Homework 8 Solution

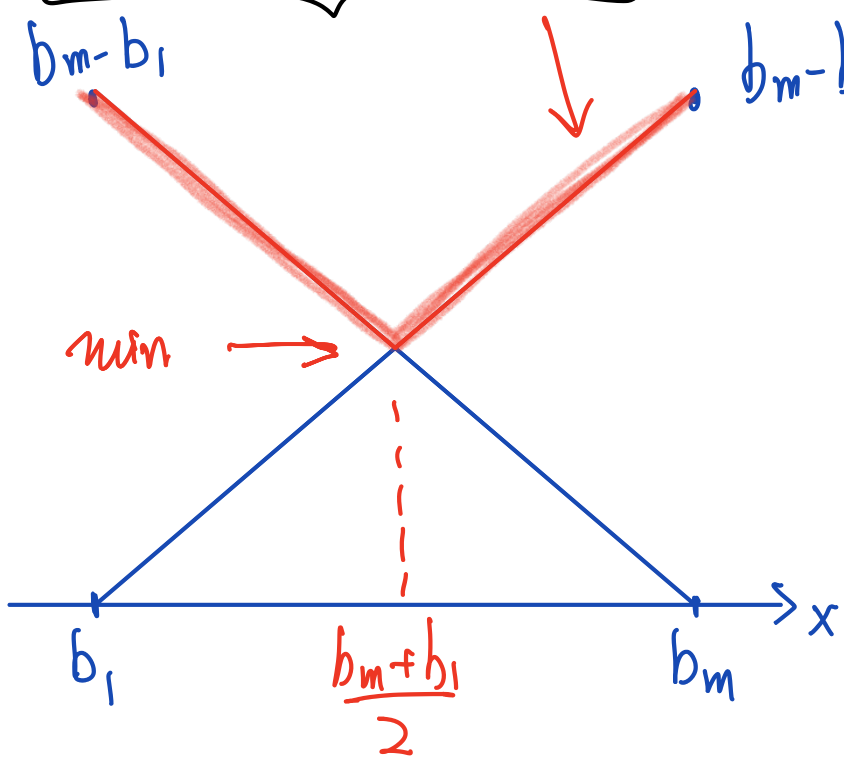
HW8 [V] #12.3

Given $\{b_1 \le b_2 \le \cdots \le b_{m-1} \le b_m\}$.

$$\min_x \max_i |x - b_i|$$

$$= \min_x \max\{|x-b_1|, |x-b_m|\}$$

$$= \min_x \max\{x-b_1, b_m - x\}$$



$$= \frac{b_m - b_1}{2} \quad \text{achieved at} \quad x = \frac{b_m + b_1}{2}$$

[V] #12.4 $\min_{X} \sum_{i=1}^{m} \|X-b_i\|^2$

$$= \min_{X} \underbrace{\sum_i \langle X-b_i, X-b_i \rangle}_{\text{take gradient wrt } X}$$

$$\Rightarrow 2\sum_i (X-b_i) = 0$$

$$X = \frac{1}{m}\sum_{i=1}^{m} b_i$$

12.7 *Sales Force Planning*. A distributor of office equipment finds that the business has seasonal peaks and valleys. The company uses two types of sales persons: (a) regular employees who are employed year-round and cost the company \$17.50/h (fully loaded for benefits and taxes) and (b) temporary employees supplied by an outside agency at a cost of \$25/h. Projections for the number of hours of labor by month for the following year are shown in Table 12.2. Let a_i denote the number of hours of labor needed for month i and let x denote the number of hours per month of labor that will be handled by regular employees. To minimize total labor costs, one needs to solve the following optimization problem:

[V]

$$\text{minimize} \sum_i (25\max(a_i - x, 0) + 17.50x). = \left(\sum_i 25\max(a_i - x, 0)\right) + 210x$$

(a) Show how to reformulate this problem as a linear programming problem.

(b) Solve the problem for the specific data given above.

(c) Use calculus to find a formula giving the optimal value for x.

Jan	390	May	310	Sep	550
Feb	420	Jun	590	Oct	360
Mar	340	Jul	340	Nov	420
Apr	320	Aug	580	Dec	600

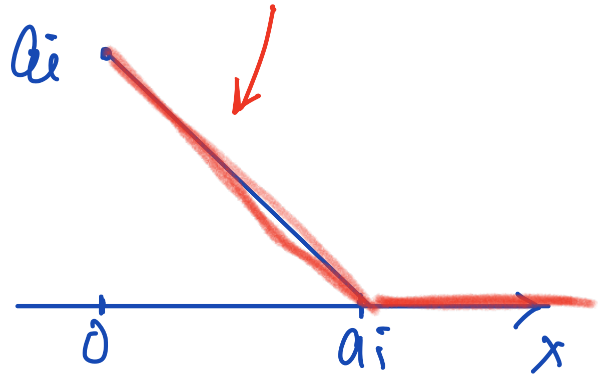
TABLE 12.2. Projected labor hours by month.

