Duality

(P) max $c^T x$

s.t. $Ax \leq b$

$x \geq 0$

y^T

$$y^T A x \leq y^T b \qquad y \geq 0$$

$$\underbrace{c^T x \leq y^T A x}_{c^T \leq y^T A} \leq y^T b$$

$(x \geq 0)$

(D) min $b^T y$

s.t. $A^T y \geq c$

$y \geq 0$

Complementary Slackness (at optimal)

$$\begin{tikzcd}
c^T x \arrow[rrrr, leftrightarrow, bend right] & \underset{=}{\cancel{\leq}} & y^T A x & \underset{=}{\cancel{\leq}} & y^T b
\end{tikzcd}$$

$$c^T x = y^T A x$$

$$0 = \underbrace{(y^T A - c^T)}_{z} x$$

$$0 = z^T x$$

$$\boxed{x_j z_j = 0}$$

$$y^T A x = y^T b$$

$$y^T \underbrace{(Ax - b)}_{w} = 0$$

$$y^T w = 0$$

$$\boxed{y_i w_i = 0}$$

THE GRAPHIC METHOD

[C] p.260

The geometric interpretation of linear programming leads to a geometric procedure for solving LP problems with two variables. For illustration, let us consider the problem

$$
\begin{array}{lrl}
\text{maximize} & x_1 + x_2 & \\
\text{subject to} & 2x_1 + x_2 \le & 14 \\
 & -x_1 + 2x_2 \le & 8 \\
 & 2x_1 - x_2 \le & 10 \\
 & x_1, x_2 \ge & 0.
\end{array}
\tag{17.4}
$$

Its region of feasibility is shown in Figure 17.7. together with the line representing all the points

