Purdue University
MA 351: Linear Algebra and Its Applications
Final Examination, Spring 2016
Instructor: Aaron N. K. Yip

- The examination consists of **SEVEN QUESTIONS**, totaling 200 points. You have 120 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is closed book and closed notes.
  No calculator nor any electronic devices are allowed.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them. **Accuracy counts. It is not enough simply writing down some general procedures and ideas.**
- When drawing curves and graphs, **label your axes** and **indicate all the necessary key features**.
- You can use both sides of the papers to write your answers. But please indicate so if you do.

Name: Answer Key (Major:)

Question	Score
1.(30 pts)	
2.(20 pts)	
3.(20 pts)	
4.(30 pts)	
5.(40 pts)	
6.(30 pts)	
7.(30 pts)	
Total (200 pts)	

1. Your Name:______________________________

Consider the matrix $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 3 & -2 & 7 \end{pmatrix}$

a) Find a basis and the dimension for the column space and null space of A. 5 5pt.

b) Is the matrix onto? 2 If not, give an explicit ~~and~~ 4 concrete condition for a vector Y such that $AX = Y$ is solvable. In addition, give an explicit numerical example of a vector Z such that $AX = Z$ is not solvable. 2

c) Is the matrix one-to-one? 2 If not, give an explicit numerical example of two vectors X and Y such that $X \neq Y$ and yet $AX = AY$. 8.

(a) $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 3 & -2 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -4 & 5 \\ 0 & -8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -5/4 \\ 0 & 0 & 0 \end{bmatrix}$

$\longrightarrow \begin{bmatrix} 1 & 0 & 6/4 \\ 0 & 1 & -5/4 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad -1 + \frac{10}{4}$

Basis for column space $= \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \right\}$

Basis for null space : $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{6}{4}\alpha \\ \frac{5}{4}\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} -\frac{6}{4} \\ \frac{5}{4} \\ 1 \end{pmatrix}$

$\alpha = 4 \Longrightarrow$ Basis $= \left\{ \begin{pmatrix} -6 \\ 5 \\ 4 \end{pmatrix} \right\}$

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(b) Rank(A) = 2 < 3. Hence A is not onto.

Solve: $AX = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$: $\left[\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 2 & 0 & 3 & b \\ 3 & -2 & 7 & c \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 0 & -4 & 5 & b-2a \\ 0 & -8 & 10 & c-3a \end{array}\right]$

$\to \left[\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 0 & -4 & 5 & b-2a \\ 0 & 0 & 0 & c-3a-2b+4a \end{array}\right]$

$\boxed{\text{Solvable}}$ if and only if $c - 2b + a = 0$

i.e. $\boxed{2b = a + c}$

When $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, for example, then $2b \neq a + c$

Hence $AX = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is <u>not solvable</u>.

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(c) Let $X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $Y = \begin{pmatrix} -6 \\ 5 \\ 4 \end{pmatrix}$

Note both $X, Y \in Null(A)$,

Hence $AX = AY = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

But $X \neq Y$.

2. Your Name:______________________________

Consider the matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Is A diagonalizable? If so, find an invertible matrix Q and a diagonal matrix D such that $A = QDQ^{-1}$. (Note: you are not required to compute Q^{-1}.)

(Hint: try to factor the polynomial $\det(A - \lambda I)$ as early as possible.)

$$\det(A-\lambda I) = \det\begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix}$$

$$= -\lambda \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & -\lambda \end{vmatrix} + \begin{vmatrix} 1 & -\lambda \\ 1 & 1 \end{vmatrix}$$

$$= -\lambda(\lambda^2-1) - (-\lambda-1) + (1+\lambda)$$

$$= -\lambda\cancel{(\lambda+1)}(\lambda-1) + 2\cancel{(\lambda+1)}$$

$$= (\lambda+1)\left[-\lambda^2+\lambda+2\right]$$

$$= (\lambda+1)(\lambda^2-\lambda-2)$$

$$= -(\lambda+1)(\lambda+1)(\lambda-2)$$

Hence $\boxed{\lambda = 2, -1, -1}$

5

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$\boxed{\lambda=2}$ $\quad (A-2I)X=\begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0\\ 1 & -2 & 1 & 0\\ 1 & 1 & -2 & 0\end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0\\ 1 & 1 & -2 & 0\\ -2 & 1 & 1 & 0\end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0\\ 0 & 3 & -3 & 0\\ 0 & -3 & 3 & 0\end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\end{array}\right], \quad X=\begin{pmatrix}\alpha\\\alpha\\\alpha\end{pmatrix}=\alpha\underbrace{\begin{pmatrix}1\\1\\1\end{pmatrix}}_{X_1}$$

