MA 351 Fall 2021 (Aaron N. K. Yip)
Homework 5
Due: Thursday, Oct. 7, in class

1. Recall the definition of a vector space: a set $V = \{u, v, w, \cdots\}$ is called a vector space if it is endowed with two operations:

   (i) scalar multiplication such that for any $\alpha \in \mathbf{R}$ and $u \in V$, we have $\alpha u \in V$;
   (ii) vector addition such that for any $u, v \in V$, we have $u + v \in V$

   which satisfy the following properties: (the Greek alphabets refer to real numbers (scalars) and the English alphabets refer to vectors from V)

   (a) $u + v = v + u$;
   (b) $(u + v) + w = u + (v + w)$;
   (c) $\alpha(u + v) = \alpha u + \alpha v$;
   (d) $(\alpha + \beta)u = \alpha u + \beta u$;
   (e) $\alpha(\beta u) = (\alpha\beta)u$;
   (f) there is a $\mathbf{0}$ such that $u + \mathbf{0} = u$ for any $u \in V$;
   (g) for any $u \in V$, there is a $-u \in V$ such that $u + (-u) = \mathbf{0}$;
   (h) $1u = u$.

   Now consider $V = \mathbf{R}^+$, the set of *positive real numbers*. On V we define the following "scalar multiplication" and "vector addition":

   (i) (scalar multiplication, $\cdot$) for any $\alpha \in \mathbf{R}$ and $u \in V$, $\alpha \cdot u = u^\alpha$, i.e. raising u to its usual α power; (note: u^α still belongs to V so that the scalar multiplication is a legitimate operation. Hence (i) above is satisfied.)
   (ii) (vector addition, $\oplus$) for any $u, v \in V$, $u \oplus v = uv$, i.e. taking the usual multiplication between u and v. (note: uv still belongs to V so that the vector addition is a legitimate operation. Hence (ii) above is satisfied.)

   Prove that V (endowed with the scalar multiplication and vector addition just defined above) is a vector space by showing that all the above properties (yeah, 8 of them) are satisfied. More explicitly,

   (a) $u \oplus v = v \oplus u$;
   (b) $(u \oplus v) \oplus w = u \oplus (v \oplus w)$;
   (c) $\alpha \cdot (u \oplus v) = \alpha \cdot u \oplus \alpha \cdot v$;
   (d) $(\alpha + \beta) \cdot u = \alpha \cdot u \oplus \beta \cdot u$;
   (e) $\alpha(\beta \cdot u) = (\alpha\beta) \cdot u$;

(f) there is a $\mathbf{0}$ such that $u \oplus \mathbf{0} = u$ for any $u \in V$;

(g) for any $u \in V$, there is a $-u \in V$ such that $u \oplus (-u) = \mathbf{0}$;

(h) $1 \cdot u = u$.

(Hint: NOTATION MATTERS.)

2. An MA351 class in Fall 2021 is asked to solve a system of linear equation. Suppose the "left-half" of the class gives an answer as:

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + s_1 \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + s_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

while the "right-half" of the class gives:

$$\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$

where s_1, s_2, t_1 and t_2 are all free variables.

(a) Show that both answers are the same.

(b) Express the following expressions from the "left"

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - 5 \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

using the "right" form.

(c) Express the following expressions from the "right"

$$\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} + 5 \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix},$$

using the "left" form.

For example,

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} - 1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}.$$