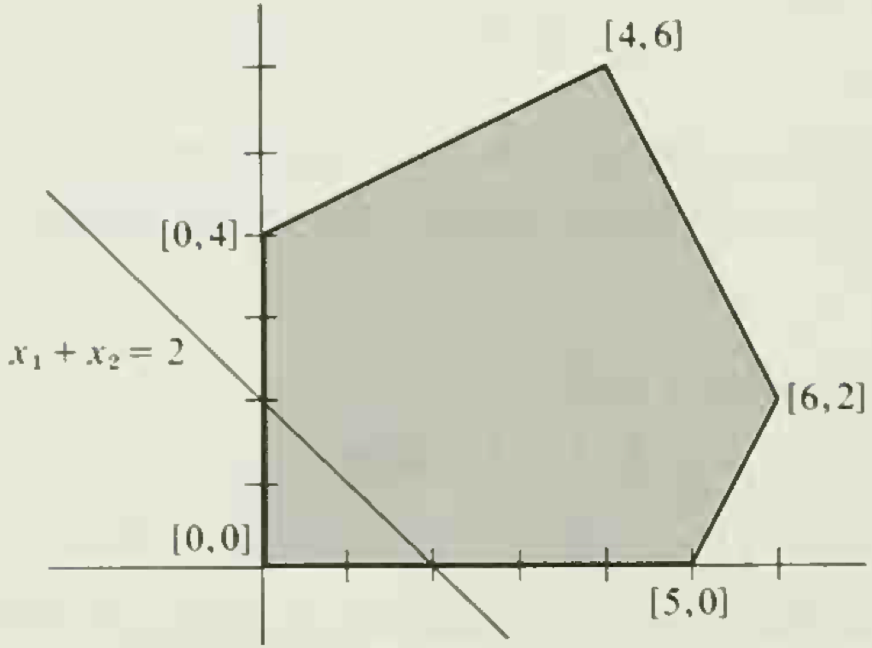


Figure 17.7

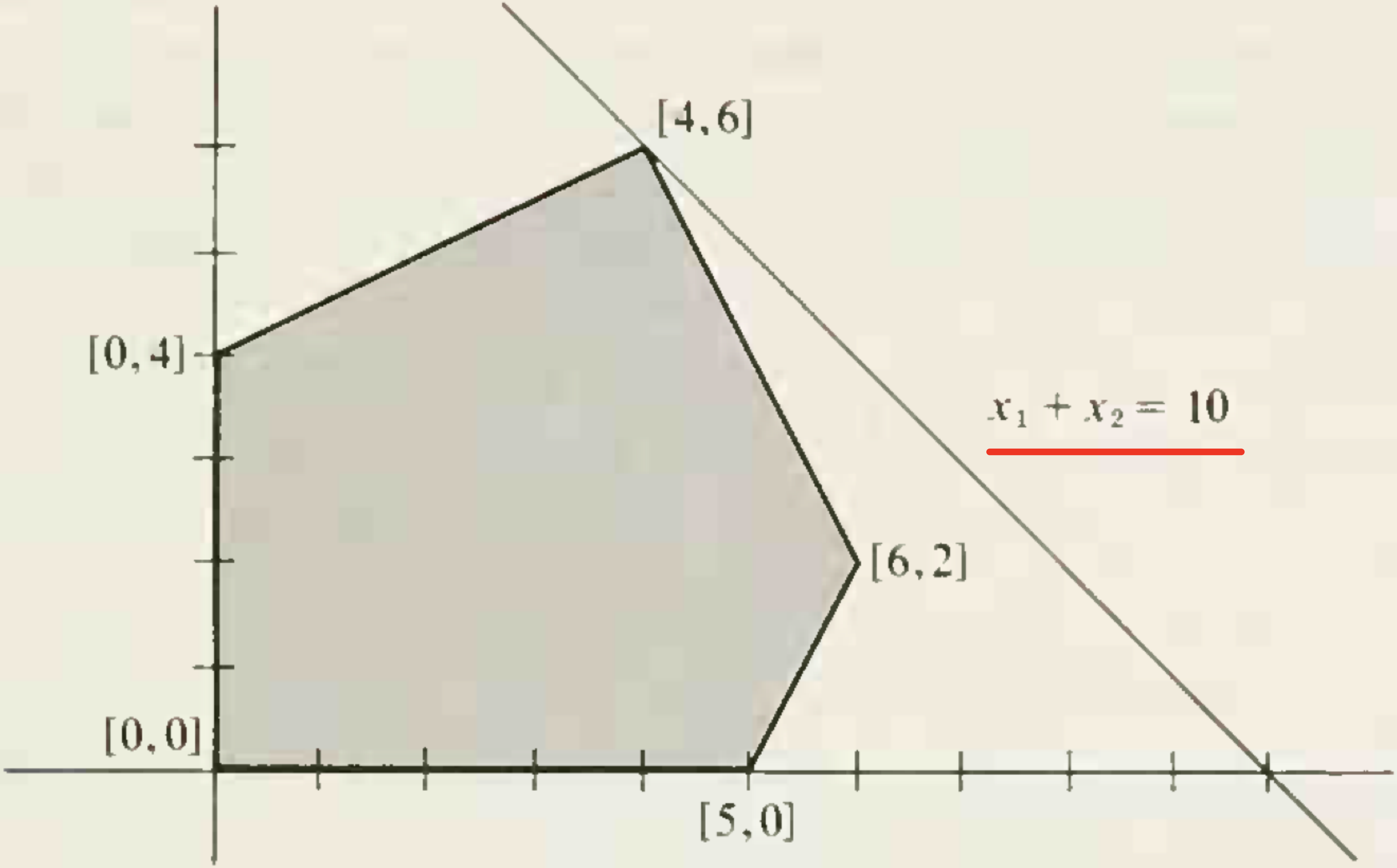


Figure 17.8

$$\max \quad x_1+x_2$$

$$(P) \quad \text{s.t.} \quad \begin{cases} 2x_1+x_2 \leqslant 14 \\ -x_1+2x_2 \leqslant 8 \\ 2x_1-x_2 \leqslant 10 \\ x_1, x_2 \geqslant 0 \end{cases}$$

$$\begin{cases} y_1(2x_1+x_2) \leqslant 14y_1 \\ y_2(-x_1+2x_2) \leqslant 8y_2 \\ y_3(2x_1-x_2) \leqslant 10y_3 \end{cases}$$

$$(2y_1-y_2+2y_3)x_1+(y_1+2y_2-y_3)x_2 \leqslant 14y_1+8y_2+10y_3$$

$$(P) \quad \begin{array}{ll} \max & x_1 + x_2 \\ \text{s.t.} & \begin{cases} 2x_1 + x_2 \leq 14 \\ -x_1 + 2x_2 \leq 8 \\ 2x_1 - x_2 \leq 10 \end{cases} \\ & x_1, x_2 \geq 0 \end{array}$$

$$(D) \quad \begin{array}{ll} \min & 14y_1 + 8y_2 + 10y_3 \\ & \begin{cases} 2y_1 - y_2 + 2y_3 \geq 1 \\ y_1 + 2y_2 - y_3 \geq 1 \end{cases} \\ & y_1, y_2, y_3 \geq 0 \end{array}$$

$x_1 = 4,\ x_2 = 6$

$$\max \quad x_1 + x_2$$

$$(P) \quad \text{s.t.} \quad \begin{cases} 2x_1 + x_2 \leq 14 \\ -x_1 + 2x_2 \leq 8 \\ 2x_1 - x_2 \leq 10 \end{cases}$$

$$x_1, x_2 \geq 0$$

$$(D) \quad \min \quad 14y_1 + 8y_2 + 10y_3$$

$$\begin{cases} 2y_1 - y_2 + 2y_3 \geq 1 \\ y_1 + 2y_2 - y_3 \geq 1 \end{cases}$$

$$y_1, y_2, y_3 \geq 0$$

$x_1 = 4, \ x_2 = 6$

$z_1 = 0, \ z_2 = 0$

$$\max \quad x_1 + x_2 = 10$$

$$(P) \quad \text{s.t.} \quad \begin{cases} 2x_1 + x_2 \leqslant 14 & \leftarrow w_1 = 0 \\ -x_1 + 2x_2 \leqslant 8 & \leftarrow w_2 = 0 \\ 2x_1 - x_2 \leqslant 10 & \leftarrow w_3 > 0 \Rightarrow \underline{y_3 = 0} \\ x_1, x_2 \geq 0 & \end{cases}$$

$$(D) \quad \min \quad 14y_1 + 8y_2 + 10y_3 = 10$$

$$\begin{cases} 2y_1 - y_2 + 2y_3 \geq 1 & \leftarrow z_1 = 0 \\ y_1 + 2y_2 - y_3 \geq 1 & \leftarrow z_2 = 0 \\ y_1, y_2, y_3 \geqslant 0 & \end{cases}$$

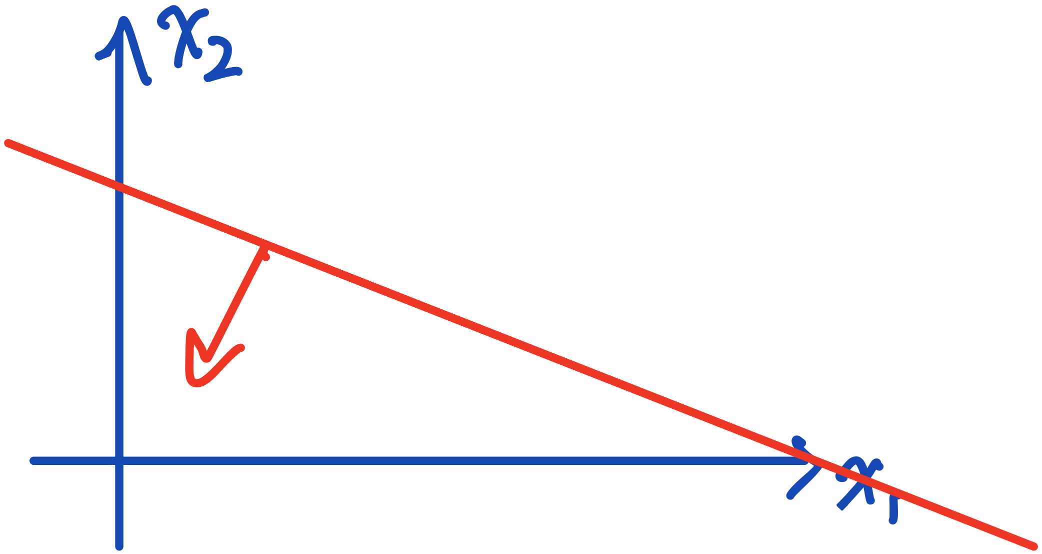
$$\underline{y_1 = \frac{3}{5}, \quad y_2 = \frac{1}{5}}$$

