Homework 11 Guidelines

#1 Network Primal Simplex

(1) Start from a spanning tree T

(2) Compute (primal) flow x_{ij} ($x_{ij} = 0$ if $(i,j) \notin T$)

Assume $x_{ij} \geq 0$ i.e. a feasible solution

(3) Compute dual (y_i) and dual slack variables (z_{ij})

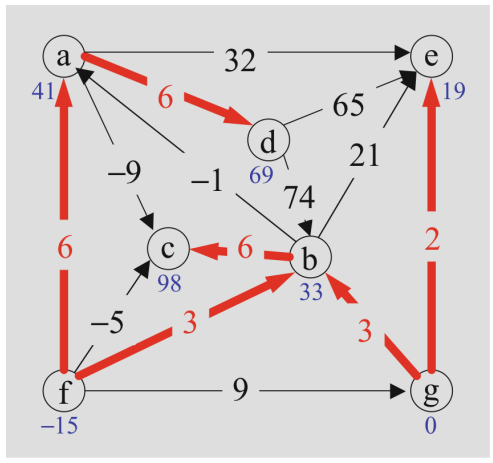
$\forall (i,j) \in T$, $y_j - y_i + z_{ij} = c_{ij}$ ($z_{ij} = 0$)

$\forall (i,j) \notin T$, $z_{ij} = c_{ij} - (y_j - y_i)$

Complementary slackness $x_{ij} z_{ij} = 0$

(4) Choose (i,j) $z_{ij} < 0$ (if none, opt.)

(5) Update tree



#2 Network Dual Simplex

(1) Start from a spanning tree T

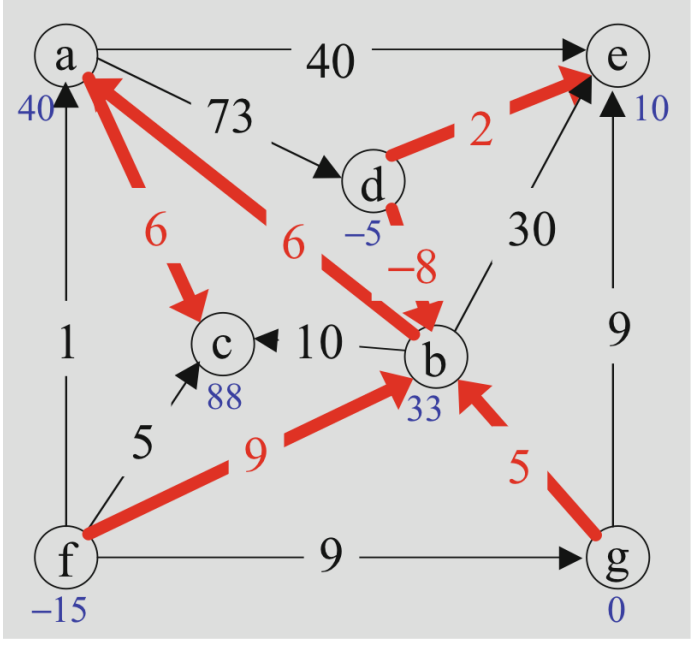
(2) Compute (primal) flow x_{ij} ($x_{ij} = 0$ if $(i,j) \notin T$)

dual variables, y_j, z_{ij}

Assume $z_{ij} \geq 0$ ie. a feasible dual solution

(3) Choose $x_{ij} < 0$ (if none, opt.)

(4) Update tree



#3 (Ex 14.12) If $\tilde{B}^{-1}$ exists, then arcs represented by B forms a spanning tree.

$\Updownarrow$

i.e. $\tilde{B}X = Y$ is uniquely solvable for any Y

Proof (by contradiction)

(1) what if the nodes are disconnected?

(2) what if the arcs form a node?

N - network, with m nodes and n (directed) arcs.

