Hw 9 Solution

[V] #10.2

$$\max \ \zeta = c^T x \quad \text{s.t.} \quad Ax \le b, \quad x \ge 0$$

Let $\zeta_1^* = \zeta^*(b_1)$, achieved at x_1^*

$\zeta_2^* = \zeta^*(b_2)$, achieved at x_2^*

Consider

$$\max \ \zeta = c^T x \quad \text{s.t.} \quad Ax \le t b_1 + (1-t) b_2, \quad x \ge 0$$

$0 \le t \le 1$

Let $x = t x_1^* + (1-t) x_2^*$

Then

$$\begin{aligned} Ax &= A(t x_1^* + (1-t) x_2^*) \\ &= t A x_1^* + (1-t) A x_2^* \\ &\le t b_1 + (1-t) b_2 \end{aligned}$$

$\leftarrow$ x is feasible

Hence

$$\begin{aligned} \underbrace{\zeta^*}_{\text{max}} \ge \underbrace{c^T x}_{\text{candidate}} &= c^T (t x_1^* + (1-t) x_2^*) \\ &= t c^T x_1^* + (1-t) c^T x_2^* \\ &= \underline{t \zeta_1^* + (1-t) \zeta_2^*} \end{aligned}$$

[V] # 10.5

The proof of Thm 10.3 makes use of Thm 3.4.
But Thm 3.4 requires LP in standard form:
$Ax \le b$ while (10.2) is not in standard form.

So we need to transform (10.2) into standard form.

$$(10.2) \qquad \begin{cases} Ax = b \\ e^T x = 1, \end{cases} \quad (x \geq 0), \quad A^{m \times n}, \; e, x \in \mathbb{R}^m$$

$$\Longleftrightarrow \quad \underbrace{\begin{bmatrix} A \\ e^T \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} x \end{bmatrix} = \underbrace{\begin{bmatrix} b \\ 1 \end{bmatrix}}_{\tilde{b}} \in \mathbb{R}^{m+1}$$

i.e. $$\tilde{A} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \tilde{b} \qquad \tilde{A}^{(m+1) \times n}$$

$$\begin{bmatrix} \tilde{u}_1 & \tilde{u}_2 & \cdots & \tilde{u}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \tilde{b}$$

cols of $\tilde{A}$

Let $\text{Rank}(\tilde{A}) = r \leq m+1 < n$

(if $n \leq m+1$, nothing to proof)

Without loss of generality, assume $\{\tilde{u}_1, \tilde{u}_2, \dots \tilde{u}_r\}$ is linearly independent.

$$\begin{bmatrix} \underbrace{\tilde{u}_1 \ \tilde{u}_2 \ \cdots \ \tilde{u}_r}_{\tilde{B}} & \underbrace{\tilde{u}_{r+1} \ \cdots \ \tilde{u}_n}_{\tilde{C}} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_r \\ x_{r+1} \\ \vdots \\ x_n \end{bmatrix} = \tilde{b}$$

$$\Longleftrightarrow \quad \underbrace{\tilde{B}}_{(m+1)\times r} \begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix} + \underbrace{\tilde{C}}_{(m+1)\times(n-r)} \begin{bmatrix} x_{r+1} \\ \vdots \\ x_n \end{bmatrix} = \tilde{b}$$

$$\underbrace{\tilde{B}^T \tilde{B}}_{r\times r} \begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix} + \underbrace{\tilde{B}^T \tilde{C}}_{r\times(n-r)} \begin{bmatrix} x_{r+1} \\ \vdots \\ x_n \end{bmatrix} = \underbrace{\tilde{B}^T \tilde{b}}_{r\times 1}$$

Note: as $\tilde{B}$ has linearly independent cols,
$\tilde{B}^T \tilde{B}$ is invertible, i.e. $(\tilde{B}^T \tilde{B})^{-1}$ exists

$$\Longrightarrow \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix}}_{\underset{\approx}{W}} + \underbrace{\left(\tilde{B}^T \tilde{B}\right)^{-1} \tilde{B}^T \tilde{C}}_{\underset{\approx}{A}} \underbrace{\begin{bmatrix} x_{r+1} \\ \vdots \\ x_n \end{bmatrix}}_{\underset{\approx}{X}} = \underbrace{\left(\tilde{B}^T \tilde{B}\right)^{-1} \tilde{b}}_{\underset{\approx}{b}}$$

i.e. $$\underset{\approx}{W} + \underset{\approx}{A}\,\underset{\approx}{X} = \underset{\approx}{b}$$

≥ 0

i.e. $$\underset{\approx}{A}\,\underset{\approx}{X} \leq \underset{\approx}{b}, \quad \underset{\approx}{X} \geq 0$$

in standard form, with $r \leq m+1$ basic vars.

