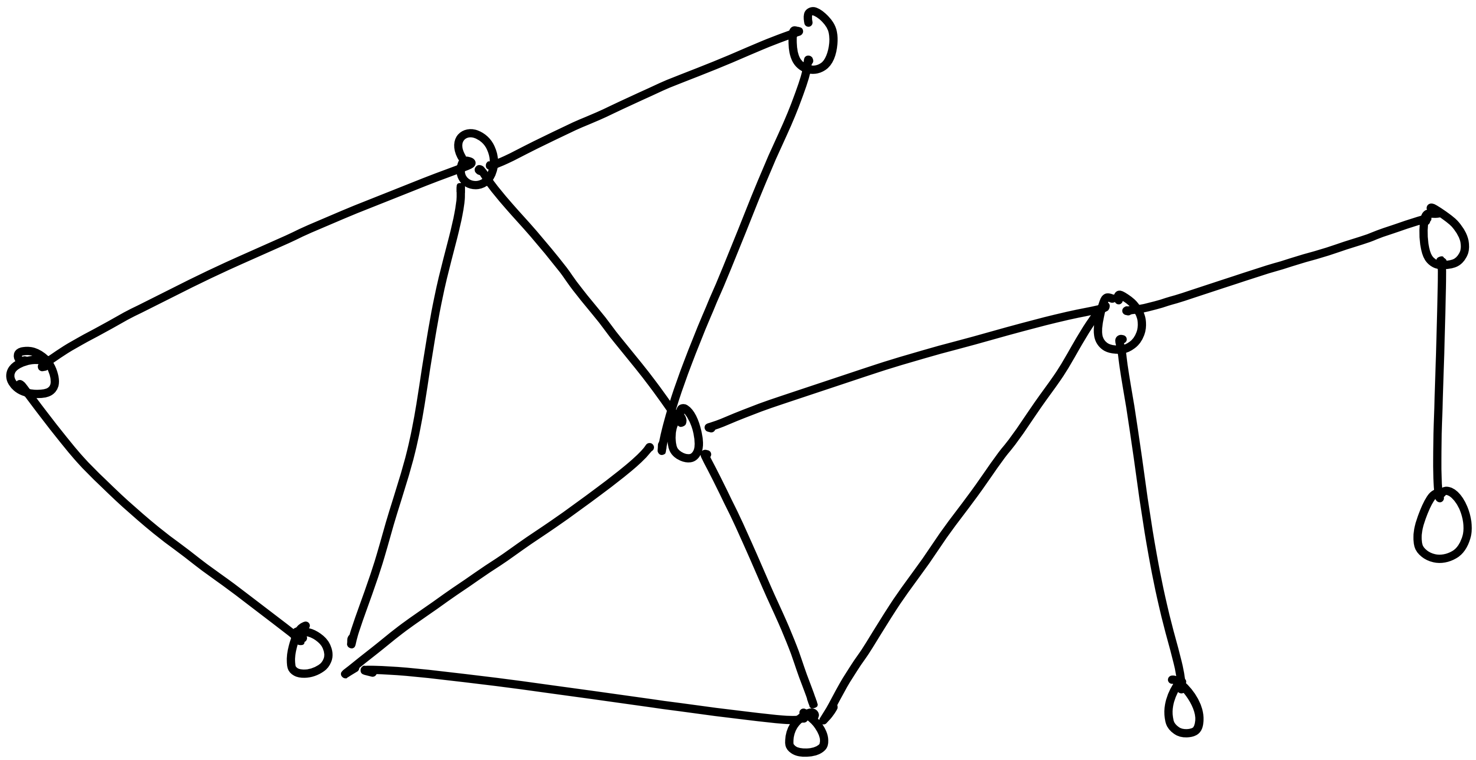
Minimum Spanning Tree (Bertsimas-Tsitsiklis)

Network / Graph $G = (N, \mathcal{E})$

(nodes) (edges/arcs) undirected



Minimum Spanning Tree (Bertsimas-Tsitsiklis)