$(n \geq m-1)$

Incidence matrix $A^{m \times n} \longrightarrow \tilde{A}^{(m-1) \times n}$

delete one row

$\tilde{A}^{(m-1)\times n} = [\,|\;|\;|\;|\;\cdots\;|\,]$ $\}\, m-1$

each col. represents an arc

choose $m-1$ col. (i.e. choose $m-1$ arcs)

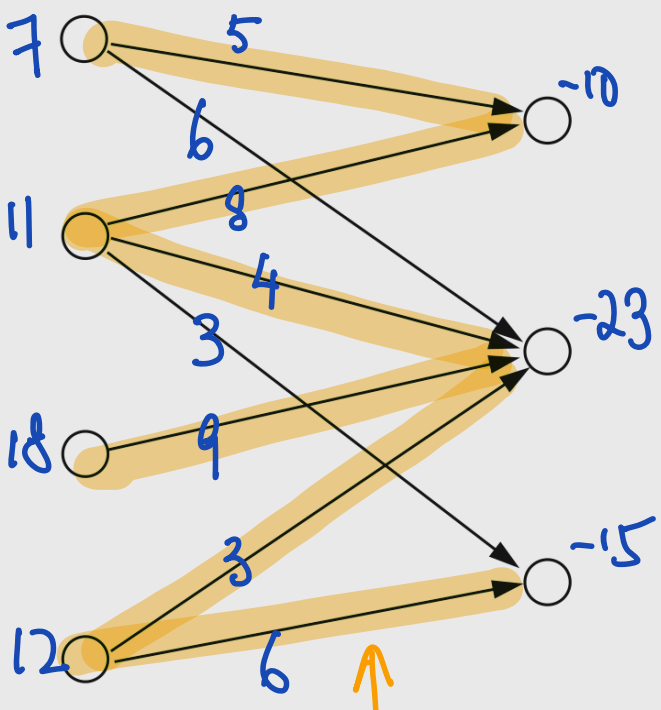
$\tilde{B}$

Thm 14.1 $\tilde{B}^{-1}$ exists if and only if the arcs form a spanning tree

Pf Spanning tree $\Longrightarrow$ $\tilde{B}$ is invertible

$\tilde{B}$ is invertible $\Longrightarrow$ Spanning tree (Ex 14.12)

#4 15.1



s_i \ d_j	-10	-23	-15
7	5	6	*
11	8	4	3
18	*	9	*
12	*	3	6

c_{ij}

$y_j - y_i = c_{ij}$

$5 - 0 = 5$

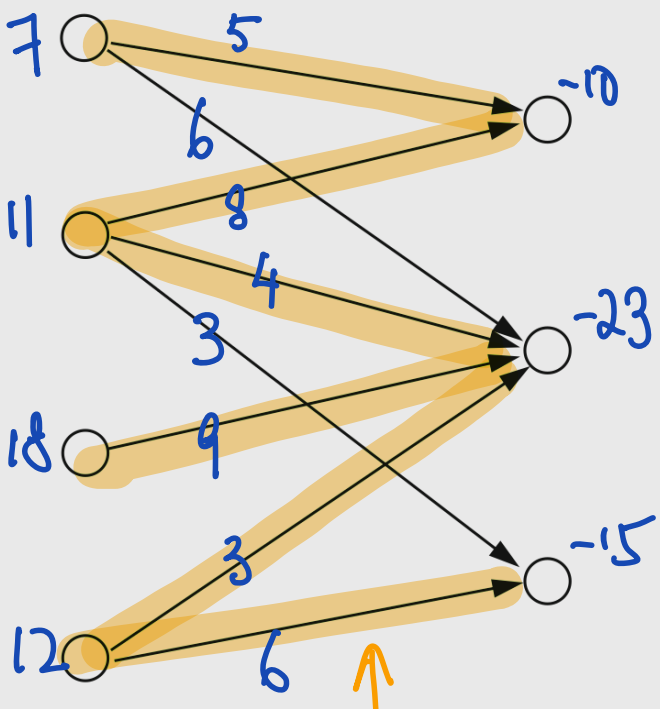
y_i \ y_j	5	1	4
0	7	5	*
-3	3	8	-4
-8	*	18	*
-2	*	-3	15

