MA351 Spring 2016 (Yip)

Test One Solution

1. Your Name:____________________________

Consider the following three matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 5 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 4 \\ -1 & -3 \end{pmatrix}.$$

a) Give an explicit condition for an arbitrary matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to be in the span of A, B, C. Your condition should be easily verifiable or checkable.

b) Give two explicit examples of matrices which are not in the span of A, B, C.

(a)

$$c_1 A + c_2 B + c_3 C = M$$

(Given a, b, c, d, find c_1, c_2, c_3)

$$c_1 \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 & 0 \\ 1 & 5 \end{pmatrix} + c_3 \begin{pmatrix} 1 & 4 \\ -1 & -3 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\left\{\begin{array}{l} c_1 + c_2 + c_3 = a \\ 2c_1 + 4c_3 = b \\ c_2 - c_3 = c \\ c_1 + 5c_2 - 3c_3 = d \end{array}\right. \iff \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 2 & 0 & 4 & b \\ 0 & 1 & -1 & c \\ 1 & 5 & 3 & d \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & -2 & 2 & b-2a \\ 0 & 1 & -1 & c \\ 0 & 4 & -4 & d-a \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & -1 & c \\ 0 & -2 & 2 & b-2a \\ 0 & 4 & -4 & d-a \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & -1 & c \\ 0 & 0 & 0 & b-2a+2c \\ 0 & 0 & 0 & d-a-4c \end{array}\right) \begin{array}{l} = 0 \\ = 0 \end{array}$$

You can use this blank page.

Hence $M \in \text{Span}\{A, B, C\}$ iff $\boxed{\begin{matrix} b = 2a - 2c \\ d = a + 4c \end{matrix}}$

(b) eg $a = 1$, $c = 0$, $b = 0$, $d = 0$, $M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$a = 0$, $c = 1$, $b = 0$, $d = 0$, $M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

in both cases, $b \neq 2a - 2c$
$d \neq a + 4c$

2. Your Name:________________________________

Let X_1, X_2, X_3 be three linearly independent vectors from some vector space and let $\mathcal{W} = \text{Span}\{X_1, X_2, X_3\}$. Define the following vectors:

$$Y_1 = X_1, \;\; Y_2 = X_1 + X_2, \;\; Y_3 = X_1 + X_2 + X_3$$

a) Show that Y_1, Y_2, Y_3 are linearly independent.

b) If possible, express an arbitrary vector $Z = aX_1 + bX_2 + cX_3$ from $\mathcal{W}$ as a linear combination of Y_1, Y_2, Y_3.

c) Is $\{Y_1, Y_2, Y_3\}$ a basis for $\mathcal{W}$?

(a) $c_1 Y_1 + c_2 Y_2 + c_3 Y_3 = 0$

$c_1 X_1 + c_2(X_1 + X_2) + c_3(X_1 + X_2 + X_3) = 0$

$\underbrace{(c_1 + c_2 + c_3)}_{= 0} X_1 + \underbrace{(c_2 + c_3)}_{= 0} X_2 + \underbrace{c_3}_{= 0} X_3 = 0$

$\boxed{\text{Since } X_1, X_2, X_3 \text{ are lin ind}}$

ie. $c_1 + c_2 + c_3 = 0$

$c_2 + c_3 = 0$

$c_3 = 0$

$c_3 = 0, \Rightarrow c_2 = 0 \Rightarrow c_1 = 0$

Hence Y_1, Y_2, Y_3 are lin. ind.

(b) $c_1 Y_1 + c_2 Y_2 + c_3 Y_3 = aX_1 + bX_2 + cX_3$ (given a, b, c, find c_1, c_2, c_3)

$c_1(X_1) + c_2(X_1 + X_2) + c_3(X_1 + X_2 + X_3) = aX_1 + bX_2 + cX_3$

$\underline{(c_1 + c_2 + c_3)} X_1 + \underline{(c_2 + c_3)} X_2 + \underline{c_3} X_3 = \underline{a} X_1 + \underline{b} X_2 + \underline{c} X_3$

You can use this blank page.

Hence

$$\begin{aligned} c_1 + c_2 + c_3 &= a \\ c_2 + c_3 &= b \\ c_3 &= c \end{aligned} \implies c_2 = b - c$$

$$c_1 = a - (b-c) - c = a - b$$

i.e. $\boxed{(a-b)Y_1 + (b-c)Y_2 + cY_3 = aX_1 + bX_2 + cX_3}$

(c) Yes, $\{Y_1, Y_2, Y_3\}$ is a basis for W as it spans and is lin ind.

3. Your Name:______________________________

Consider the following system which involves a parameter a in the coefficient matrix:

$$\begin{aligned} x+y-z &= 2 \\ x+2y+z &= 7 \\ x+y+(a^2-5)z &= a \end{aligned}$$

a) Find all those a such that the system has a unique solution.

b) Find all those a such that the system has infinitely many solutions and SOLVE the system.

c) Find all those a such that the system has no solution.

(Hint: perform row reduction and deduce the consistency condition as a function of a.)