Network / Graph $G = (N, \mathcal{E})$

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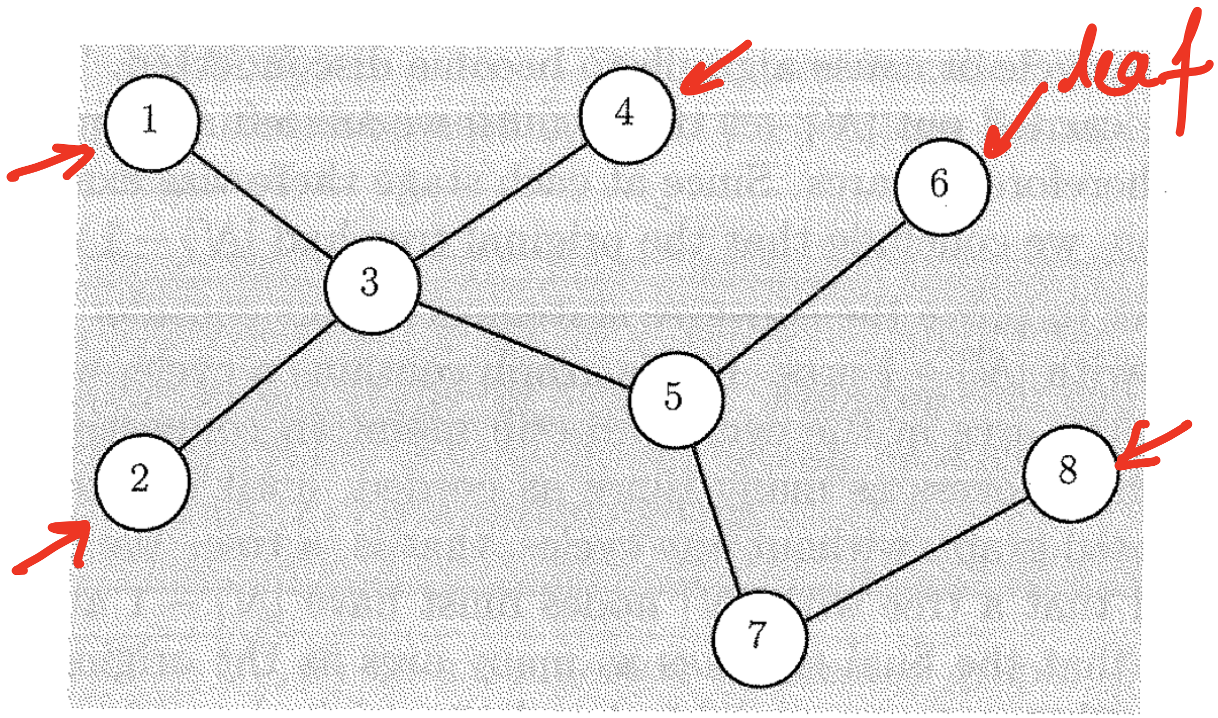
Trees

An undirected graph $G = (\mathcal{N}, \mathcal{E})$ is called a *tree* if it is connected and has no cycles. If a node of a tree has degree equal to 1, it is called a *leaf*. See Figure 7.3 for an illustration.



Minimum Spanning Tree (Bertsimas-Tsitsiklis)

Theorem 7.1

(a) *Every tree with more than one node has at least one leaf.*

(b) *An undirected graph is a tree if and only if it is connected and has $|\mathcal{N}| - 1$ arcs.*

(c) *For any two distinct nodes i and j in a tree, there exists a unique path from i to j.*

(d) *If we start with a tree and add a new arc, the resulting graph contains exactly one cycle (as long as we do not distinguish between cycles involving the same set of nodes).*

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(d) *If we start with a tree and add a new arc, the resulting graph contains exactly one cycle (as long as we do not distinguish between cycles involving the same set of nodes).*