Hence Thm 3.4 can be used.

[c] 16.10 Derive the following theorems (with the vector inequality $\mathbf{v} > \mathbf{w}$ meaning, as usual, $v_k > w_k$ for all k) from the result of problem 16.9.

Only do the -if parts

(i) P. Gordan (1873): The system $\mathbf{Ax} < \mathbf{0}$ is unsolvable if and only if the system $\mathbf{yA} = \mathbf{0}$, $\mathbf{y} \geq \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$ is solvable.

(ii) J. Farkas (1902): The system $\mathbf{Ax} \leq \mathbf{0}$, $\mathbf{bx} > 0$ is unsolvable if and only if the system $\mathbf{yA} = \mathbf{b}$, $\mathbf{y} \geq \mathbf{0}$ is solvable.

(iii) E. Stiemke (1915): The system $\mathbf{Ax} = \mathbf{0}$, $\mathbf{x} > \mathbf{0}$ is unsolvable if and only if the system $\mathbf{yA} \geq \mathbf{0}$, $\mathbf{yA} \neq \mathbf{0}$ is solvable.

(iv) J. A. Ville (1938): The system $\mathbf{Ax} < \mathbf{0}$, $\mathbf{x} \geq \mathbf{0}$ is unsolvable if and only if the system $\mathbf{yA} \geq \mathbf{0}$, $\mathbf{y} \geq \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$ is solvable.

(v) A. W. Tucker (1956): The system $\mathbf{Ax} \geq \mathbf{0}$, $\mathbf{x} \geq \mathbf{0}$ has no solution with $x_k > 0$ if and only if the system $\mathbf{yA} \leq \mathbf{0}$, $\mathbf{y} \geq \mathbf{0}$ has a solution with

$$\sum_{i=1}^{m} a_{ik} y_i < 0.$$

(i) If there is a y s.t. $yA=0$, $y \geq 0$, $y \neq 0$.

then $AX < 0$ is not solvable.

pf Suppose $AX<0$ is solvable, i.e. such an X exists.

$AX < 0 \Rightarrow y\cancel{A}X < y0 \quad (y \neq 0)$

$\Rightarrow 0 < 0$ Contradiction!

(ii) Suppose $AX \leq 0$, $bX > 0$ is solvable

$y \geq 0 \quad \underbrace{yA}_{b}X \leq 0 \Rightarrow bX \leq 0$ contradiction!

(iii) Suppose $AX=0$, $X>0$ is solvable

contradiction

$y \quad \underbrace{yA}_{\geq 0,\text{ but } \neq 0}X = 0 \Rightarrow (yA)X = 0$

$\Rightarrow$ Some component of X must be zero

(iv) Suppose $AX<0,\ X\geq 0$ is solvable

$y\geq 0$, $y\neq 0$

$\underbrace{(yA)}_{\geq 0}\underbrace{X}_{\geq 0} < 0$

$\Rightarrow \quad 0<0$ contradiction!

(v) Suppose $AX\geq 0,\ X\geq 0$ has a solution with $x_k>0$

y: $yA\leq 0,\ y\geq 0,\ \sum_i a_{ik}y_i<0$

$$yAX \geq 0$$

$$0\leq \sum_i y_i (AX)_i = \sum_i y_i \sum_j A_{ij}X_j$$

$$= \sum_j \sum_i y_i A_{ij} X_j$$

$$= \sum_j \underbrace{\left(\sum_i y_i A_{ij}\right)}_{\leq 0}\underbrace{X_j}_{\geq 0} \leq 0$$

but for index $j=k$, $\underbrace{\left(\sum_i y_i A_{ik}\right)}_{<0}\underbrace{X_k}_{>0} < 0$

