Solution

1. Your Name:________________________________

Consider the matrix $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$.

a) Find A^{-1}.

b) Solve for X which satisfies: $AX = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

c) Solve for Y which satisfies: $YA = (1\ \ 1)$.

d) Solve for B which satisfies: $AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

e) Solve for C which satisfies: $CA = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

(a) $\left(\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 2 & 5 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right)$

$\longrightarrow \left(\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & 3 & -5 \\ 0 & 1 & -1 & 2 \end{array}\right)$

$A^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$ #

(b) $AX = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Longleftrightarrow X = A^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ #

(c) $YA = (1\ \ 1) \Longleftrightarrow Y = (1\ \ 1)A^{-1} = (1\ \ 1)\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$

$= (2\ \ -3)$ #

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(d) $AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Longleftrightarrow B = A^{-1}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$= \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$= \begin{pmatrix} -12 & -14 \\ 5 & 6 \end{pmatrix}$ #

(e) $CA = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Longleftrightarrow C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} A^{-1}$

$= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1 & -1 \\ 5 & -7 \end{pmatrix}$

$-15+8$

2. Your Name:______________________________

Consider the following matrix:

$$A = \begin{pmatrix} 1 & 1 & -1 & -7 \\ 2 & 3 & 0 & 1 \\ 5 & 7 & -1 & -5 \end{pmatrix}$$

Find a basis and the dimension for the column, null, and row spaces of A.

$$A = \begin{pmatrix} 1 & 1 & -1 & -7 \\ 2 & 3 & 0 & 1 \\ 5 & 7 & -1 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & -7 \\ 0 & 1 & 2 & 15 \\ 0 & 2 & 4 & 30 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & -1 & -7 \\ 0 & 1 & 2 & 15 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -3 & -22 \\ 0 & 1 & 2 & 15 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivots free

$$Col(A) = Span\left\{ \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} \right\}, \quad \dim(Col(A)) = 2$$

$$Row(A) = Span\left\{ \begin{matrix} (1 & 0 & -3 & -22) \\ (0 & 1 & 2 & 15) \end{matrix} \right\}, \quad \dim(Row(A)) = 2$$

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For Null(A): solve $AX=0$:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3\alpha+22\beta \\ -2\alpha-15\beta \\ \alpha \\ \beta \end{pmatrix} = \alpha\begin{pmatrix} 3 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 22 \\ -15 \\ 0 \\ 1 \end{pmatrix}$$

$$= \mathrm{Span}\left\{\begin{pmatrix} 3 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 22 \\ -15 \\ 0 \\ 1 \end{pmatrix}\right\}$$

basis,

$\dim(\mathrm{Null}(A)) = \underline{\underline{2}}$

3. Your Name:______________________________

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear transformation such that

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \end{pmatrix} \quad \text{and} \quad T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

a) What is the value of $T\begin{pmatrix} 1 \\ 0 \end{pmatrix}$?

b) What is the value of $T\begin{pmatrix} 0 \\ 1 \end{pmatrix}$?

c) What is the value of $T\begin{pmatrix} 10 \\ 3 \end{pmatrix}$?

d) What is the matrix representation of T?

e) Is T onto? Is T one-to-one? Is T invertible?

(Hint: for (a), (b), (c), it might be easier if you consider a "general case".)

(a) $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2T\begin{pmatrix} 1 \\ 1 \end{pmatrix} - T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 7 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 12 \end{pmatrix}$

(b) $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = T\begin{pmatrix} 1 \\ 2 \end{pmatrix} - T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$

(c) $T\begin{pmatrix} 10 \\ 3 \end{pmatrix} = 10T\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 10\begin{pmatrix} 1 \\ 12 \end{pmatrix} + 3\begin{pmatrix} 1 \\ -5 \end{pmatrix} = \begin{pmatrix} 13 \\ 105 \end{pmatrix}$

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(d) $T\begin{pmatrix} a \\ b \end{pmatrix} = aT\begin{pmatrix} 1 \\ 0 \end{pmatrix} + bT\begin{pmatrix} 0 \\ 1 \end{pmatrix} = a\begin{pmatrix} 1 \\ 12 \end{pmatrix} + b\begin{pmatrix} 1 \\ -5 \end{pmatrix}$

$$= \underbrace{\begin{pmatrix} 1 & 1 \\ 12 & -5 \end{pmatrix}}_{\text{matrix of } T.}\begin{pmatrix} a \\ b \end{pmatrix}$$