(b) " $\Longrightarrow$ " delete a leaf, by induction

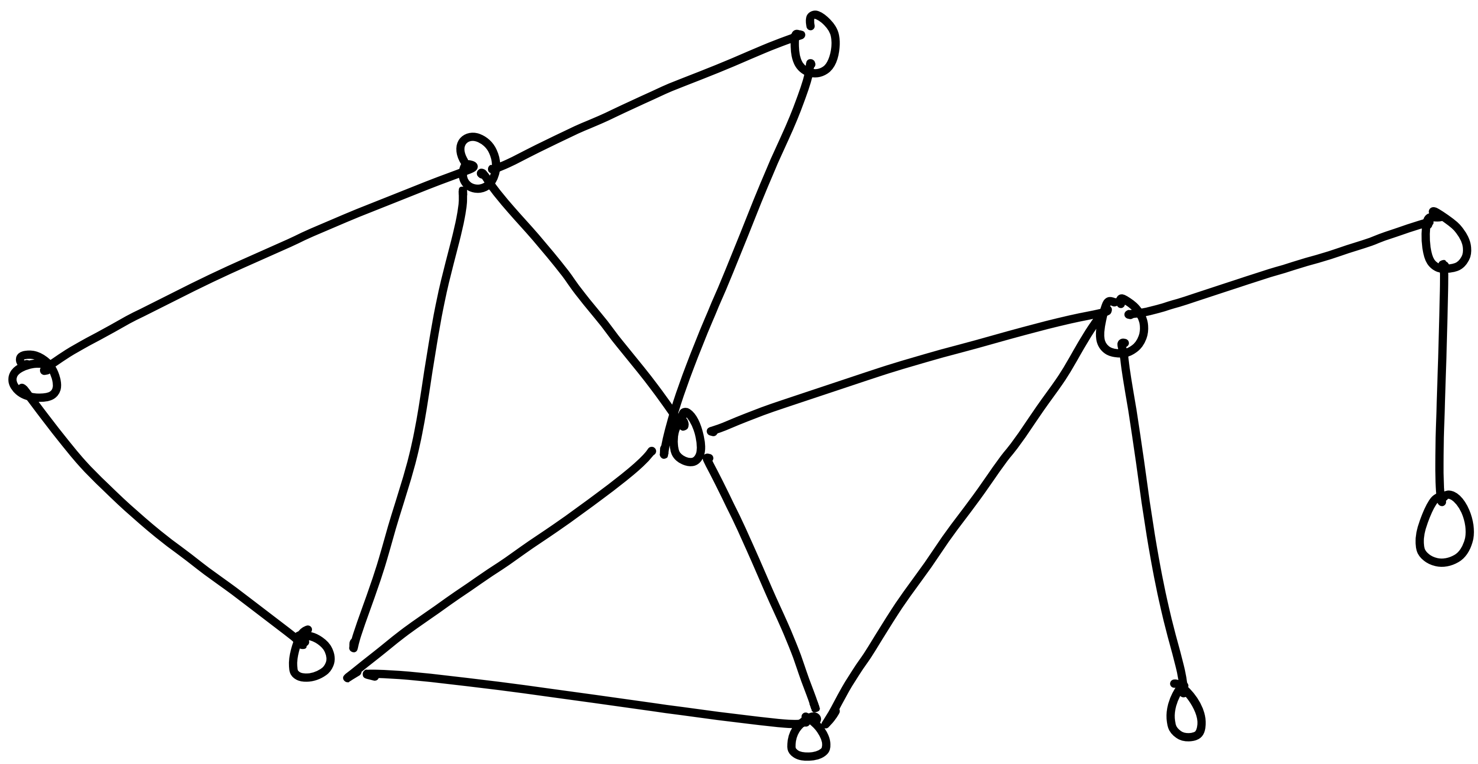
" $\Longleftarrow$ " if there is a cycle, delete an arc.

Minimum Spanning Tree (Bertsimas-Tsitsiklis)

Spanning trees

Given a connected undirected graph $G = (\mathcal{N}, \mathcal{E})$, let $\mathcal{E}_1$ be a subset of $\mathcal{E}$ such that $T = (\mathcal{N}, \mathcal{E}_1)$ is a tree. Such a tree is called a *spanning tree*.

connects all the nodes



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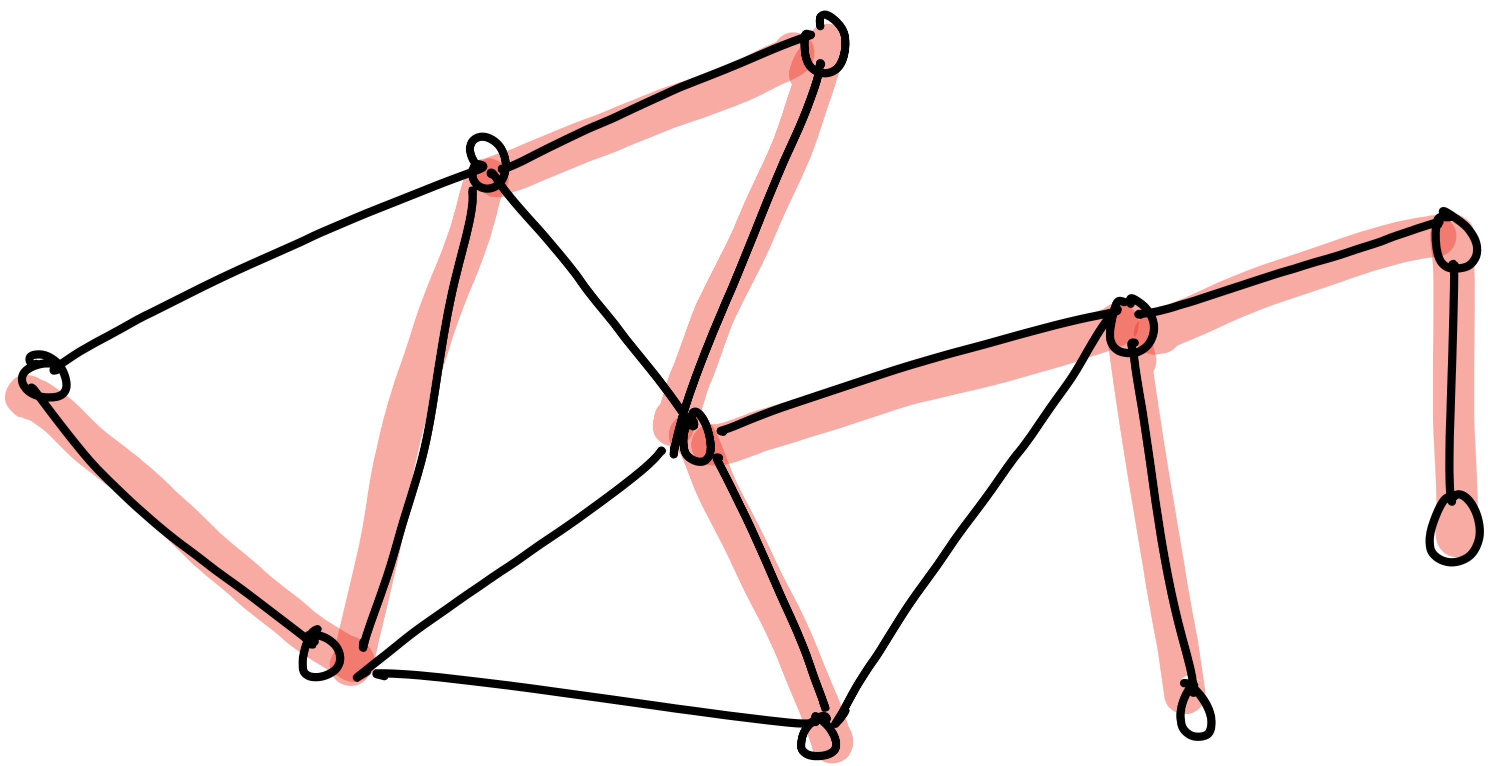