Interpretation of

$$(\underline{2y_1 - y_2 + 2y_3})x_1 + (\underline{y_1 + 2y_2 - y_3})x_2 \le \underline{14y_1 + 8y_2 + 10y_3}$$

$$\boxed{p(y_1,y_2,y_3)x_1 + q(y_1,y_2,y_3)x_2 \le r(y_1,y_2,y_3)}$$

half space



$$y_3=0 \Rightarrow (2y_1-y_2)x_1+(y_1+2y_2)x_2 \le 14y_1+8y_2$$

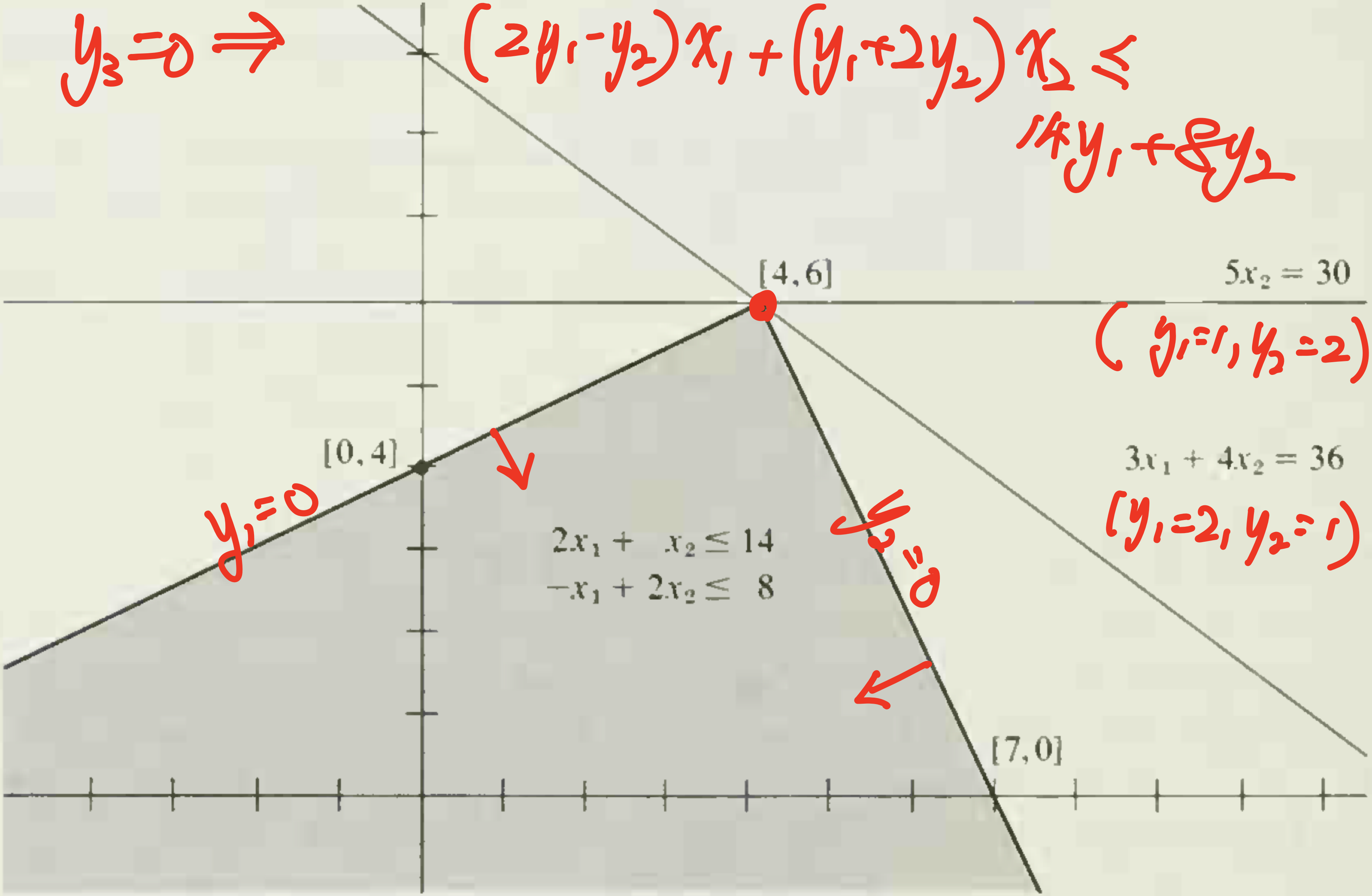


Figure 17.9

Sensitivity Analysis (what if c, b change)

(P) max $\zeta = c^T x$

s.t. $Ax \leq b$

$x \geq 0$

(D) min $\xi = b^T y$

s.t. $A^T y \geq c$

$y \geq 0$

Sensitivity Analysis (what if c, b change)

$$(P) \quad \max \ \zeta = c^T x \qquad (D) \quad \min \ \xi = b^T y$$

$$\text{s.t.} \quad Ax \le b \qquad \qquad \text{s.t.} \quad A^T y \ge c$$

$$x \ge 0 \qquad \qquad y \ge 0$$

$$\zeta(x_B, x_N) = c_B^T B^{-1} b - \left((B^{-1} N)^T c_B - c_N \right)^T x_N$$

$$x_B = B^{-1} b - (B^{-1} N) x_N$$

at opt.