Hence $0<0$ (Contradiction)

#2 (See also note on duality on Dec, 3, 2024)

$\boxed{\begin{array}{lll} (P) & & \min\ t_1+t_2+t_3+t_4 \\ (0,0) \Rightarrow & & -t_1 \leq -b \leq t_1 \\ (2,1) \Rightarrow & & -t_2 \leq 1-2a-b \leq t_2 \\ (4,2) \Rightarrow & & -t_3 \leq 2-4a-b \leq t_3 \\ (1,p) \Rightarrow & & -t_4 \leq p-a-b \leq t_4 \end{array}}$

$y = ax+b$

$a = \frac{1}{2},\ b = 0$

$(0,0)$
$(2,1)$
$(4,2)$
$(1,p)$

Proposed solution : $a = \frac{1}{2},\ b = 0$

$$\Rightarrow \quad \begin{array}{lcl} -t_1 \leq 0 \leq t_1 & \Rightarrow & t_1 = 0 \\ -t_2 \leq 0 \leq t_2 & \Rightarrow & t_2 = 0 \\ -t_3 \leq 0 \leq t_3 & \Rightarrow & t_3 = 0 \\ -t_4 \leq p-\frac{1}{2} \leq t_4 & \Rightarrow & t_4 = \left|p-\frac{1}{2}\right| \end{array}$$

min value

$$\boxed{\min\ t_1+t_2+t_3+t_4 = \left|p-\tfrac{1}{2}\right|}$$

$$\boxed{\text{Proposed solution: } a=\tfrac{1}{2},\ b=0,\ t_1=t_2=t_3=0,\ t_4=\left|p-\tfrac{1}{2}\right|}$$

Find (D)

$$-t_1 \le -b \le t_1$$
$$-t_2 \le 1-2a-b \le t_2$$
$$-t_3 \le 2-4a-b \le t_3$$
$$-t_4 \le p-a-b \le t_4$$

Standard form:

$$\Rightarrow \begin{array}{ll}
u_1 (& b + t_1 \ge 0) \\
u_2 (& -b + t_1 \ge 0) \\
u_3 (& 2a + b + t_2 \ge 1) \\
u_4 (& -2a - b + t_2 \ge -1) \\
u_5 (& 4a + b + t_3 \ge 2) \\
u_6 (& -4a - b + t_3 \ge -2) \\
u_7 (& a + b + t_4 \ge p) \\
u_8 (& -a - b + t_4 \ge -p)
\end{array}$$

$u_i \ge 0$
$i = 1, 2 \cdots, 8$

$$t_1 + t_2 + t_3 + t_4$$
$$\ge \left(2(u_3-u_4) + 4(u_5-u_6) + (u_7-u_8) \right) a$$
$$+ \left((u_1-u_2) + (u_3-u_4) + (u_5-u_6) + (u_7-u_8) \right) b$$
$$+ (u_1+u_2) t_1 + (u_3+u_4) t_2 + (u_5+u_6) t_3 + (u_7+u_8) t_4$$
$$\ge (u_3-u_4) + 2(u_5-u_6) + p(u_7-u_8)$$

(D)
$$\max \ (u_3-u_4)+2(u_5-u_6)+p(u_7-u_8)$$
$$\text{s.t.}\quad 2(u_3-u_4)+4(u_5-u_6)+(u_7-u_8)=0$$
$$(u_1-u_2)+(u_3-u_4)+(u_5-u_6)+(u_7-u_8)=0$$
$$u_1+u_2=1,\ u_3+u_4=1,\ u_5+u_6=1,\ u_7+u_8=1$$
$$u_1,u_2,\dots,u_8\geq 0$$

Let

$V_1=u_1-u_2$, $V_2=u_3-u_4$, $V_3=u_5-u_6$, $V_4=u_7-u_8$

(D):
$$\max\ \xi = V_2+2V_3+pV_4$$
$$\text{s.t.}\quad 2V_2+4V_3+V_4=0 \quad ①$$
$$V_1+V_2+V_3+V_4=0 \quad ②$$