Minimum Spanning Tree (Bertsimas-Tsitsiklis)

Spanning trees

Given a connected undirected graph $G = (\mathcal{N}, \mathcal{E})$, let $\mathcal{E}_1$ be a subset of $\mathcal{E}$ such that $T = (\mathcal{N}, \mathcal{E}_1)$ is a tree. Such a tree is called a *spanning tree*.

connects all the nodes

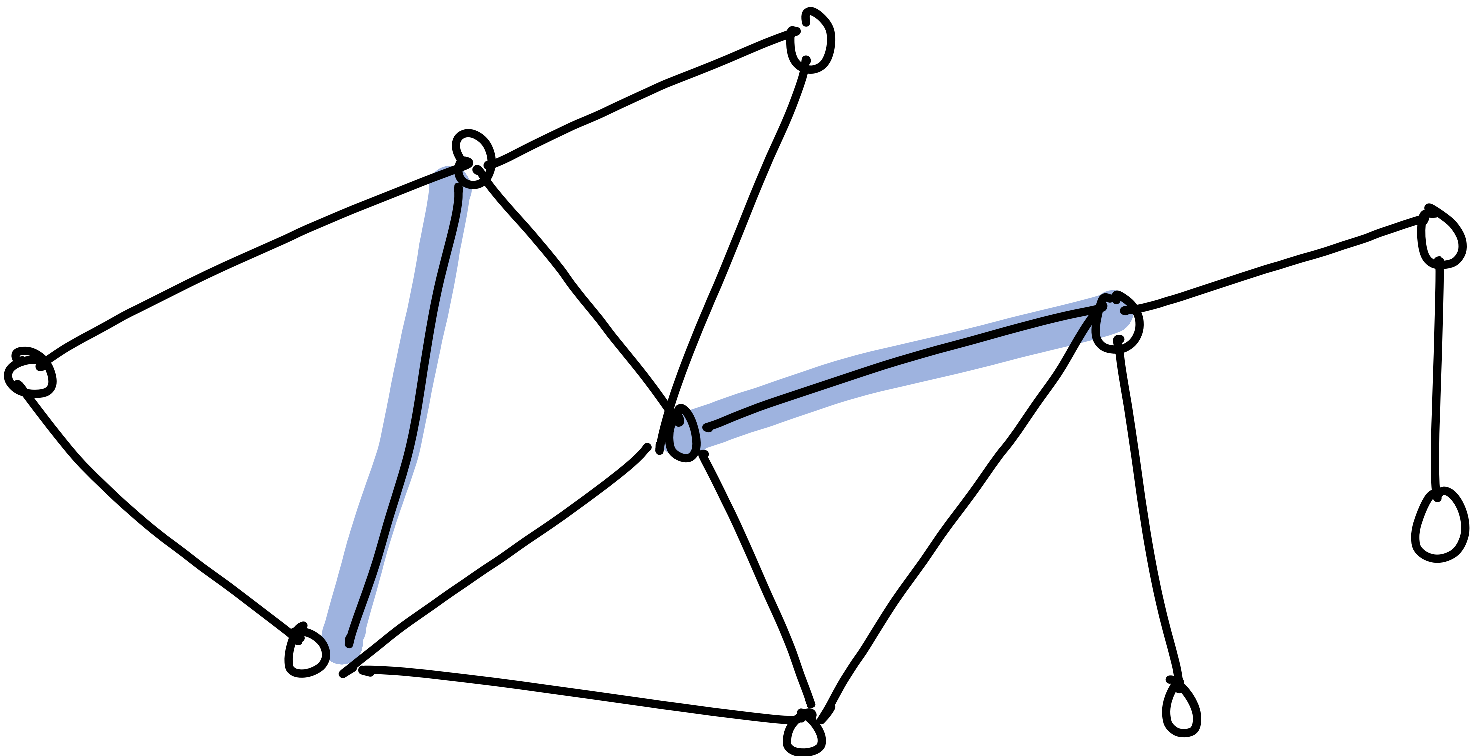
Theorem 7.2 *Let $G = (\mathcal{N}, \mathcal{E})$ be a connected undirected graph and let $\mathcal{E}_0$ be some subset of the set $\mathcal{E}$ of arcs. Suppose that the arcs in $\mathcal{E}_0$ do not form any cycles. Then, the set $\mathcal{E}_0$ can be augmented to a set $\mathcal{E}_1 \supset \mathcal{E}_0$ so that $(\mathcal{N}, \mathcal{E}_1)$ is a spanning tree.*