x_{ij}

$y_j - y_i + z_{ij} = c_{ij}$

$1 - 0 + 5 = 6$

#4 15.1



s_i \ d_j	−10	−23	−15
7	5	6	*
11	8	4	3
18	*	9	*
12	*	3	6

c_{ij}

$y_j - y_i = c_{ij}$

$5 - 0 = 5$

y_i \ y_j	5	1	4
0	7	5	*
−3	3	8	−4
−8	*	18	*
−2	*	−3	15

x_{ij}

z_{ij}

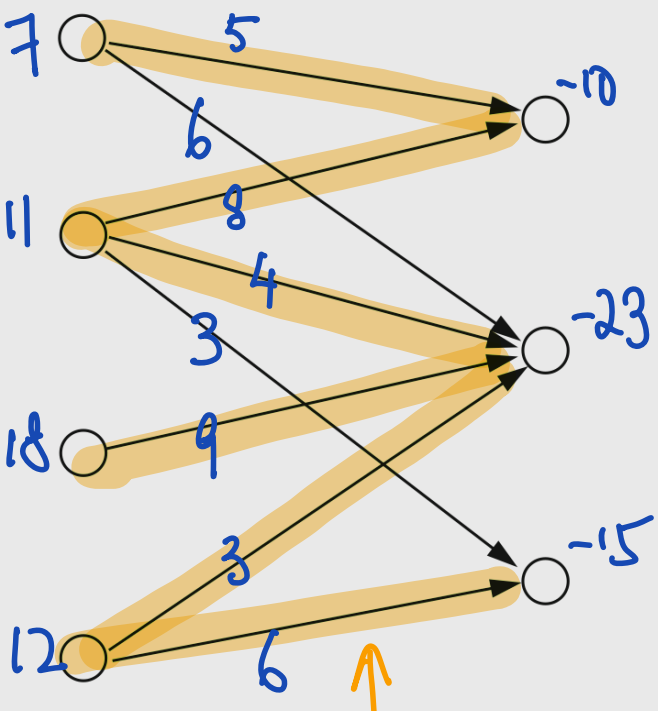
$y_j - y_i + z_{ij} = c_{ij}$

$1 - 0 + 5 = 6$

dual infeasible

primal feasible

#4 15.1



y_i \ y_j	5	1	4
0	7	5	*
−3	3	8	−4
−8	*	18	*
−2	*	−3	15

$(i=2, j=3)$