(a) $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 7 \\ 1 & 1 & a^2-5 & a \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & a^2-4 & a-2 \end{array}\right)$

We need $a^2-4 \neq 0$ ie. $a \neq 2, -2$.

(b) The REF is

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & a^2-4 & a-2 \end{array}\right)$$

We need both a^2-4 & $a-2=0$ ie. $a=2$

In this case, the matrix becomes

You can use this blank page.

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x \quad y \quad z = \alpha \text{ (free)}$

$$\boxed{\begin{pmatrix} x = -3 + 3\alpha \\ y = 5 - 2\alpha \\ z = \alpha \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}}$$

(c) $$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & a^2-4 & a-2 \end{array}\right)$$

We need $a^2-4=0$ and $a-2 \neq 0 \Rightarrow$ inconsistent.

$\Updownarrow \qquad \Updownarrow$

$a = 2, -2 \qquad a \neq 2$

Hence we need $\underline{\underline{a = -2}}$

4. Your Name:________________________________

This question explores the solution of the equation $AX = \lambda X$ which involves a parameter λ.

a) Solve the following system:

$$\begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$$

b) Solve the following system:

$$\begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

(Hint: note that the unknown variables x and y appear on both the left and right hand sides. Re-write the system in such a way that x and y only appear in the left hand side.)

Be ware of notation!

(a) $\begin{cases} -y = 3x \\ 2x + 3y = 3y \end{cases} \iff \begin{cases} -y = 3x \\ 2x = 0 \end{cases} \Rightarrow \begin{pmatrix} x = 0 \\ y = 0 \end{pmatrix}$

(b) $\begin{cases} -y = 2x \\ 2x + 3y = 2y \end{cases} \iff \begin{cases} 2x + y = 0 \\ 2x + y = 0 \end{cases}$,

$y = \alpha$, $x = -\frac{\alpha}{2}$, $\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$ (α free).

5. Your Name:______________________

a) Prove that if $\{X_1, X_2, \ldots, X_n\}$ is linear independent and $Y \notin \text{Span}\{X_1, X_2, \ldots, X_n\}$, then $\{X_1, X_2, \ldots, X_n, Y\}$ is also linearly independent.

b) Prove that if $\{X_1, X_2, \ldots, X_n\}$ contains the zero vector, i.e. one of the X_i is the zero vector, then $\{X_1, X_2, \ldots, X_n\}$ is linearly dependent.

c) Prove that if $Y_1, Y_2 \in \text{Span}\{X_1, X_2\}$, then

$$\text{Span}\{Y_1, Y_2\} \subseteq \text{Span}\{X_1, X_2\}.$$

d) Give an example of a list of vector(s) that spans $\mathbf{R}^2$ but is linearly dependent.

e) Given an example of a list of vector(s) from $\mathbf{R}^2$ that is linearly independent but does not span $\mathbf{R}^2$.

(a) Test for lin ind for $X_1, X_2, \ldots, X_n, Y$

$$c_1X_1 + c_2X_2 + \cdots + c_nX_n + c_{n+1}Y_n = 0$$

If $c_{n+1} \neq 0$, then $Y_n = \left(-\frac{c_1}{c_{n+1}}\right)X_1 + \cdots + \left(-\frac{c_n}{c_{n+1}}\right)X_n$

$\in \text{Span}\{X_1, \ldots, X_n\}$

Contradiction.

Hence c_{n+1} must be 0. Then

$c_1X_1 + c_2X_2 + \cdots + c_nX_n = 0 \Rightarrow c_1, c_2, \ldots, c_n = 0$ (Since $X_1, \ldots, X_n$ are lin ind)

Hence $c_1 = c_2 = \cdots = c_n = c_{n+1} = 0$, i.e. $X_1, X_2, \ldots, X_n, Y$ is also lin ind

You can use this blank page.

(b) Suppose $X_1 = \vec{0}$ ~~[illegible]~~

Then $1 \cdot X_1 + 0X_2 + 0X_3 + \cdots + 0X_n = 0$

non-trivial number $(1 \neq 0)$

Hence $\{X_1, X_2, \ldots, X_n\}$ is <u>lin dep.</u>

(c) $Y_1 \in \text{Span}\{X_1, X_2\}$, i.e. $Y_1 = aX_1 + bX_2$

$Y_2 \in \text{Span}\{X_1, X_2\}$ i.e. $Y_2 = cX_1 + dX_2$

Note $c_1Y_1 + c_2Y_2 = c_1(aX_1 + bX_2) + c_2(cX_1 + dX_2)$

$= (\underline{c_1a + c_2c})X_1 + (\underline{c_1b + c_2d})X_2$

$= \tilde{c}_1 X_1 + \tilde{c}_2 X_2 \in \text{Span}\{X_1, X_2\}$

You can use this blank page.

(d) $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ spans $\mathbb{R}^2$

but are <u>lin dep</u>.

(any 3 vectors in $\mathbb{R}^2$ must be lin dep.)

(e) $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ by itself is lin indep

but it cannot span the whole $\mathbb{R}^2$.

(In fact any non-zero vector will do)