Minimum Spanning Tree (Bertsimas-Tsitsiklis)

Spanning trees

Given a connected undirected graph $G = (\mathcal{N}, \mathcal{E})$, let $\mathcal{E}_1$ be a subset of $\mathcal{E}$ such that $T = (\mathcal{N}, \mathcal{E}_1)$ is a tree. Such a tree is called a *spanning tree*.

connects all the nodes



(just keep adding)

Minimum Spanning Tree (Bertsimas-Tsitsiklis)

Spanning trees

Given a connected undirected graph $G = (\mathcal{N}, \mathcal{E})$, let $\mathcal{E}_1$ be a subset of $\mathcal{E}$ such that $T = (\mathcal{N}, \mathcal{E}_1)$ is a tree. Such a tree is called a *spanning tree*.

connects all the nodes

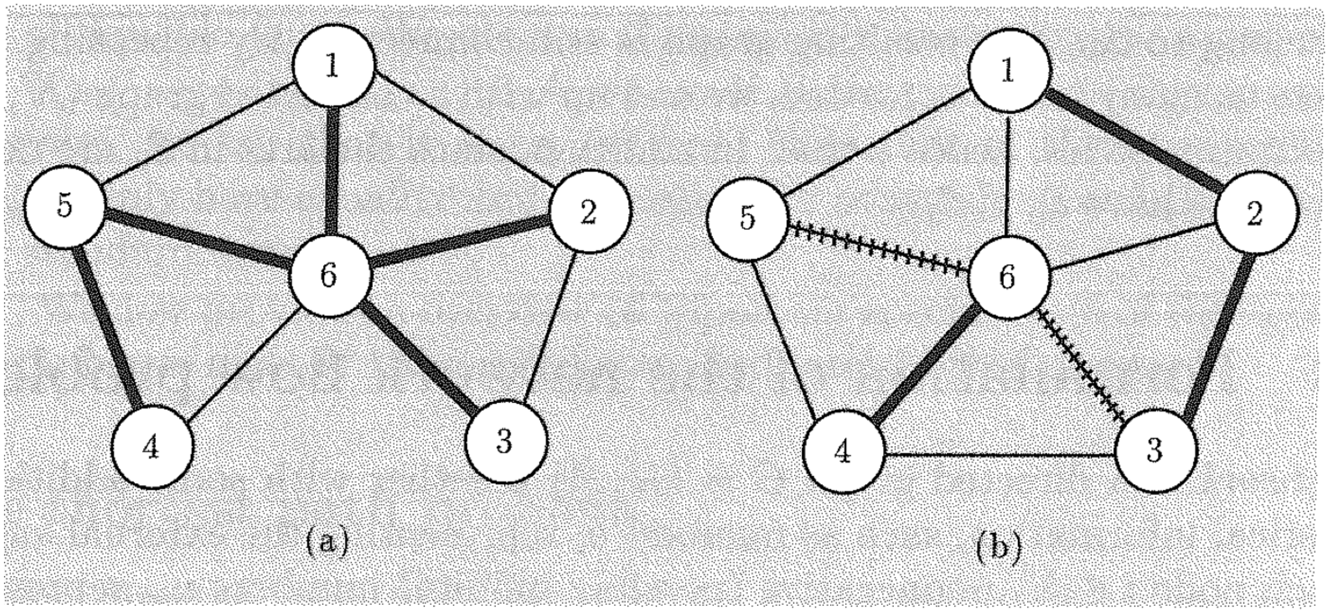
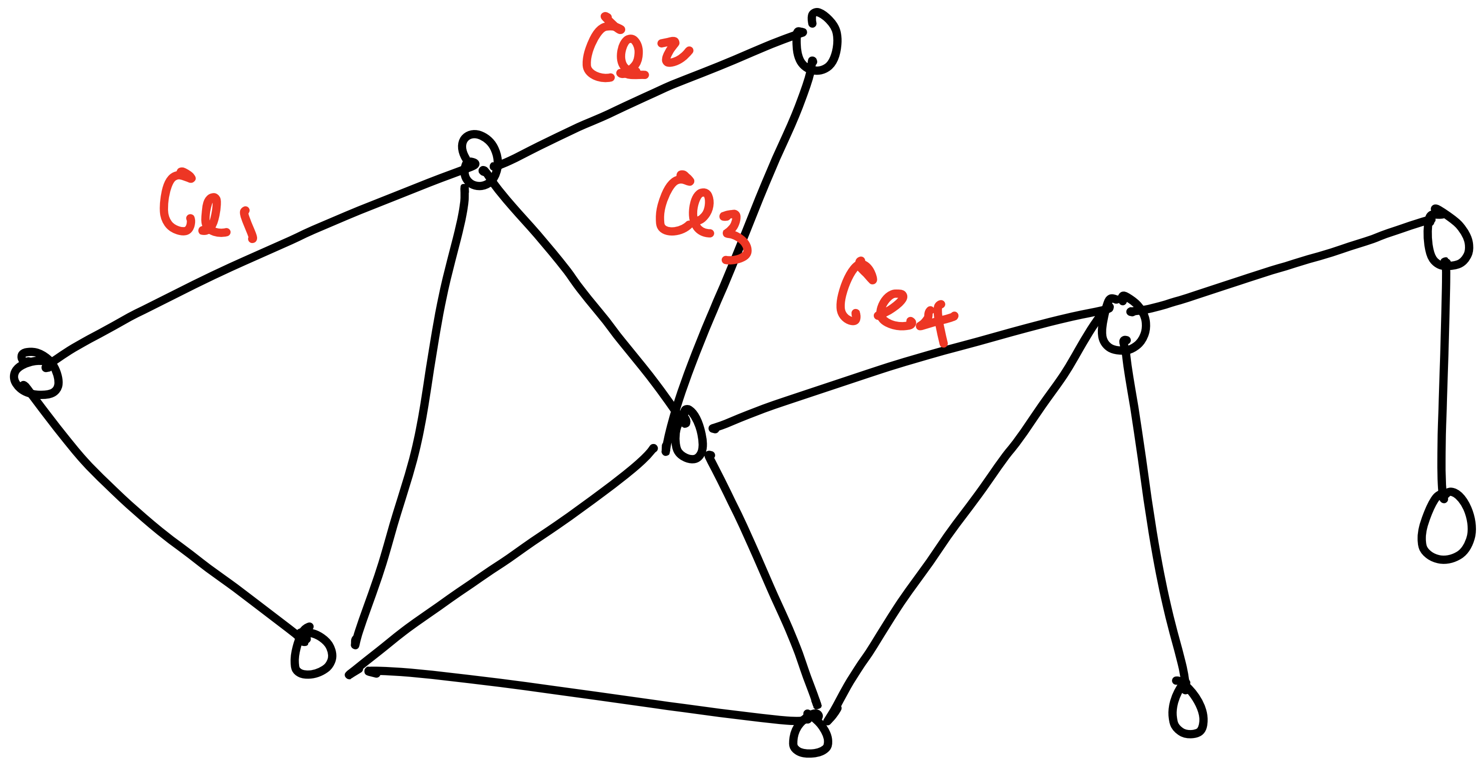


Figure 7.4: (a) An undirected graph. The thicker arcs form a spanning tree. (b) Another undirected graph. The arcs $\{1, 2\}$, $\{2, 3\}$, $\{4, 6\}$ do not form any cycle. They can be augmented to form a spanning tree, e.g., by adding arcs $\{3, 6\}$ and $\{5, 6\}$.

Minimum Spanning Tree (Bertsimas-Tsitsiklis)

7.10 The minimum spanning tree problem

We are given a connected *undirected* graph $G = (\mathcal{N}, \mathcal{E})$, with n nodes. For each edge $e \in \mathcal{E}$, we are also given a cost coefficient c_e. (Recall that an edge in an undirected graph is an unordered pair $e = \{i, j\}$ of distinct nodes in $\mathcal{N}$.) A *minimum spanning tree* (MST) is defined as a spanning tree such that the sum of the costs of its edges is as small as possible.