(1) $y_j - y_i + z_{ij} = c_{ij}$

$4 - (-3) + (-4) = 3$

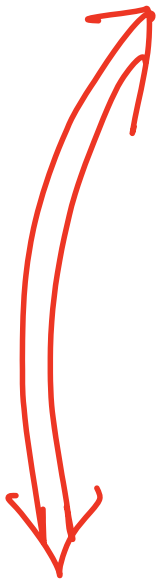
(2) Change c_{23} to 8. Then

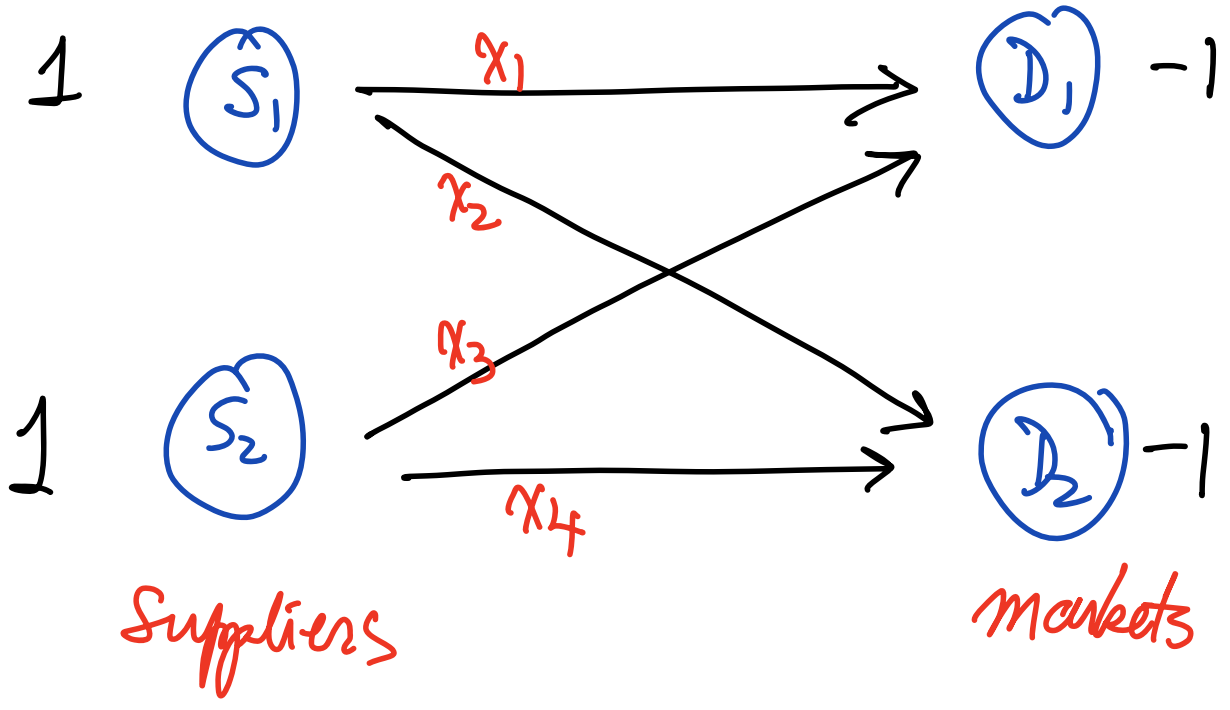
$z_{ij} = 1 > 0$

dual feasible →

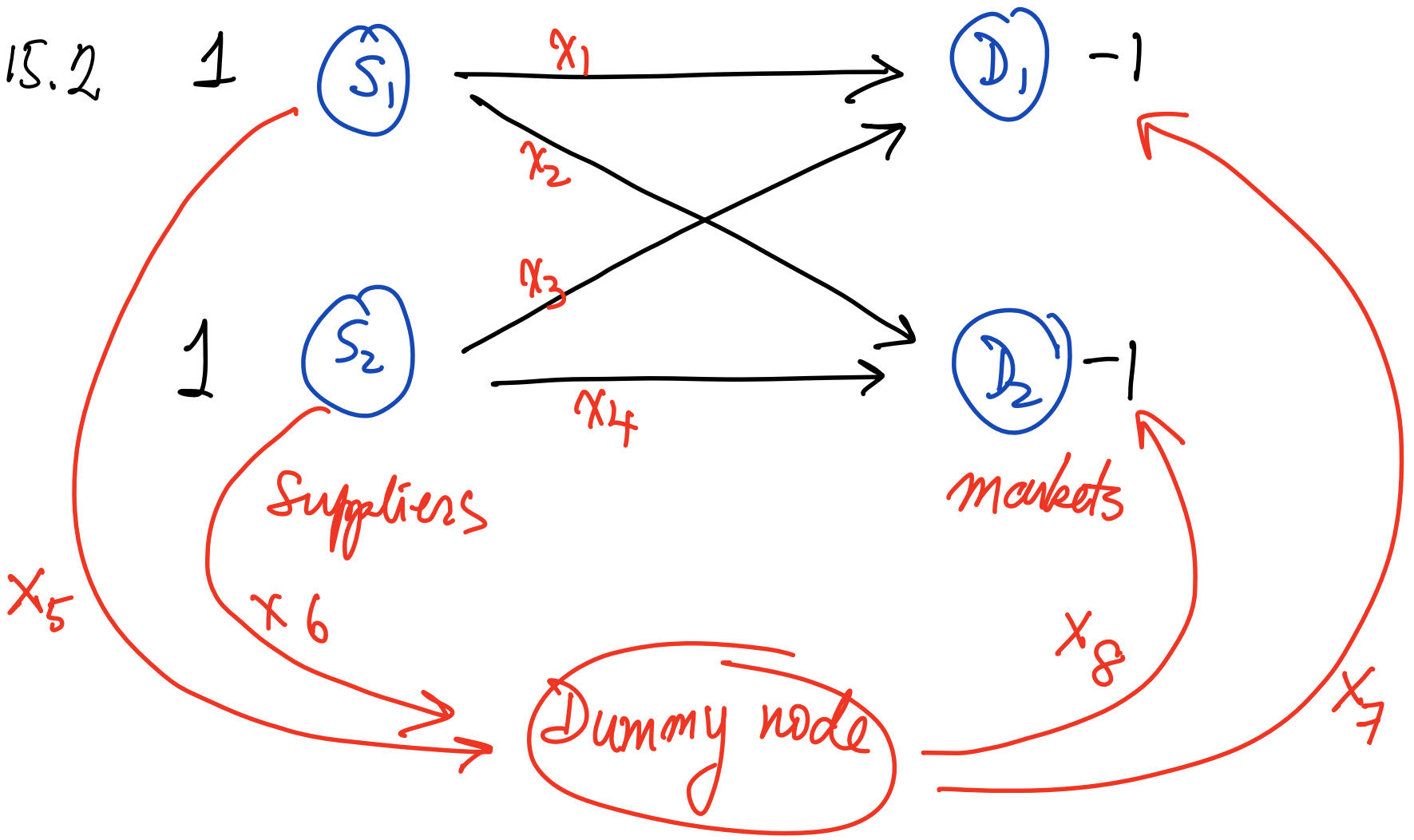
#4 15.2

$$\begin{aligned}
\text{maximize} \quad & 7x_1 - 3x_2 + 9x_3 + 2x_4 \\
\text{subject to} \quad & x_1 + x_2 \le 1 \quad \leftarrow S_1 \\
& x_3 + x_4 \le 1 \quad \leftarrow S_2 \\