$V_1=1-2u_2,\quad 0\leq u_2\leq 1 \Leftarrow (u_1\geq 0)$

$V_2=1-2u_4,\quad 0\leq u_4\leq 1 \Leftarrow (u_3\geq 0)$

$V_3=1-2u_6,\quad 0\leq u_6\leq 1 \Leftarrow (u_5\geq 0)$

$V_4=1-2u_8,\quad 0\leq u_8\leq 1 \Leftarrow (u_7\geq 0)$

$$① \Longrightarrow \quad V_2 + 2V_3 = -\frac{V_4}{2}$$

$$\Longrightarrow \quad \xi = -\frac{V_4}{2} + p\,V_4 = \left(p - \frac{1}{2}\right) V_4$$

$$\Longrightarrow \quad \boxed{\max \left(p - \frac{1}{2}\right) V_4}$$

①,② $\Longrightarrow$ solve for V_2, V_3

$$V_2 + 2V_3 = -\frac{V_4}{2}$$

$$V_2 + V_3 = -V_1 - V_4$$

$$\Longrightarrow \quad \boxed{\begin{aligned} V_3 &= V_1 + \frac{V_4}{2} \\ V_2 &= -2V_1 - \frac{3V_4}{2} \end{aligned}} \quad \leftarrow \text{eliminate } V_2, V_3$$

$$\left.\begin{aligned} V_1 &= 1 - 2u_2 & \quad 0 &\le u_2 \le 1 \\ V_2 &= 1 - 2u_4 & \quad 0 &\le u_4 \le 1 \\ V_3 &= 1 - 2u_6 & \quad 0 &\le u_6 \le 1 \\ V_4 &= 1 - 2u_8, & \quad 0 &\le u_8 \le 1 \end{aligned}\right\} \leftarrow \text{eliminate } V_2, V_3$$

$$V_2 = 1 - 2u_4 \Rightarrow \quad u_4 = \frac{1}{2} - \frac{1}{2}V_2$$

$$= \frac{1}{2} - \frac{1}{2}\left(-2V_1 - \frac{3V_4}{2}\right)$$

$$u_4 = \frac{1}{2} + V_1 + \frac{3}{4}V_4$$

$$\boxed{\begin{aligned} u_4 &= \frac{1}{2} + 1 - 2u_2 + \frac{3}{4}(1 - 2u_8) \\ &= \frac{9}{4} - 2u_2 - \frac{3}{2}u_8 \end{aligned}}$$

$$0 \leq u_4 \leq 1$$

$$0 \leq \frac{9}{4} - 2u_2 - \frac{3}{2}u_8 \leq 1$$

$$\boxed{\frac{5}{8} \leq u_2 + \frac{3}{4}u_8 \leq \frac{9}{8}} \quad 0 \leq u_4 \leq 1$$

$$V_3 = 1 - 2u_6 \Rightarrow \quad u_6 = \frac{1}{2} - \frac{1}{2} V_3$$

$$= \frac{1}{2} - \frac{1}{2}\left(V_1 + \frac{V_4}{2}\right)$$

$$u_6 = \frac{1}{2} - \frac{1}{2} V_1 - \frac{V_4}{4}$$

$$\boxed{\begin{aligned} u_6 &= \frac{1}{2} - \frac{1}{2}(1 - 2u_2) - \frac{1}{4}(1 - 2u_8) \\ &= -\frac{1}{4} + u_2 + \frac{u_8}{2} \end{aligned}}$$

$$0 \leq u_6 \leq 1$$

$$0 \leq -\frac{1}{4} + u_2 + \frac{u_8}{2} \leq 1$$

$$\boxed{\frac{1}{4} \leq u_2 + \frac{u_8}{2} \leq \frac{5}{4}} \qquad 0 \leq u_6 \leq 1$$