Minimum Spanning Tree (Bertsimas-Tsitsiklis)

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LP formulation

Minimize

$$z = \sum_{i-j \in E} w_{ij} x_{ij}$$

Subject to

$$\sum_{i-j \in E} x_{ij} = |V| - 1$$

$$\sum_{\substack{i-j \in E \\ i,j \in S}} x_{ij} \le |S| - 1 \quad \text{for each } S \subset V,\ S \neq \emptyset$$

$$x_{ij} \ge 0 \qquad \text{for each } i - j \in E$$

Minimum Spanning Tree (Bertsimas-Tsitsiklis)

Greedy Algorithm: at every step, find the next minimum arcs

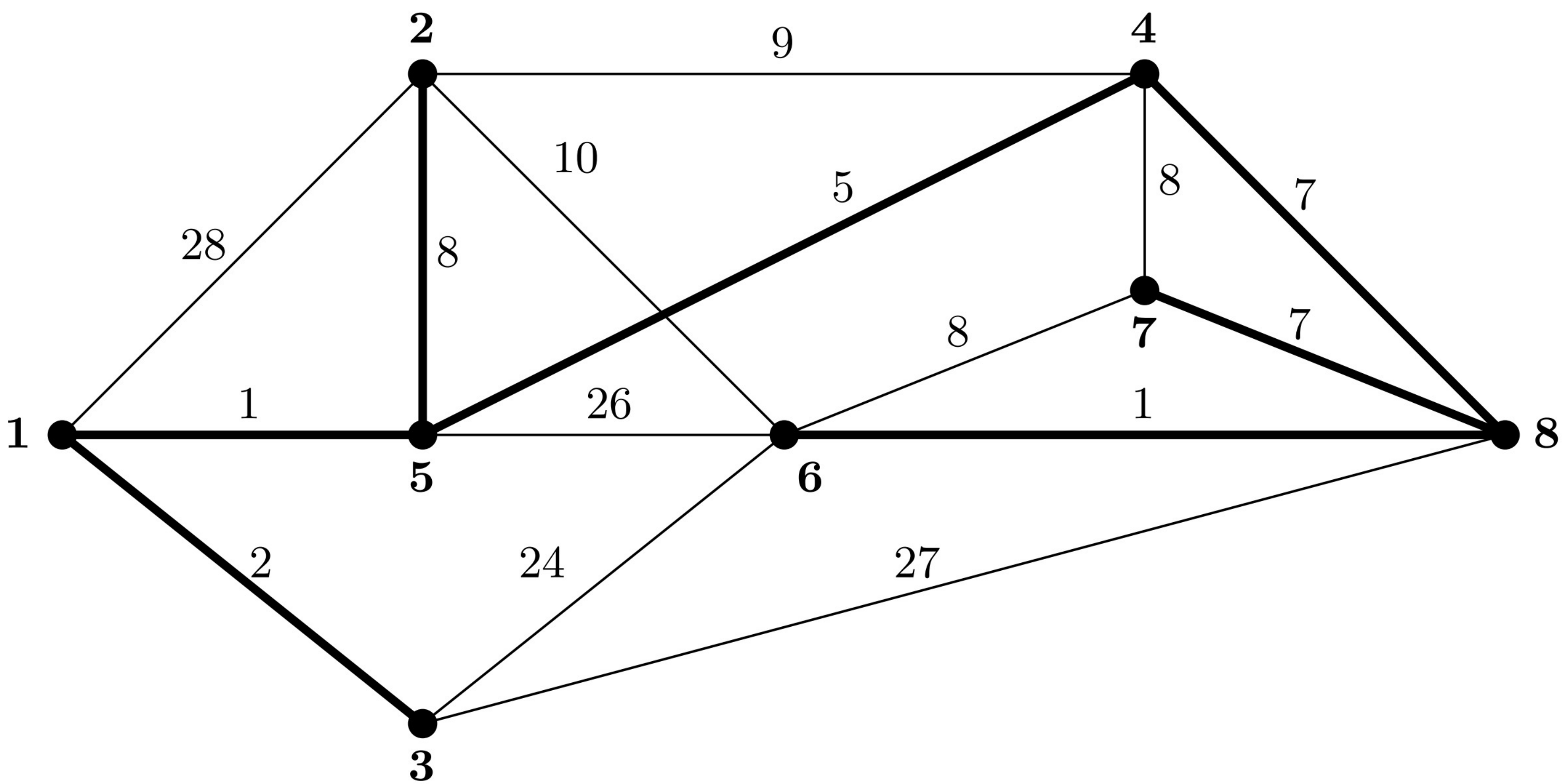
Greedy algorithm for the minimum spanning tree problem

1. The input to the algorithm is a connected undirected graph $G = (\mathcal{N}, \mathcal{E})$ and a coefficient c_e for each edge $e \in \mathcal{E}$. The algorithm is initialized with a tree $(\mathcal{N}_1, \mathcal{E}_1)$ that has a single node and no edges ($\mathcal{E}_1$ is empty).
2. Once $(\mathcal{N}_k, \mathcal{E}_k)$ is available, and if $k < n$, we consider all edges $\{i, j\} \in \mathcal{E}$ such that $i \in \mathcal{N}_k$ and $j \notin \mathcal{N}_k$. Choose an edge $e^* = \{i, j\}$ of this type whose cost is smallest. Let

$$\mathcal{N}_{k+1} = \mathcal{N}_k \cup \{j\}, \qquad \mathcal{E}_{k+1} = \mathcal{E}_k \cup \{e^*\}.$$