$$x_N^* = 0, \quad x_B^* = B^{-1} b \ge 0$$

$$(B^{-1} N)^T c_B - c_N \ge 0$$

$$\zeta^* = c_B^T x_B^* = c^T x^*$$

Sensitivity Analysis (what if c, b change)

$$(P)\quad \max\ \zeta = c^T x \qquad\qquad (D)\quad \min\ \xi = b^T y$$

$$\text{s.t.}\quad Ax \leq b \qquad\qquad \text{s.t.}\quad A^T y \geq c$$

$$x \geq 0 \qquad\qquad y \geq 0$$

$$-\xi(z_B, z_N) = -c_B^T B^{-1} b - (B^{-1} b)^T z_B$$

$$z_N = \left((B^{-1}N)^T c_B - c_N\right) + (B^{-1}N)^T z_B$$

at opt. $z_B^* = 0,\quad z_N^* = (B^{-1}N)^T c_B - c_N \geq 0$

$$B^{-1} b \geq 0$$

$$\xi^* = c_B^T B^{-1} b = b^T y^* \quad \left(y^* = (B^{-1})^T c_B\right)$$

Sensitivity Analysis (what if c, b change)

(P) max $\zeta = c^T x$

s.t. $Ax \le b$

$x \ge 0$

(D) min $\xi = b^T y$

s.t. $A^T y \ge c$

$y \ge 0$

at Opt. $\zeta^* = c^T x^* = b^T y^* = \xi^*$

if only c changes

$$\Delta \zeta^* = (\Delta c)^T x^*$$

Sensitivity Analysis (what if c, b change)

(P) max $\zeta = c^T x$

s.t. $Ax \leq b$

$x \geq 0$

(D) min $\xi = b^T y$

s.t. $A^T y \geq c$

$y \geq 0$

at Opt. $\zeta^* = c^T x^* = b^T y^* = \xi^*$

if only b changes

$$\Delta \zeta^* (= \Delta \xi^*) = (\Delta b)^T y^*$$

y^*: shadow prices

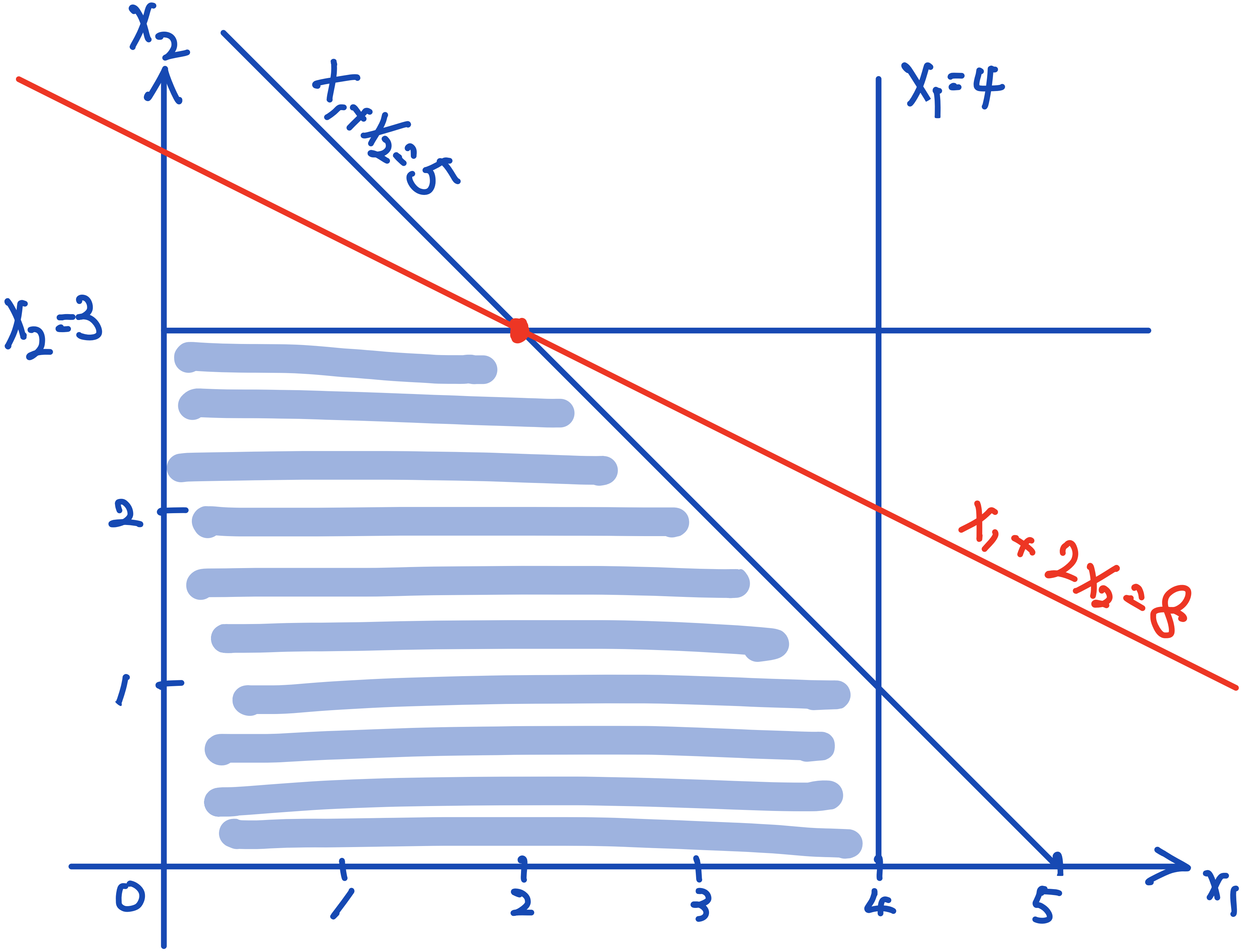
4. Consider the following linear programming problem:

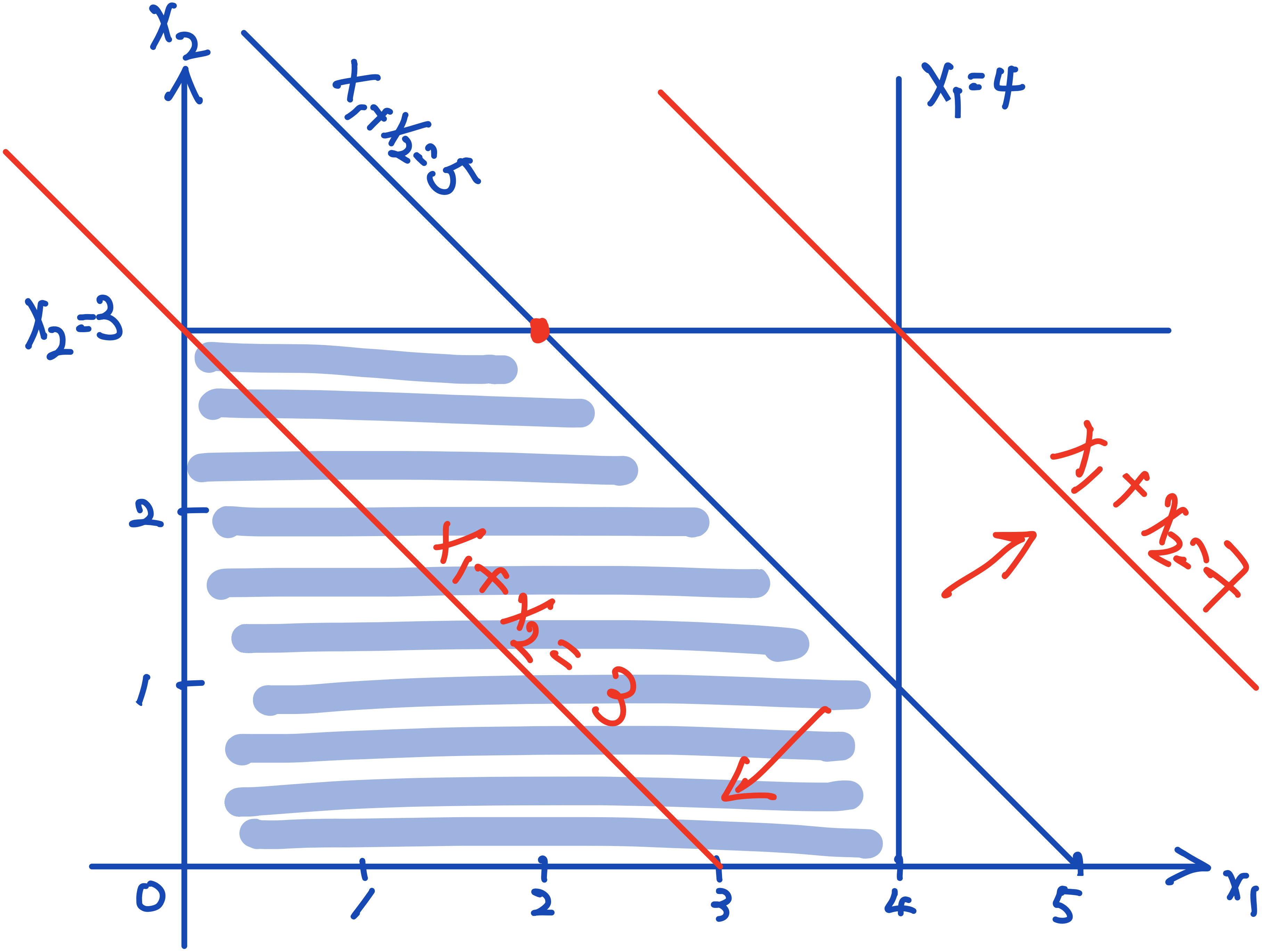
$$\begin{array}{lrl} \text{maximize} & x_1 + 2x_2 & \\ \text{subject to} & x_1 + x_2 & \leq 5 \\ & x_1 & \leq 4 \\ & x_2 & \leq 3 \\ & x_1, x_2 & \geq 0 \end{array}$$

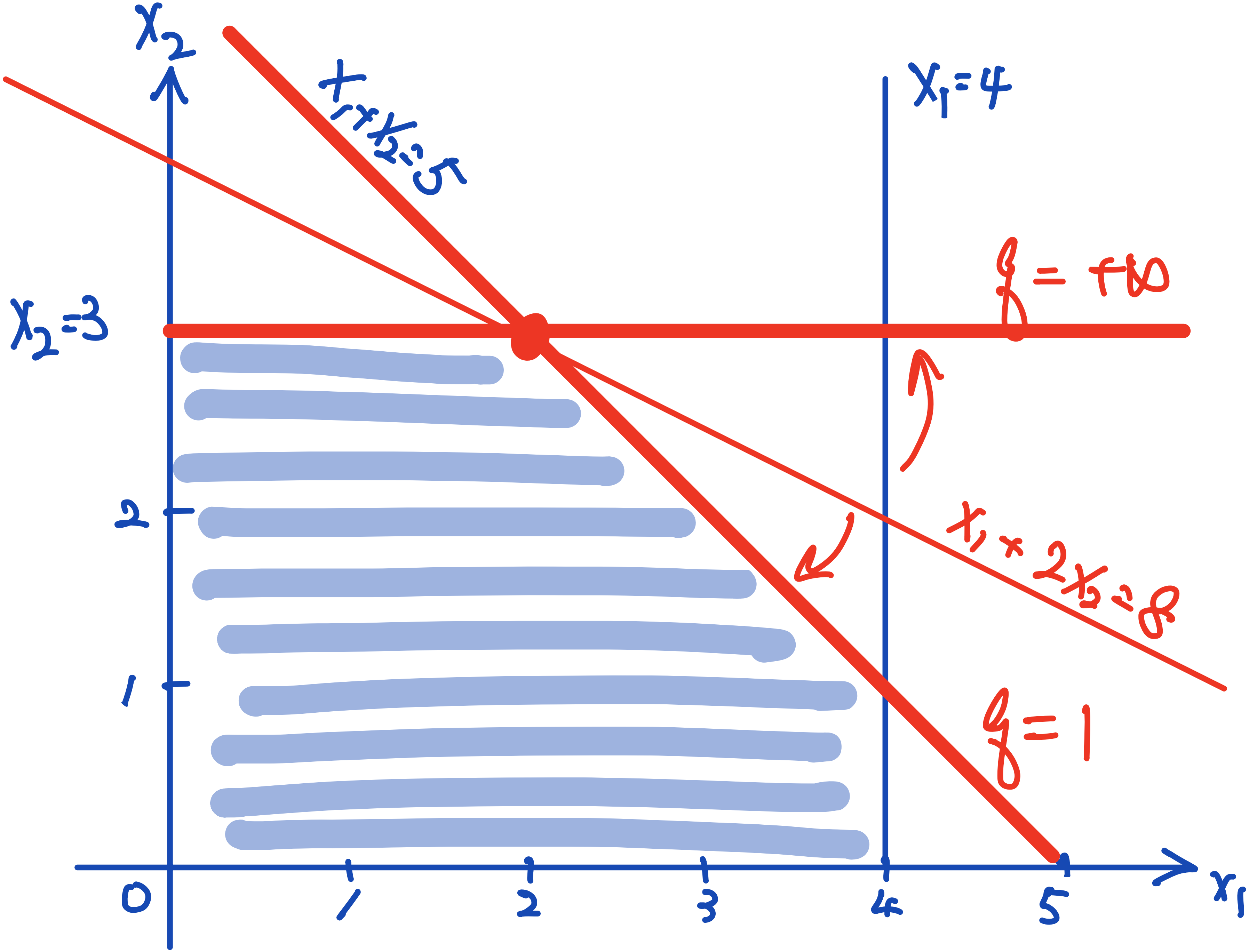
(a) Solve the above problem and give also the dictionary at the optimal point.