5

$\boxed{\lambda=-1}$ $\quad (A+I)X=\begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0\\ 1 & 1 & 1 & 0\\ 1 & 1 & 1 & 0\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

5

$$X=\begin{pmatrix}-\alpha-\beta\\\alpha\\\beta\end{pmatrix}=\alpha\begin{pmatrix}-1\\1\\0\end{pmatrix}+\beta\begin{pmatrix}-1\\0\\1\end{pmatrix}, \quad X_2=\begin{pmatrix}-1\\1\\0\end{pmatrix}, \quad X_3=\begin{pmatrix}-1\\0\\1\end{pmatrix}.$$

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A has 3 lin ind eigenvector, hence A is diagonalizable.

$$A = \underbrace{\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 2 & & 0 \\ & -1 & \\ 0 & & -1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1}}_{Q^{-1}}$$

5

3. Your Name:______________________________

Consider the matrix $A = \begin{pmatrix} 2 & -2 & a & b \\ 0 & 1 & c & d \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$. Determine conditions on a, b, c and d such that A is diagonalizable.

Since A is upper triangular, the eigenvalues of A are simply the diagonal entries. Hence $\lambda = 2, 2, 2, 1$.

$\lambda = 1$ has no problem as it only appears once. 10

For $\lambda=2$, we need $\underline{\underline{3}}$ free variables for $(A-2I)X=0$

$$(A-2I)X=0 \Rightarrow \left(\begin{array}{cccc|c} 0 & -2 & a & b & 0 \\ 0 & -1 & c & d & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 0 & 1 & -c & -d & 0 \\ 0 & 2 & -a & -b & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

pivot

$$\rightarrow \left(\begin{array}{cccc|c} 0 & 1 & -c & -d & 0 \\ 0 & 0 & 2c-a & 2d-b & 0 \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \end{array}\right)$$

free

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In order to have 3 free variables, we need

$2c - a = 0$ & $2d - b = 0$

ie. $\boxed{a = 2c}$ & $\boxed{b = 2d}$

10

4. Your Name:______________________________

Consider the space of 2×2 matrices. Given $M = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, recall that $M^T = \begin{pmatrix} x & z \\ y & w \end{pmatrix}$. A matrix A is called *symmetric* if $A = A^T$. For example, $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$. A matrix B is called *skew-symmetric* if $B = -B^T$. For example, $B = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$.

10 a) Find a basis and the dimension of the space of symmetric matrices.

10 b) Find a basis and the dimension of the space of skew-symmetric matrices.

10 c) Show that for any matrix M, it can be written in a unique way as the sum of a symmetric and skew-symmetric matrices, i.e. $M = A + B$ where $A^T = A$ and $B^T = -B$. For example,

$$\begin{pmatrix} 2 & 6 \\ 4 & 8 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 5 & 8 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(Hint: what is the most general form of a 2×2 symmetric and skew-symmetric matrices?)

(a) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$A = A^T \iff \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$\iff b = c$

Symmetric matrices $= \left\{ \begin{pmatrix} a & b \\ b & d \end{pmatrix} \right\}$

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$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{Basis} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \quad \boxed{\dim = 3}$$

(b) $$A = -A^T \iff \begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$\iff \begin{cases} a = -a \\ b = -c \\ c = -b \\ d = -d \end{cases} \iff a = 0,\ d = 0,\ b = -c$$

$$\text{Skew-symmetric matrices} = \left\{ \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \right\} = \left\{ b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

$$\text{Basis} = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}, \quad \boxed{\dim = 1}$$

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? ?