Alternative for (a), (b), (c), (d)
<u>method</u>

$\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad c_1 + c_2 = x$
$\quad c_1 + 2c_2 = y$

$\Rightarrow c_2 = y - x$
$c_1 = x - y + x = 2x - y$

$\begin{pmatrix} x \\ y \end{pmatrix} = (2x-y)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (y-x)\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Hence $T\begin{pmatrix} x \\ y \end{pmatrix} = (2x-y)T\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (y-x)T\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

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$$T\begin{pmatrix} x \\ y \end{pmatrix} = (2x-y)\begin{pmatrix} 2 \\ 7 \end{pmatrix} + (y-x)\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4x-2y+3y-3x \\ 14x-7y+2y-2x \end{pmatrix} \Rightarrow \begin{pmatrix} x+y \\ 12x-5y \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 12 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

matrix for T.

(a) $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 12 & -5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 12 \end{pmatrix}$

(b) $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 12 & -5 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$

(c) $T\begin{pmatrix} 10 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 12 & -5 \end{pmatrix}\begin{pmatrix} 10 \\ 3 \end{pmatrix} = \begin{pmatrix} 13 \\ 105 \end{pmatrix}$

You can use this blank page.

(e) $A = \begin{pmatrix} 1 & 1 \\ 12 & -5 \end{pmatrix} \leftarrow$ a square matrix.

Rank(A) = 2 since columns of A are lin. ind.

Hence A is onto and also one-to-one.

Thus ~~A~~ A is invertible.

4. Your Name:________________________________

In the following, an example means an example with explicit numbers. You also need to determine the values of m and n.

a) Give an example of an $m \times n$ matrix A such that the linear transformation given by the matrix A is onto but not one-to-one. In your example, m and n must at least 2. You need to explain to me that your matrix is indeed onto but not one-to-one.

For your example, give an example of vectors X_1 and X_2 such that $X_1 \neq X_2$ but $AX_1 = AX_2$.

b) Give an example of an $m \times n$ matrix A such that the linear transformation given by the matrix A is one-to-one but not onto. In your example, m and n must at least 2. You need to explain to me that your matrix is indeed one-to-one but not onto.

For your example, give an example of a vector Y such that there is no X that satisfies $AX = Y$.

(a) Need rank$(A) = m < n$.

eg. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $m = 2 < n = 3$

$A\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = A\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\parallel$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

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(b) Need $\operatorname{rank}(A) = n < m$

eg $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ $n = 2 < 3 = m$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ has no solution.

$\leftarrow$ (inconsistent)

5. Your Name:______________________________

This question is to fill in some proofs related to the properties of inverses. It only concerns square matrices of dimension $n \times n$.

a) (Uniqueness of inverses)
Suppose B satisfies $AB = I$ and $BA = I$ and C satisfies $AC = I$ and $CA = I$. Prove that $B = C$.
(Hint: consider multiplying C to the first set of conditions.)

b) (Equivalence of onto and one-to-one)
Suppose $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear transformation given by a A. Prove that the property that T is onto is equivalent to the property that it is one-to-one.

c) (Equivalence of $AB = I$ and $BA = I$.)
Suppose A and B are two matrices such that $AB = I$. Prove that $BA = I$.
(Hint: what can you say about the solvability of the equation $AX = Y$ for any value of Y?)

(a) $AB = I$

$CAB = CI$

$IB = CI$

$B = C$

(b) A is $n \times n$, a square matrix.

A is onto $\Leftrightarrow$ rank$(A) =$ # of rows

$\Leftrightarrow$ rank$(A) = n$

$\Leftrightarrow$ nullity$(A) = 0$ rank + nullity = # of cols.

$\Leftrightarrow$ A is one-to-one.

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(c) One solution of $AX=Y$ is given by

$X=BY$.

(Check: $AX=ABY=IY=Y$).

Hence $AX=Y$ is always soluble for any Y.

Hence A is onto.

By (b), A is also one-to-one.

Hence A^{-1} exists.

$AB=I$.

$\Rightarrow A^{-1}AB=A^{-1}I$

$\Rightarrow B=A^{-1}$

Thus $BA=A^{-1}A=I=AB$