(b) Suppose the constraint $x_1 + x_2 \leq 5$ is changed to $x_1 + x_2 \leq p$. Find the range of p such that the dictionary you have found in (a) remains optimal.

(c) Suppose the objective function $x_1 + 2x_2$ is changed to $x_1 + qx_2$. Find the range of q such that the dictionary you have found in (a) remains optimal.


x_2
$x_1 + x_2 = 5$
$x_1 = 4$
$x_2 = 3$
$x_1 + 2x_2 = 8$
2
1
0
1
2
3
4
5
x_1


x_2
$x_1 + x_2 = 5$
$x_1 = 4$
$x_2 = 3$
2
1
$x_1 + x_2 = 3$
$x_1 + x_2 = 7$
0
1
2
3
4
5
x_1


x_2
x_1+x_2=5
x_1=4
x_2=3
g=+∞
2
1
x_1+2x_2=8
g=1
0
1
2
3
4
5
x_1

Sensitivity Analysis (what if ~~c, b~~ A changes)

$$\zeta^* = c_B^T B^{-1} b = \xi^*$$

$$(= c^T x^*) \qquad (= b^T y^*)$$

$$\Delta \zeta^* = c_B^T (\Delta B^{-1}) b$$

$$\Delta (B^{-1}) = - B^{-1} (\Delta B) B^{-1}$$

$$\Delta \zeta^* = - c_B^T B^{-1} (\Delta B) B^{-1} b$$

Sensitivity Analysis (what if ~~c, b~~ A changes)

$$\zeta^* = c_B^T B^{-1} b = \xi^*$$

$$(= c^T x^*) \qquad\qquad (= b^T y^*)$$

$$\Delta \zeta^* = - \underbrace{c_B^T B^{-1}}_{(y^*)^T} (\Delta B) \underbrace{B^{-1} b}_{x_B^*}$$

$$= - (y^*)^T (\Delta B) x_B^* = - \langle (\Delta B) x_B^*, y^* \rangle$$

Ex (Midterm #4)

$$\max \quad x_1 + 2x_2$$

$$\begin{aligned} \text{s.t.} \quad & x_1 + x_2 \leq 5 \\ & x_1 \leq 4 \\ & x_2 \leq 3 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

$$\left[\underbrace{\begin{matrix}1 & 1\\ 1 & 0\\ 0 & 1\end{matrix}}_{A}\;\; \underbrace{\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}}_{N}\right] \begin{bmatrix} x_1 \\ x_2 \\ w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$$

Ex (Midterm #4)

$$\begin{aligned} \max \quad & x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 5 \\ & x_1 \leq 4 \\ & x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

at Opt:

$x_1^* = 2, \; x_2^* = 3$

$w_1^* = 0$

$w_2^* > 0$

$w_3^* = 0$

$$\left[\; \underbrace{\begin{matrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{matrix}}_{A} \quad \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{N} \;\right] \begin{bmatrix} x_1 \\ x_2 \\ w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$$

$$B = \begin{array}{c} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ \begin{matrix} 1 & \;\; 2 & \;\; 4 \end{matrix} \end{array}, \qquad \mathcal{B} = \{1, 2, 4\}$$

$$x_B^* = B^{-1} b$$

$$= \begin{pmatrix} x_1 \\ x_2 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \begin{array}{l} \leftarrow x_1^* \\ \leftarrow x_2^* \\ \leftarrow w_2^* \end{array}$$

$$B = \begin{array}{c} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ \begin{array}{ccc} 1 & \;\;2 & \;\;4 \end{array} \end{array}, \qquad \mathcal{B} = \{1, 2, 4\}$$

$$y^* = (B^{-1})^T c_B = \left[\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1} \right]^T \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{array}{l} \leftarrow y_1^* \\ \leftarrow y_2^* \\ \leftarrow y_3^* \end{array}$$

$$\zeta = c_B^T B^{-1} b$$

$$\begin{aligned}
\Delta \zeta &= c_B^T \, \Delta\left(B^{-1}\right) b \\
&= - c_B^T B^{-1} (\Delta B) B^{-1} b \\
&= - \left(\left(B^{-1}\right)^T c_B\right)^T \Delta B \left(B^{-1} b\right) \\
&= - \left(y^*\right)^T \Delta B \, x_B^* \\
&= - \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} (\Delta B) \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}
\end{aligned}$$

$$\Delta \xi = -\begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta a_{11} & \Delta a_{12} & 0 \\ \Delta a_{21} & \Delta a_{22} & 0 \\ \Delta a_{31} & \Delta a_{23} & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$

$$= -\begin{pmatrix} \Delta a_{11} + \Delta a_{31} & \Delta a_{12} + \Delta a_{23} & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$

$$= -\left(2\Delta a_{11} + 2\Delta a_{31} + 3\Delta a_{12} + 3\Delta a_{23}\right)$$