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} + \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}$$

symmetric skew-symmetric

$$\begin{cases} x = a \\ y = d+b \\ z = b-d \\ w = c \end{cases} \iff \begin{array}{l} a = x \\ b = \frac{y+z}{2} \\ d = \frac{y-z}{2} \\ c = w \end{array}$$

unique solution

i.e. $\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & \frac{y+z}{2} \\ \frac{y+z}{2} & w \end{pmatrix} + \begin{pmatrix} 0 & \frac{y-z}{2} \\ -\left(\frac{y-z}{2}\right) & 0 \end{pmatrix}$

symmetric skew-symmetric

5. Your Name:______________________________

Consider the line $L : y = x$.

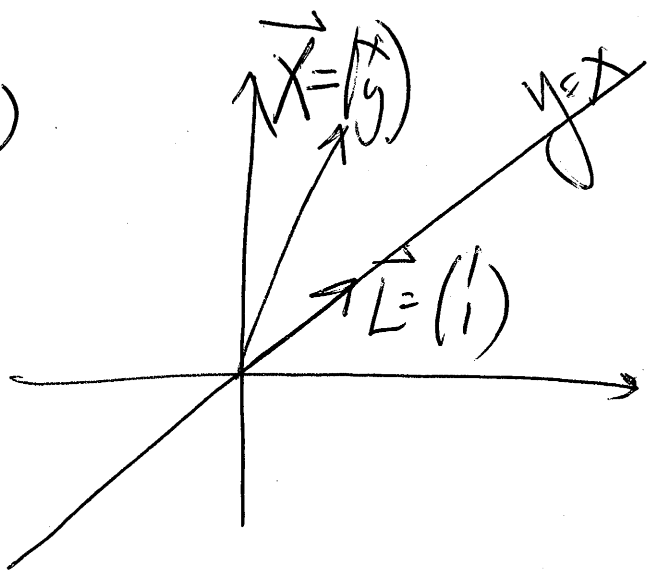
10 a) Compute the matrix P that projects any vector X onto L.

10 b) Find the column and null spaces and the eigenvalues and eigenvectors of P.

10 c) Compute the matrix R that reflects any vector X with respect to L, i.e. L is viewed as a mirror.

10 d) Find the column and null spaces and the eigenvalues and eigenvectors of R.

(a)



$$\mathrm{Proj}_{\vec{L}} X = \frac{\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}}{1^2 + 1^2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{x+y}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{x+y}{2} \\ \frac{x+y}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(b) $\left(\begin{array}{cc|c} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\mathrm{Col}(P) = \mathrm{Span}\left\{\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}\right\} = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$, Null Space$(P) = \left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$

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From ~~[illegible]~~ geometry, it is clear that,

$$P\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix} \quad \& \quad P\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}=0\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\ \frac{1}{2}&\frac{1}{2}\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=1\begin{pmatrix}1\\1\end{pmatrix}, \quad \begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\ \frac{1}{2}&\frac{1}{2}\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}=0\begin{pmatrix}1\\-1\end{pmatrix}$$

Hence $\left\{\lambda=1,\ X=\begin{pmatrix}1\\1\end{pmatrix}\right\}$ & $\left\{\lambda=0,\ X=\begin{pmatrix}1\\-1\end{pmatrix}\right\}$

(or alternatively solve $\det(P-\lambda I)=\det\begin{pmatrix}\frac{1}{2}-\lambda&\frac{1}{2}\\ \frac{1}{2}&\frac{1}{2}-\lambda\end{pmatrix}$

$$=\left(\lambda-\frac{1}{2}\right)^2-\frac{1}{4}$$

$$=\lambda^2-\lambda=0 \Rightarrow \lambda=1,0.$$

$$\lambda=1 \Rightarrow (P-I\,|\,0) \to \left(\begin{array}{cc|c}-\frac{1}{2}&\frac{1}{2}&0\\ \frac{1}{2}&-\frac{1}{2}&0\end{array}\right) \to \left(\begin{array}{cc|c}-1&1&0\\0&0&0\end{array}\right) \Rightarrow X=\begin{pmatrix}1\\1\end{pmatrix}$$

$$\lambda=0 \Rightarrow (P-0I\,|\,0) \to \left(\begin{array}{cc|c}\frac{1}{2}&\frac{1}{2}&0\\ \frac{1}{2}&\frac{1}{2}&0\end{array}\right) \to \left(\begin{array}{cc|c}1&1&0\\0&0&0\end{array}\right) \Rightarrow X=\begin{pmatrix}1\\-1\end{pmatrix}.$$