& x_1 + x_3 \ge 1 \quad \leftarrow D_1 \\
& x_2 + x_4 \ge 1 \quad \leftarrow D_2 \\
& x_1,\ x_2,\ x_3,\ x_4 \ge 0.
\end{aligned}$$





#4 15.2



$$x_1 + x_2 + x_5 = 1 \quad \leftarrow S_1$$

$$x_3 + x_4 + x_6 = 1 \quad \leftarrow S_2$$

$$x_1 + x_3 + x_7 = 1 \quad \leftarrow D_1$$

$$x_2 + x_4 + x_8 = 1 \quad \leftarrow D_2$$

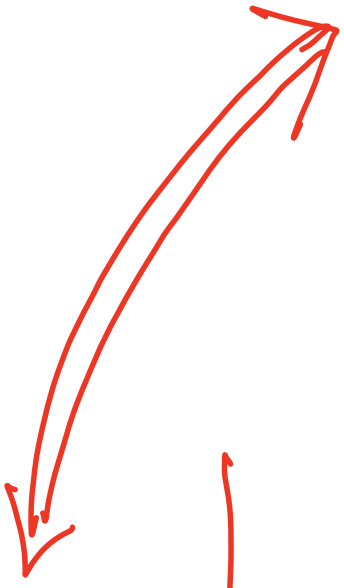
$$x_5 + x_6 = x_7 + x_8 \quad \leftarrow \text{Dummy node}$$