$$x_B^* = B^{-1} b$$

$$\Delta x_B^* = \Delta(B^{-1}) b$$

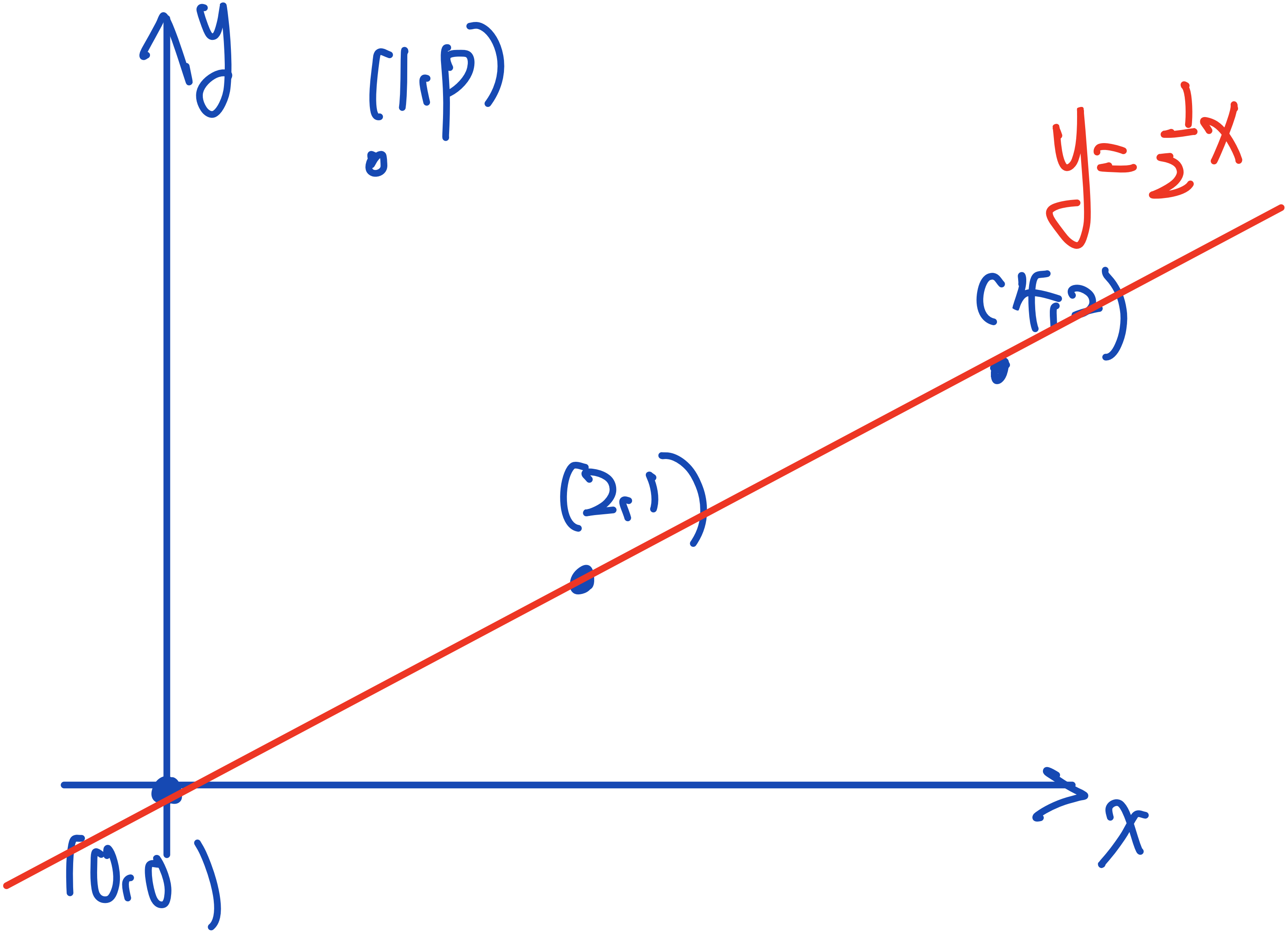
$$= - B^{-1} (\Delta B) B^{-1} b$$

$$y^* = (B^{-1})^T c_B$$

$$\Delta y^* = - \left(B^{-1} (\Delta B) B^{-1} \right)^T c_B$$

$$= - (B^{-1})^T (\Delta B)^T (B^{-1})^T c_B$$

Hw 9 #2



$$y = ax + b \qquad \left(a = \frac{1}{2},\ b = 0 \right)$$

$$(P) \quad \min \quad t_1 + t_2 + t_3 + t_4$$

$$\text{s.t.} \quad |\, 0 - (a(0) + b) \,| \leq t_1$$

$$|\, 1 - (a(2) + b) \,| \leq t_2$$

$$|\, 2 - (a(4) + b) \,| \leq t_3$$

$$|\, p - (a(1) + b) \,| \leq t_4$$

$$y = ax + b \qquad \left(a = \frac{1}{2},\ b = 0 \right)$$

$$
\begin{aligned}
(P)\quad & \min \quad t_1 + t_2 + t_3 + t_4 \\
\text{s.t.}\quad & -t_1 \leq -b \leq t_1 \\
& -t_2 \leq 1 - 2a - b \leq t_2 \\
& -t_3 \leq 2 - 4a - b \leq t_3 \\
& -t_4 \leq p - a - b \leq t_4
\end{aligned}
$$