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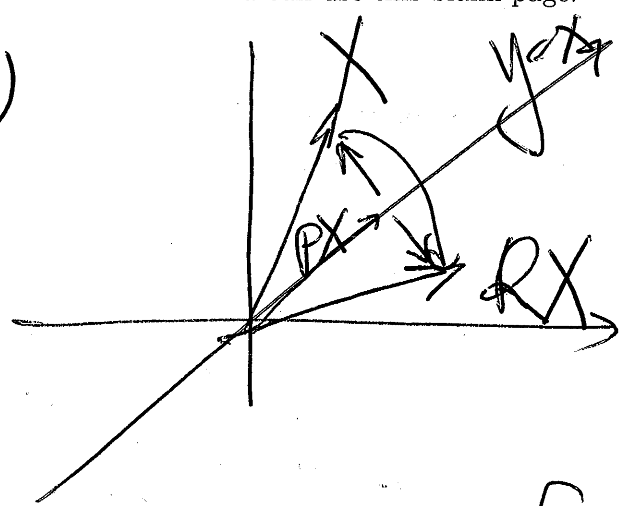
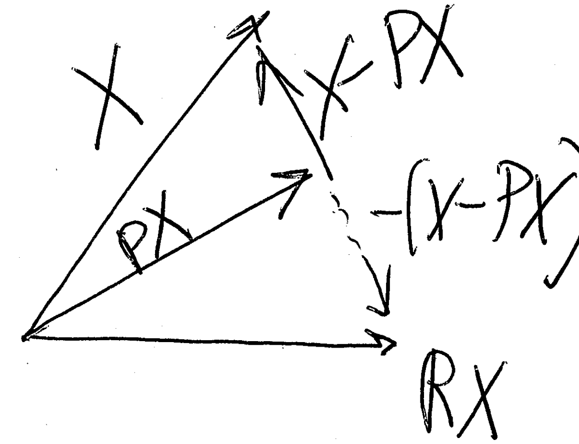
(c)





$$RX = PX + [-(X-PX)] = 2PX - X$$
$$= (2P-I)X$$

Hence $R = 2P - I = 2\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(d) $\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$

$Col(R) = Span\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$, $Null(R) = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$, (Since no free var.)

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From geometry it is clear that

$$R\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}, \quad R\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}-1\\1\end{pmatrix}=-1\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}, \quad \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}-1\\1\end{pmatrix}=-1\begin{pmatrix}1\\-1\end{pmatrix}$$

ie. $\left\{\lambda=1,\ X=\begin{pmatrix}1\\1\end{pmatrix}\right\}$ & $\left\{\lambda=-1,\ X=\begin{pmatrix}-1\\1\end{pmatrix}\right\}$

~~Alter~~ Alternatively $\det(R-\lambda I)=\det\begin{pmatrix}-\lambda&1\\1&-\lambda\end{pmatrix}$

$$=\lambda^2-1=0$$

$$\boxed{\lambda=1,-1}$$

$$\lambda=1 \Rightarrow (R-I|\begin{smallmatrix}0\\0\end{smallmatrix}) \to \left(\begin{array}{cc|c}-1&1&0\\1&-1&0\end{array}\right) \to \left(\begin{array}{cc|c}1&-1&0\\0&0&0\end{array}\right) \Rightarrow X=\begin{pmatrix}1\\1\end{pmatrix}$$

$$\lambda=-1 \Rightarrow (R+I|\begin{smallmatrix}0\\0\end{smallmatrix}) \to \left(\begin{array}{cc|c}1&1&0\\1&1&0\end{array}\right) \to \left(\begin{array}{cc|c}1&1&0\\0&0&0\end{array}\right) \to X=\begin{pmatrix}1\\-1\end{pmatrix}.$$

6. Your Name:______________________________

Consider the plane $\Pi : x - y + z = 0$ in $\mathbb{R}^3$.