#4 15.2

maximize $7x_1 - 3x_2 + 9x_3 + 2x_4$

subject to

$$\begin{aligned}
x_1 + x_2 &\le 1 \\
x_3 + x_4 &\le 1 \\
x_1 + x_3 &\ge 1 \\
x_2 + x_4 &\ge 1 \\
x_1, x_2, x_3, x_4 &\ge 0.
\end{aligned}$$



$$\begin{aligned}
x_1 + x_2 &\le 1 \\
x_3 + x_4 &\le 1 \\
-x_1 - x_3 &\le -1 \\
-x_2 - x_4 &\le -1
\end{aligned}$$

$+$ (first two rows) $+$ (last two rows)

$$0 \le 0$$

#4 15.2

maximize $7x_1 - 3x_2 + 9x_3 + 2x_4$

subject to

$$\begin{aligned}
x_1 + x_2 &\leq 1 \\
x_3 + x_4 &\leq 1 \\
x_1 + x_3 &\geq 1 \\
x_2 + x_4 &\geq 1 \\
x_1, x_2, x_3, x_4 &\geq 0.
\end{aligned}$$

$$\begin{aligned}
x_1 + x_2 &\leq 1 \\
x_3 + x_4 &\leq 1 \\
-x_1 - x_3 &\leq -1 \\
-x_2 - x_4 &\leq -1 \\
\hline
0 &\leq 0
\end{aligned}$$

#15.4

Job	Duration (weeks)	Must be preceded by
0. Sign contract with buyer	0	–
1. Framing	2	0
2. Roofing	1	1
3. Siding	3	1
4. Windows	2.5	3
5. Plumbing	1.5	3
6. Electrical	2	2,4
7. Inside finishing	4	5,6
8. Outside painting	3	2,4
9. Complete the sale to buyer	0	7,8

(a), (b)

$\min t_9$

s.t.

$t_2 - t_1 \geq 2$

$t_3 - t_1 \geq 2$

$t_4 - t_3 \geq 3$

$t_5 - t_3 \geq 3$

$t_6 - t_2 \geq 1$

$t_6 - t_4 \geq 2.5$

$t_7 - t_5 \geq 1.5$

$t_7 - t_6 \geq 2$

$t_8 - t_2 \geq 1$

$t_8 - t_4 \geq 2.5$

$t_9 - t_7 \geq 4$

$t_9 - t_8 \geq 3$

15.6 (a) $\min t_9$

(P)

x_{12} ($t_2 - t_1 \geq 2$)
x_{13} ($t_3 - t_1 \geq 2$)
x_{34} ($t_4 - t_3 \geq 3$)
x_{35} ($t_5 - t_3 \geq 3$)
x_{26} ($t_6 - t_2 \geq 1$)
x_{46} ($t_6 - t_4 \geq 2.5$)
x_{57} ($t_7 - t_5 \geq 1.5$)
x_{67} ($t_7 - t_6 \geq 2$)
x_{28} ($t_8 - t_2 \geq 1$)
x_{48} ($t_8 - t_4 \geq 2.5$)
x_{79} ($t_9 - t_7 \geq 4$)
x_{89} ($t_9 - t_8 \geq 3$)