(a) $\min \ 210X + 25\sum_i \varepsilon_i$

$$\varepsilon_i \geq a_i - X$$

$$X, \ \varepsilon_i \geq 0$$

(c) Let $f(x) = 210x + 25\sum_i \max(a_i - x, 0)$

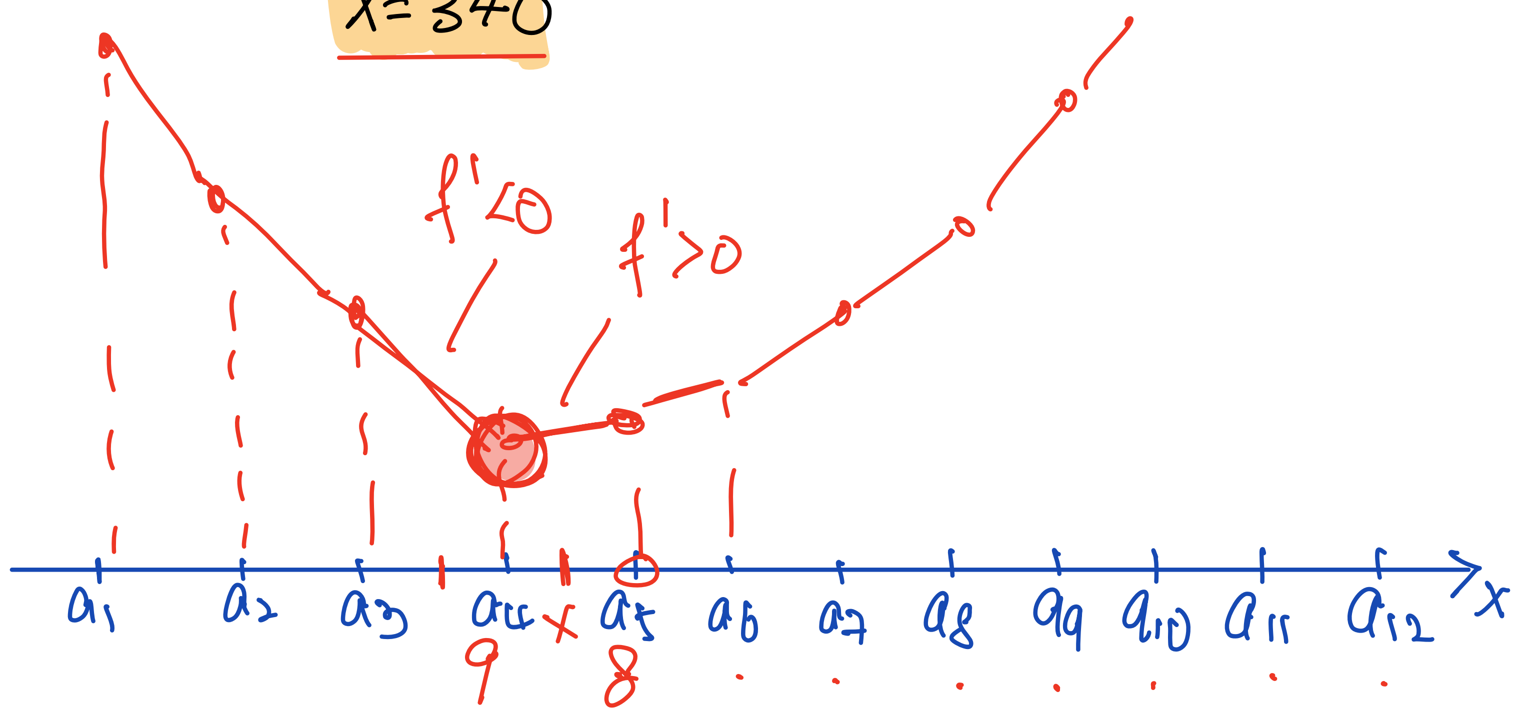


$$f'(x) = 210 + 25\sum_i (-1)\,\mathbb{1}_{\{x \le a_i\}}$$
$$= 210 - 25\sum_i \mathbb{1}_{\{x \le a_i\}} = 0$$
$$\Longrightarrow \quad \sum_i \mathbb{1}_{\{x \le a_i\}} = \frac{210}{25} = \underline{\underline{8.4}}$$