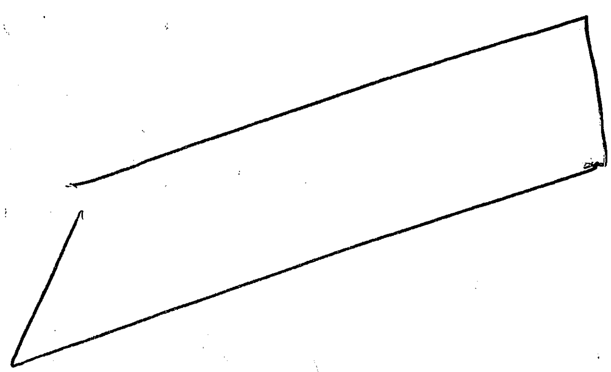
5 a) Find a basis for Π.

20 b) Find the orthogonal projection of any vector $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ onto Π.

(Hint: draw a picture and write the projected image as a linear combination of your basis. Then determine the coefficients in the linear combination.)

5 c) Find the matrix P corresponding to the above projection.

(a)



$\Pi : x - y + z = 0$

$\left(1 \quad -1 \quad 1 \,\middle|\, 0\right)$

free free

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha - \beta \\ \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Basis $\rightarrow \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

X_1 X_2

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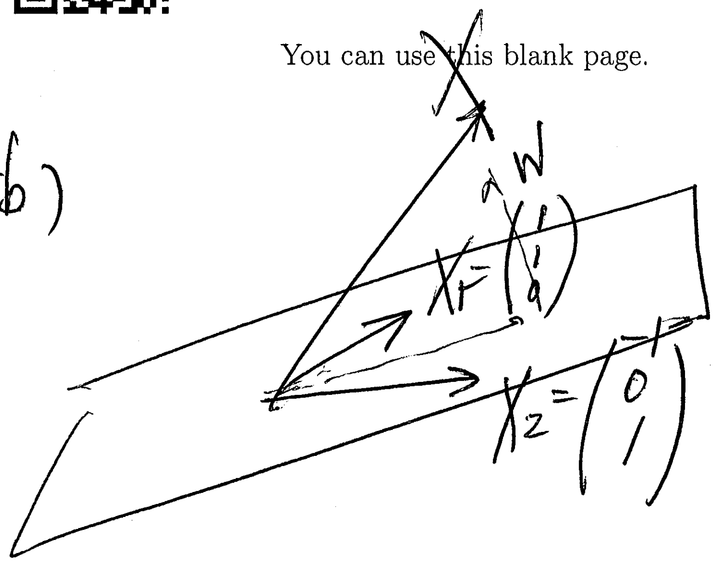
(b)



$$X = \mathrm{Proj}_{\Pi} X + W$$

$$X = c_1 X_1 + c_2 X_2 + W$$

$$W \perp X_1 \ \& \ W \perp X_2.$$

$$X = c_1 X_1 + c_2 X_2 + W$$

$$X \cdot X_1 = c_1 X_1 \cdot X_1 + c_2 X_2 \cdot X_1 + \cancel{W \cdot X_1}^{\,0}$$

$$X \cdot X_2 = c_1 X_1 \cdot X_2 + c_2 X_2 \cdot X_2 + \cancel{W \cdot X_2}^{\,0}$$

Let $X = \begin{pmatrix}x\\y\\z\end{pmatrix}$. Then,

$$\begin{cases} x+y = c_1 2 + c_2(-1) \\ -x+z = -c_1 + 2c_2 \end{cases} \iff \begin{array}{l} 2c_1 - c_2 = x+y \\ -c_1 + 2c_2 = -x+\cancel{y}\,z \end{array}$$

$$\Rightarrow 3c_1 = x+2y+z \Rightarrow c_1 = \frac{x}{3} + \frac{2}{3}y + \frac{z}{3}$$

$$\Rightarrow 3c_2 = -x+y+2z \Rightarrow c_2 = -\frac{x}{3} + \frac{y}{3} + \frac{2}{3}z.$$