Proposed Solution

$a = \frac{1}{2}, \quad b = 0$

$$\Rightarrow \quad -t_1 \leq 0 \leq t_1 \Rightarrow t_1 = 0$$

$$-t_2 \leq 0 \leq t_2 \Rightarrow t_2 = 0$$

$$-t_3 \leq 0 \leq t_3 \Rightarrow t_3 = 0$$

$$-t_4 \leq p - \frac{1}{2} \leq t_4$$

$$\Rightarrow t_4 = \left|p - \frac{1}{2}\right|$$

$$\delta^* = \left|p - \frac{1}{2}\right|$$

$$u_1 \left(t_1 + b \geq 0 \right)$$

$$u_2 \left(t_1 - b \geq 0 \right)$$

$$u_3 \left(t_2 + 2a + b \geq 1 \right)$$

$$u_4 \left(t_2 - 2a - b \geq -1 \right)$$

$$u_5 \left(t_3 + 4a + b \geq 2 \right)$$

$$u_6 \left(t_3 - 4a - b \geq -2 \right)$$

$$u_7 \left(t_4 + a + b \geq \phi \right)$$

$$u_8 \left(t_4 - a - b \geq -\phi \right) \qquad u_i \geq 0$$

$$\max \ (u_3 - u_4) + 2(u_5 - u_6) + p(u_7 - u_8)$$

$$\text{s.t.} \quad u_1 + u_2 = 1,$$

$$u_3 + u_4 = 1$$

$$u_5 + u_6 = 1$$

$$u_7 + u_8 = 1$$

$$2(u_3 - u_4) + 4(u_5 - u_6) + (u_7 - u_8) = 0$$

$$\underbrace{(u_1 - u_2)}_{v_1} + \underbrace{(u_3 - u_4)}_{v_2} + \underbrace{(u_5 - u_6)}_{v_3} + \underbrace{(u_7 - u_8)}_{v_4} = 0$$

$$\begin{cases} u_1\left(t_1 + \cancel{b} \geq \cancel{0}\right) \\ u_2\left(t_1 - \cancel{b} \geq \cancel{0}\right) \\ u_3\left(t_2 + \cancel{2a+b} \geq \cancel{1}\right) \\ u_4\left(t_2 - \cancel{2a-b} \geq \cancel{-1}\right) \\ u_5\left(t_3 + \cancel{4a+b} \geq \cancel{2}\right) \\ u_6\left(t_3 - 4a\cancel{-b} \geq -\cancel{2}\right) \end{cases} \qquad a = \frac{1}{2},\ b = 0$$

$$\begin{cases} u_7\left(t_4 + a + b \geq \phi\right) \\ u_8\left(t_4 - a - b \geq -\phi\right) \end{cases} \qquad u_i \geq 0$$

(Cannot say anything about $u_1, u_2, u_3, u_4, u_5, u_6$, other than $u_i \geq 0$)

For u_7, u_8:

$$u_7 \left(t_4 + \frac{1}{2} \geq p \right)$$

$$u_8 \left(t_4 - \frac{1}{2} \geq -p \right)$$

If $p > \frac{1}{2}$, then $t_4 = p - \frac{1}{2}$, $\Rightarrow u_8 = 0, u_7 = 1$

If $p < \frac{1}{2}$, then $t_4 = \frac{1}{2} - p \Rightarrow u_7 = 0, u_8 = 1$

$$\left(p > \frac{1}{2},\ u_7 = 1,\ u_8 = 0\right)$$

$$\max \ (u_3 - u_4) + 2(u_5 - u_6) + p(\underset{1}{\cancel{u_7}} - \underset{0}{\cancel{u_8}})$$

s.t.

$$u_1 + u_2 = 1,$$

$$u_3 + u_4 = 1$$

$$u_5 + u_6 = 1$$

$$u_7 + u_8 = 1$$

$$2(u_3 - u_4) + 4(u_5 - u_6) + (u_7 - u_8) = 0$$

$$\underbrace{(u_1 - u_2)}_{v_1} + \underbrace{(u_3 - u_4)}_{v_2} + \underbrace{(u_5 - u_6)}_{v_3} + \underbrace{(u_7 - u_8)}_{v_4} = 0$$

$$\left(p > \frac{1}{2},\ u_7 = 1,\ u_8 = 0\right)$$

$$\max \quad v_2 + 2v_3 + p$$

$$\text{s.t.} \quad 2v_2 + 4v_3 + 1 = 0$$

$$v_1 + v_2 + v_3 + 1 = 0$$

$$u_1 + u_2 = 1,\ u_3 + u_4 = 1.\ u_5 + u_6 = 1.\ \overset{1}{\cancel{u_7}} + \overset{0}{\cancel{u_8}} = 1$$

$$\max \quad p - \frac{1}{2}$$

$$\text{s.t.} \quad v_2 + 2v_3 = -\frac{1}{2}$$

$$v_1 + v_2 + v_3 = -1$$

$$u_1 + u_2 = 1,\ u_3 + u_4 = 1,\ u_5 + u_6 = 1$$

Need to show this is feasible

$$\left(p < \frac{1}{2},\ u_7 = 0,\ u_8 = 1\right)$$

$$\max \quad (u_3 - u_4) + 2(u_5 - u_6) + p(\underset{0}{\cancel{u_7}} - \underset{1}{\cancel{u_8}})$$

$$\text{s.t.} \quad u_1 + u_2 = 1,$$

$$u_3 + u_4 = 1$$

$$u_5 + u_6 = 1$$

$$u_7 + u_8 = 1$$

$$2(u_3 - u_4) + 4(u_5 - u_6) + (u_7 - u_8) = 0$$

$$\underbrace{(u_1 - u_2)}_{v_1} + \underbrace{(u_3 - u_4)}_{v_2} + \underbrace{(u_5 - u_6)}_{v_3} + \underbrace{(u_7 - u_8)}_{v_4} = 0$$

$$\left(p < \tfrac{1}{2},\ u_7 = 0,\ u_8 = 1 \right)$$

$$\max \quad v_2 + 2v_3 - p$$

$$\text{s.t.} \quad 2v_2 + 4v_3 - 1 = 0$$

$$v_1 + v_2 + v_3 - 1 = 0$$

$$u_1 + u_2 = 1,\quad u_3 + u_4 = 1.\quad u_5 + u_6 = 1.\quad \overset{0}{\cancel{u_7}} + \overset{1}{\cancel{u_8}} = 1$$

$$\boxed{\begin{aligned} &\max \quad \tfrac{1}{2} - p \\ &\text{s.t.} \quad v_2 + 2v_3 = \tfrac{1}{2} \\ &\qquad\quad v_1 + v_2 + v_3 = 1 \\ &u_1 + u_2 = 1,\ u_3 + u_4 = 1,\ u_5 + u_6 = 1 \end{aligned}}$$

Need to show this is feasible