for simplicity, re-order a_i s.t. $a_1 \le a_2 \le \cdots \le a_{12}$

310, 320, 340, $\underbrace{340, 360}$, 390, 420, 420, 550, 580, 590, 600

$\boxed{x^* = 340}$



Recall

L^2- Regression (Least Square)

$$\min_{(x_1,\dots,x_n)} \sum_i \left[\left(b_i - \sum_j a_{ij} x_j\right)^2\right]$$

$$\min_X \|b - AX\|_2^2$$

$AX = b$ (might not be solvable)

$A^T A \hat{X} = A^T b$ (normal equation, always solvable)

$\hat{X} = (A^T A)^{-1} A^T b$ (if $(A^T A)^{-1}$ exists)

L^1 - regression

$$\min_{(x_1,\dots,x_n)} \sum_i \left|b_i - \sum_j a_{ij} x_j\right|$$

$$\min_X \|b - AX\|_1$$

$$\boxed{\begin{aligned} &\min_i \sum_{i=1}^{m} t_i \\ &\text{s.t.} \quad -t_i \le b_i - \sum_j a_{ij} x_j \le t_i \end{aligned}}$$

$|b_i - \sum_j a_{ij} x_j| \le t_i$

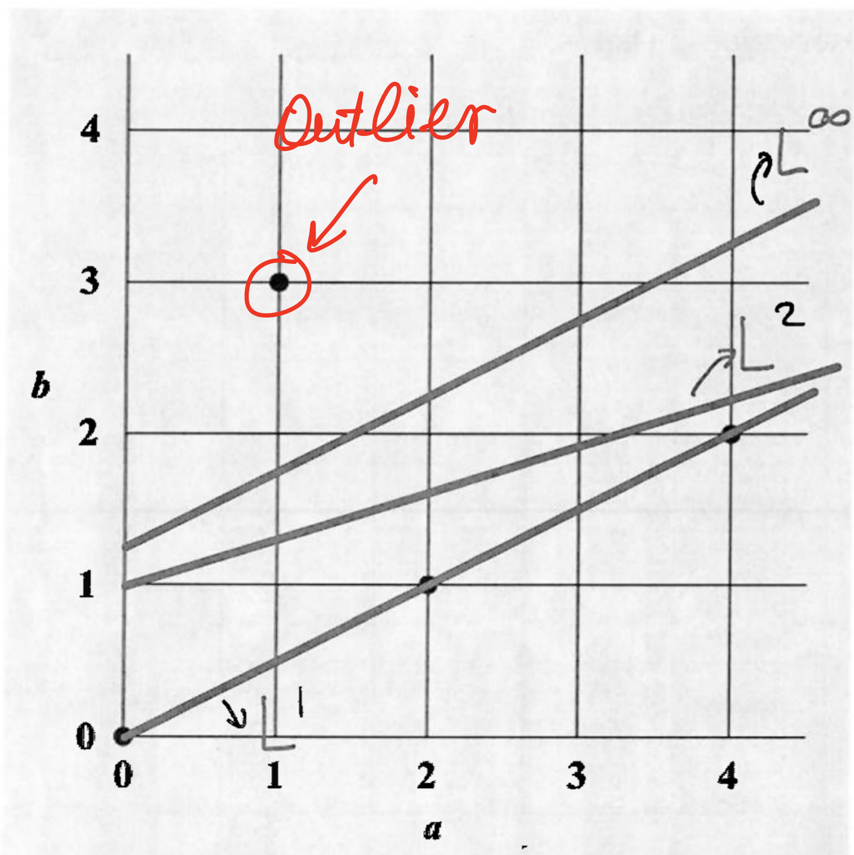
L^∞ - Regression

$$\min_{x}\left(\max_i \left|b_i - \sum_j a_{ij}x_j\right|\right)$$

$$\min_X \|b - AX\|_\infty$$

$$\boxed{\begin{aligned} &\min \ \delta \\ &\text{s.t.} \quad -\delta \le b_i - \sum_j a_{ij}x_j \le \delta \end{aligned}}$$

$\left|b_i - \sum_j a_{ij}x_j\right| \le \delta$ for all i



(Solution from student)

Note that
L^1 is least sensitive to outlier.

L^∞ is most sensitive to outlier