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Hence $\mathrm{Proj}_{\Pi} X = c_1 X_1 + c_2 X_2$

$$= \left(\frac{x}{3} + \frac{2}{3}y + \frac{z}{3}\right)\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$+ \left(-\frac{x}{3} + \frac{y}{3} + \frac{2z}{3}\right)\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{3}x + \frac{y}{3} - \frac{z}{3} \\ \frac{x}{3} + \frac{2y}{3} \quad \frac{z}{3} \\ -\frac{x}{3} + \frac{y}{3} + \frac{2z}{3} \end{pmatrix}$$

(c)

$$= \underbrace{\begin{pmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}}_{P} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

7. Your Name:______________________________

A matrix A is said to be *similar to* another matrix B if there is some invertible matrix Q such that $A = QBQ^{-1}$. (Beware that the matrix Q can be different for different pairs of matrices.)

5 a) Show that if A is similar to B, than B is similar to A.

5 b) Show that if A is similar to B and B is similar to C, then A is similar to C.

10 c) Show that if A is similar to B, then $\det(A - \lambda I) = \det(B - \lambda I)$ and hence A and B have the same set of eigenvalues, even counting multiplicities.
(Hint: note that $QQ^{-1} = I$.)

10 d) Show that if A is similar to B, then for each eigenvalue λ, A and B have the *same number* of (linearly independent) eigenvectors.
(Hint: relate the eigenvectors for A and B.)

(a) $A = QBQ^{-1}$, ~~$B = \ldots$~~

$\Longleftrightarrow Q^{-1}AQ = B \Longleftrightarrow \boxed{B = (Q^{-1})A(Q^{-1})^{-1}}$

Hence B is similar to A.

(b) $A = QBQ^{-1}$, $B = RCR^{-1}$

$\Longrightarrow A = Q(RCR^{-1})Q^{-1}$

$= (QR)C(R^{-1}Q^{-1})$

$= \underbrace{(QR)}_{\tilde{Q}}C\underbrace{(QR)^{-1}}_{\tilde{Q}^{-1}} = \boxed{\tilde{Q}C\tilde{Q}^{-1}}$

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(c)
$$\begin{aligned}\det(A-\lambda I) &= \det(QBQ^{-1}-\lambda QQ^{-1})\\ &= \det[Q(B-\lambda I)Q^{-1}]\\ &= (\det Q)\det(B-\lambda I)(\det Q^{-1})\\ &= \det(B-\lambda I)\end{aligned}$$

$= 1$

Hence $\det(A-\lambda I)=0 \iff \det(B-\lambda I)=0$

& A & B have the same <u>characteristic polynomial</u> & thus the <u>same eigenvalues</u>, (counting multiplicities.)

(d)
$$\begin{aligned} AX=\lambda X &\iff QBQ^{-1}X=\lambda X\\ &\iff B\underbrace{(Q^{-1}X)}_{Y}=\lambda\underbrace{(Q^{-1}X)}_{Y}\end{aligned}$$

Hence $AX=\lambda X \iff BY=\lambda Y$, where $Y=Q^{-1}X$.

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Now suppose for λ, A has k lin ind eigenvectors $X_1, X_2, \ldots, X_k$, i.e. $AX_i = \lambda X_i$, $i = 1, 2, \ldots, k$.

$$\boxed{\text{Then } Y_1 = Q^{-1}X_1,\ Y_2 = Q^{-1}X_2, \ldots, Y_k = Q^{-1}X_k \text{ are eigenvectors of } B}$$

Prove that $Y_1, Y_2, \ldots, Y_k$ are lin ind.

Pf: $c_1 Y_1 + c_2 Y_2 + \cdots + c_k Y_k = 0$

$\Rightarrow c_1 Q^{-1} X_1 + c_2 Q^{-1} X_2 + \cdots + c_k Q^{-1} X_k = \vec{0}$

$\Rightarrow Q^{-1}[c_1 X_1 + c_2 X_2 + \cdots + c_k X_k] = \vec{0}$) multiply by Q.

$\Rightarrow c_1 X_1 + c_2 X_2 + \cdots + c_k X_k = \vec{0}$) Since $X_1, X_2, \ldots, X_k$ are lin ind.

$\Rightarrow c_1 = c_2 = \cdots = c_k = 0$

$\Rightarrow Y_1, Y_2, \ldots, Y_k$ are lin ind.