## MA 351 Fall 2025 (Aaron N. K. Yip) Homework 5

Due: Thursday, Oct. 23rd, in class

Penney, Linear Algebra: Ideas and Applications (4th edition)

p.112 EXERCISES: 2.25(a,b,c,d) (We have done exercises showing functions are *linearly dependent*. This exercise demonstrates how to show they are *linearly independent*. One method, among others, is shown in p.107, (2.9).)

p.123 EXERCISES: 2.32, 2.33, 2.35, 2.36, 2.37, 2.46

(For 2.46, in addition to what is asked in the textbook, write every linear combination of X and Y as a linear combination of A and B and vice versa.)

Additional Problems.

- 1. Consider  $V = \text{span}\{X_1, X_2, X_3\}$  where  $X_1, X_2, X_3$  are linearly independent. (Hence  $A = \{X_1, X_2, X_3\}$  is a basis of V and  $\dim(V) = 3$ .)
  - (a) Let  $B = \{Y_1 = X_1 + 2X_2 X_3, Y_2 = X_1 + X_2, Y_3 = 7X_2\}$ . Show that B is also a basis for V by *explicitly* showing that: (i) B is linearly independent and, (ii) B spans V. (For (ii), express any/every vector from V not just  $X_1, X_2, X_3$  as a linear combination of the  $Y_i$ 's.
  - (b) Let  $C = \{Y_1 = X_1 + 2X_2 X_3, Y_2 = X_1 X_2, Y_3 = X_1 + X_2 + X_3, Y_4 = X_1 3X_2 + X_3\}$ . Show that C is linearly dependent (and hence cannot be a basis for V).

Do not use any theorem. Do the above from first principle, i.e. from scratch.

2. [P, p.125] 2.41(a,c)

Remark: In class, we have done a lot of examples of throwing away or eliminating redundant vectors. But what if we want to add vectors as in 2.41? There are many ways you can do this. For 2.41(a), simply find one vector  $X_3 \in \mathbb{R}^3$  that is not from  $\text{Span}\{X_1, X_2\}$ . Then show explicitly that  $\{X_1, X_2, X_3\}$  forms a basis for  $\mathbb{R}^3$ . More generally, for 2.41(c), you can expand the list  $\mathcal{L} = \{X_1, X_2\}$  to be  $\mathcal{L}' = \{X_1, X_2, Y_1, Y_2, Y_3, Y_4\}$  where  $\{Y_1, Y_2, Y_3, Y_4\}$  is "any of your favorite basis of  $\mathbb{R}^4$ ". Clearly  $\mathcal{L}'$  spans  $\mathbb{R}^4$  but it is unfortunately linearly dependent. Now, you can throw away two redundant vectors from  $\mathcal{L}'$ . Make sure that you put  $X_1, X_2$  at the beginning of the list so that they are not being thrown away. Feel free to think about 2.41(e).

3. Given

$$\mathcal{L} = \left\{ X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, X_3 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\}, \text{ and } Y = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Find constants  $c_1, c_2, c_3$  such that  $Y = c_1 X_1 + c_2 X_2 + c_3 X_3$  and  $c_1^2 + c_2^2 + c_3^3$  is minimized.

<u>Hint</u>: express  $c_1, c_2, c_3$  in terms of (some) free variable(s). Then determine the free variable(s) so that  $c_1^2 + c_2^2 + c_3^3$  is minimized. Recall your Calculus I.

Remark/motivation about this problem. Clearly, the expression  $Y = c_1X_1 + c_2X_2 + c_3X_3$  is "inefficient" as we don't need all the three vectors  $X_1, X_2, X_3$  to express Y because  $X_3$  can be written as a linear combination of  $X_1$  and  $X_2$  and hence can be thrown away. However, there might be "other notion(s)" of "efficiency". For example, some applications might want to keep the coefficients as "small" as possible. In this problem, the "smallness" is measured by  $c_1^2 + c_2^2 + c_3^2$ .