$t_9 \geq$ → x_{67}

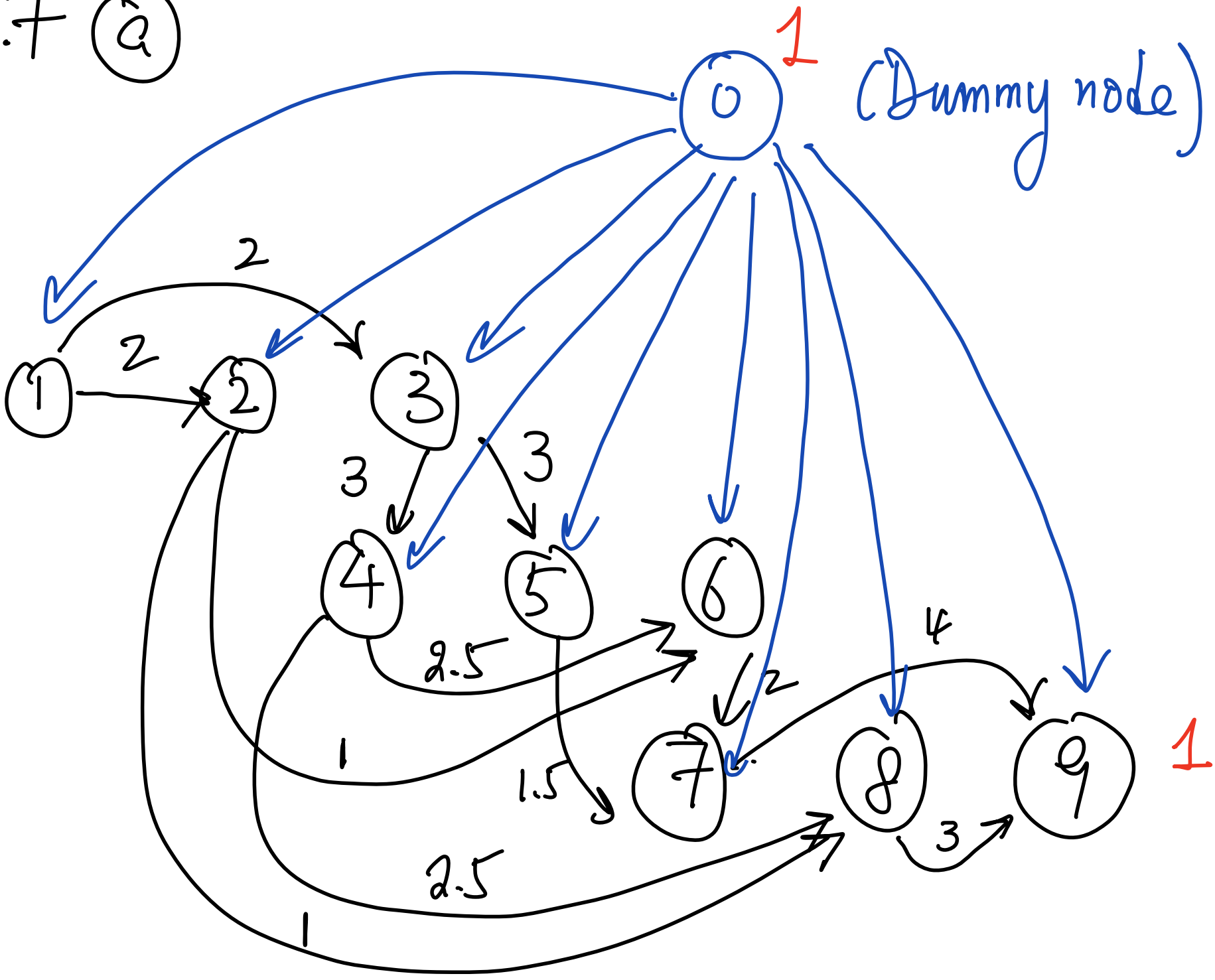
(D)

max $2x_{12} + 2x_{13} + \cdots + 4x_{79} + 3x_{89}$

s.t.

$-x_{12} - x_{13} \leq 0$
$x_{12} - x_{26} - x_{28} \leq 0$
$x_{13} - x_{34} - x_{35} \leq 0$
$x_{34} - x_{46} - x_{48} \leq 0$
$x_{35} - x_{57} \leq 0$
$x_{26} + x_{46} - x_{67} \leq 0$
$x_{57} + x_{67} - x_{79} \leq 0$
$x_{28} + x_{48} - x_{89} \leq 0$
$x_{79} + x_{89} \leq 1$

15.7 (a)



(b) $x_{ij} z_{ij} = 0$

i.e. $x_{ij}(t_j - t_i - d_i) = 0$

Hence if $t_j > t_i + d_i$, then $x_{ij} = 0$

or if $x_{ij} = 1$, then $t_j = t_i + d_i$