_4 = \beta,\ x_5 = \gamma,\quad x_1 = -2\alpha + 8\beta + 7\gamma$

$x_2 = \alpha + \beta - 5\gamma$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2\alpha + 8\beta + 7\gamma \\ \alpha + \beta - 5\gamma \\ \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$= \alpha \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 8 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 7 \\ -5 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Basis

2. Consider the following 3×3 system of linear equation:

$$\begin{pmatrix} 1 & 2 & -3 \\ 3 & -1 & 5 \\ 4 & 1 & (a^2-14) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ a+2 \end{pmatrix}.$$

where a is some parameter.

Find the value(s) of a such that the system has (i) unique solution, (ii) infinitely many solutions, and (iii) no solution.

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & \underline{a^2-16} & \underline{a-4} \end{array}\right)$$

(i) Unique solution: $a^2-16 \neq 0 \Rightarrow \underline{a \neq 4, -4}$.

(ii) inf. many solns: $a^2-16=0$ & $a-4=0$

$\Longrightarrow \underline{a=4}$

(iii) No soln: $a^2-16=0$ & $a-4 \neq 0$

$\Longrightarrow \underline{a=-4}$

3. Consider the following 2×2 system of linear equation:

$$\begin{aligned} x + (\lambda - 3)y &= 0, \\ (\lambda - 3)x + y &= 0, \end{aligned}$$

where λ is a parameter. For what values of λ is such that the system has infintely many solutions?

$$\left(\begin{array}{cc|c} 1 & \lambda-3 & 0 \\ \lambda-3 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & \lambda-3 & 0 \\ 0 & \underbrace{1-(\lambda-3)^2} & 0 \end{array}\right)$$

For infinitely many solutions, we need:

$$1-(\lambda-3)^2=0 \iff \lambda-3 = \pm 1$$

$$\underline{\lambda = 4 \text{ or } 2}$$

(Penney, p.111, #2.11, 2.13, 2.17, 2.18)

4. You are given the list of vectors $\{X_1, X_2, X_3\}$ which is linearly independent. For each of the following lists, investigate if it is linearly (in)dependent. If dependent, write down an example of linear relationship between the vectors.

   (a) $\{Y_1, Y_2\}$ where $Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$.

   (b) $\{Z_1, Z_2, Z_3\}$ where $Z_1 = X_1 + X_2 + X_3$, $Z_2 = X_1 - X_2 + 2X_3$, $Z_3 = 2X_1 - X_2 - X_3$.

   (c) $\{W_1, W_2, W_3\}$ where $W_1 = X_1 - X_2 + X_3$, $W_2 = X_1 + X_2 + 2X_3$, $W_3 = X_1 - 3X_2$.

(a) $c_1Y_1 + c_2Y_2 = 0 \Longleftrightarrow c_1(X_1+X_2) + c_2(X_1-X_2) = 0$

$\Longleftrightarrow (c_1+c_2)X_1 + (c_1-c_2)X_2 = 0$

Since X_1, X_2 – lin ind. $\Longrightarrow \begin{cases} c_1+c_2=0 \\ c_1-c_2=0 \end{cases} \Longrightarrow \begin{matrix} c_1=0 \\ c_2=0 \end{matrix}$

Hence Y_1, Y_2 are lin ind.

(b) $c_1Z_1 + c_2Z_2 + c_3Z_3 = 0 \overset{?}{\Longrightarrow} c_1=0, c_2=0, c_3=0$

$c_1(X_1+X_2+X_3) + c_2(X_1-X_2+2X_3) + c_3(2X_1-X_2-X_3) = 0$

$\underbrace{(c_1+c_2+2c_3)}_{=0}X_1 + \underbrace{(c_1-c_2-c_3)}_{=0}X_2 + \underbrace{(c_1+2c_2-c_3)}_{=0}X_3 = 0$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & 2 & -1 & 0 \end{array}\right)$$

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$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -2 & -3 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -2 & -3 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -9 & 0 \end{array}\right)$$

No free var.

Hence Z_1, Z_2, Z_3 - lin ind.

(c) $c_1 W_1 + c_2 W_2 + c_3 W_3 = 0$

$$c_1(X_1 - X_2 + X_3) + c_2(X_1 + X_2 + 2X_3) + c_3(X_1 - 3X_2) = 0$$

$$\underbrace{(c_1 + c_2 + c_3)}_{=0} X_1 + \underbrace{(-c_1 + c_2 - 3c_3)}_{=0} X_2 + \underbrace{(c_1 + 2c_2)}_{=0} X_3 = 0$$

(Since X_1, X_2, X_3 - lin ind)

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -1 & 1 & -3 & 0 \\ 1 & 2 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

free

W_1, W_2, W_3 lin. dep

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$c_3 = \alpha, \quad c_2 = \alpha, \quad c_1 = -2\alpha$

$$(-2\alpha) W_1 + \alpha W_2 + \alpha W_3 = 0$$

$\underline{\alpha = 1} \implies \boxed{-2W_1 + W_2 + W_3 = 0}$

(Check:

$$-2(X_1 - X_2 + X_3) + (X_1 + X_2 + 2X_3) + (X_1 - 3X_2) = 0 \,)$$

(See Homework #4, p. 88 #106, #107.)

5. Suppose the solution of a 2×4 system $AX = B$ is given by

$$X = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -1 \\ 3 \end{pmatrix}$$

where s and t are free variables.

Find A and B. Is there a unique answer? If not, write down two explicit examples.

Spanning vectors for Null sp. of A.

$$A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}$$

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \underline{a + b + 2c} = 0$$

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \underline{a - c + 3d = 0}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 0 \\ 1 & 0 & -1 & 3 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 0 \\ 0 & -1 & -3 & 3 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & -1 & 3 & 0 \\ 0 & 1 & \underbrace{3 \quad -3}_{\text{free}} & & 0 \end{array}\right)$$

$c = \alpha, \quad d = \beta$

$a = \alpha - 3\beta$

$b = -3\alpha + 3\beta$

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$\alpha = 1, \beta = 0, \quad (a \;\; b \;\; c \;\; d) = (1 \;\; -3 \;\; 1 \;\; 0)$

$\alpha = 0, \beta = 1 \quad (a, \; b, \; c, \; d) = (-3 \;\; 3 \;\; 0 \;\; 1)$

Hence $A = \begin{pmatrix} 1 & -3 & 1 & 0 \\ -3 & 3 & 0 & 1 \end{pmatrix}$

$AX = B$: $\begin{pmatrix} 1 & -3 & 1 & 0 \\ -3 & 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$

Make use of the translation vector $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

$\Longrightarrow \quad -3 + 1 = p \qquad p = -2$

$\qquad 3 + 0 = q \qquad q = 3$

Hence an example of $AX = B$ is:

$$\boxed{\begin{pmatrix} 1 & -3 & 1 & 0 \\ -3 & 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}}$$

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Another example: (multiply first row by 2)

$$\boxed{\begin{pmatrix} 2 & -3 & 2 & 0 \\ -3 & 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}}$$

or (add the 2 rows together)

$$\boxed{\begin{pmatrix} 1 & -3 & 1 & 0 \\ -2 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}}$$