Minimum Spanning Tree (Bertsimas-Tsitsiklis)

Greedy Algorithm: at every step, find the next minimum arcs



Minimum Spanning Tree (Bertsimas-Tsitsiklis)

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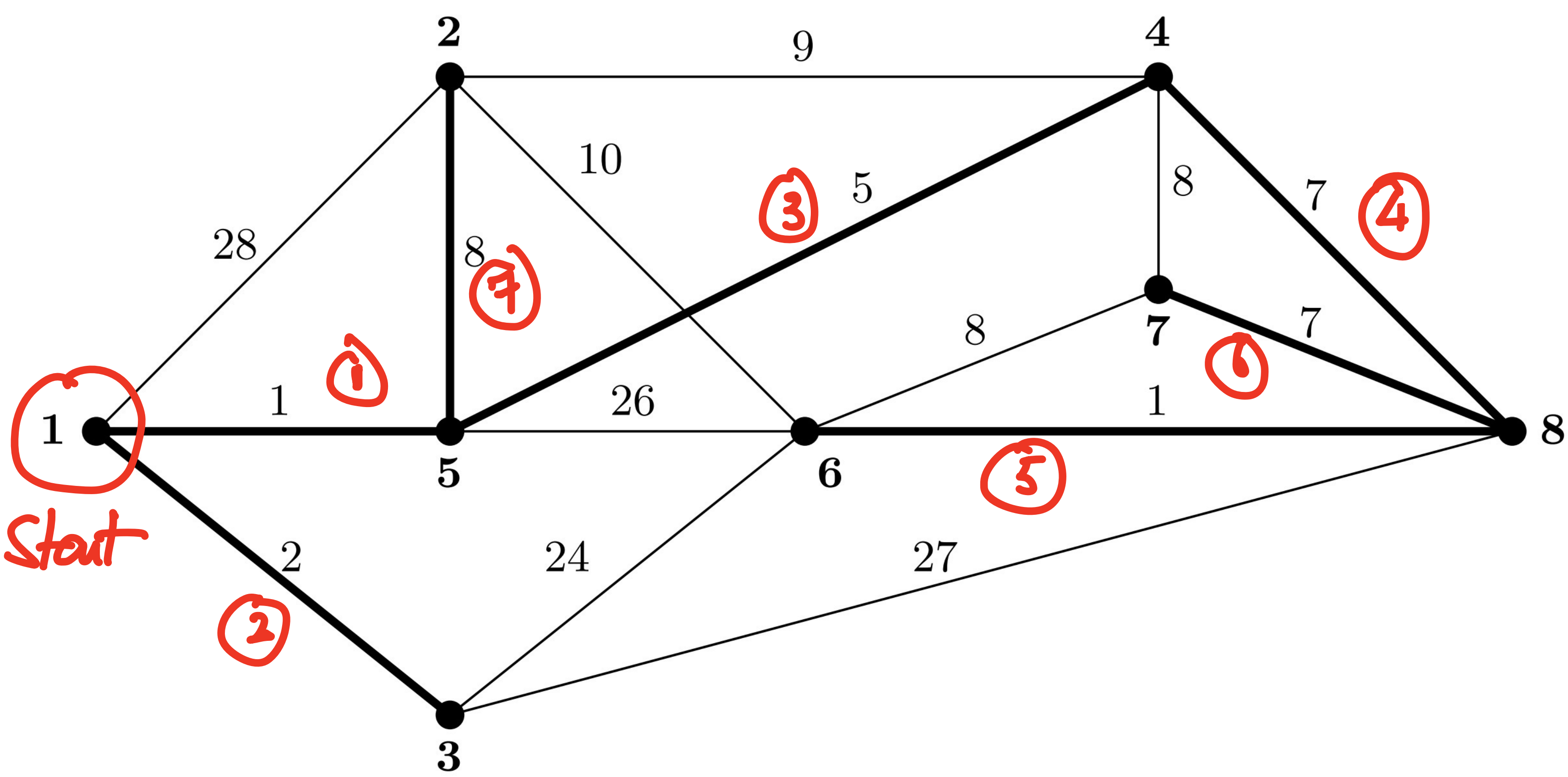

Minimum Spanning Tree (Bertsimas-Tsitsiklis)

Greedy Algorithm: at every step, find the next minimum arcs

Theorem 7.20 *For $k = 1, \ldots, n$, the tree $(\mathcal{N}_k, \mathcal{E}_k)$ is part of some MST. That is, there exists an MST $(\mathcal{N}, \overline{\mathcal{E}}_k)$ such that $\mathcal{E}_k \subset \overline{\mathcal{E}}_k$.*

Pf (by induction on k)

$(\mathcal{N}_k, \mathcal{E}_k)$

(1) $k=1$

(2) $k<n \Rightarrow k+1$