D

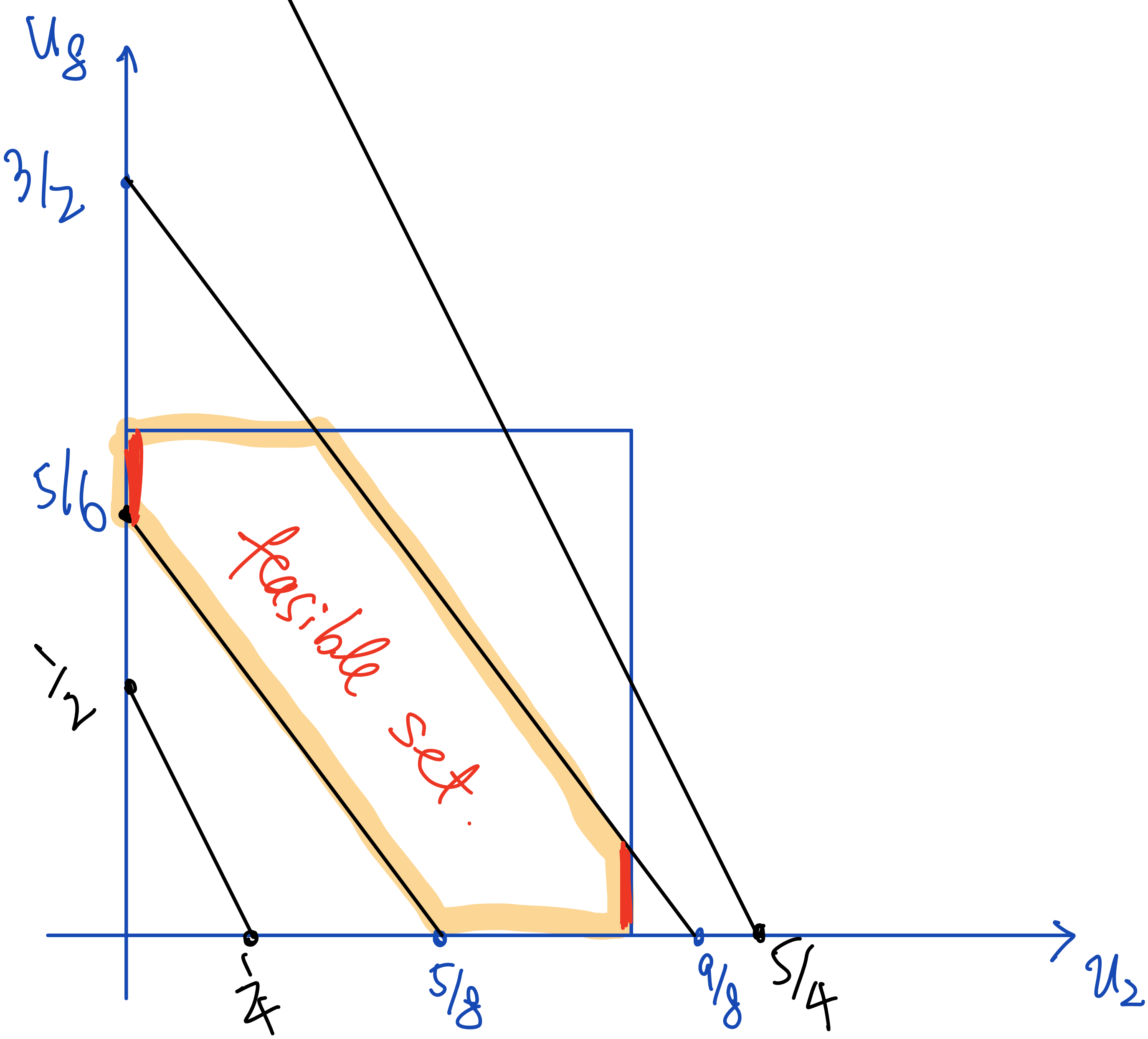
$$\max \quad \left(p - \tfrac{1}{2}\right)\left(1 - 2u_8\right)$$

$$\text{s.t.} \quad \frac{5}{8} \leq u_2 + \frac{3}{4}u_8 \leq \frac{9}{8} \qquad (0 \leq u_4 \leq 1)$$

$$\frac{1}{4} \leq u_2 + \frac{u_8}{2} \leq \frac{5}{4} \qquad (0 \leq u_6 \leq 1)$$

$$0 \leq u_2,\ u_8 \leq 1$$

5/2



If $p > \frac{1}{2}$, $\max \left(p-\frac{1}{2}\right)\left(1-2u_8\right)$

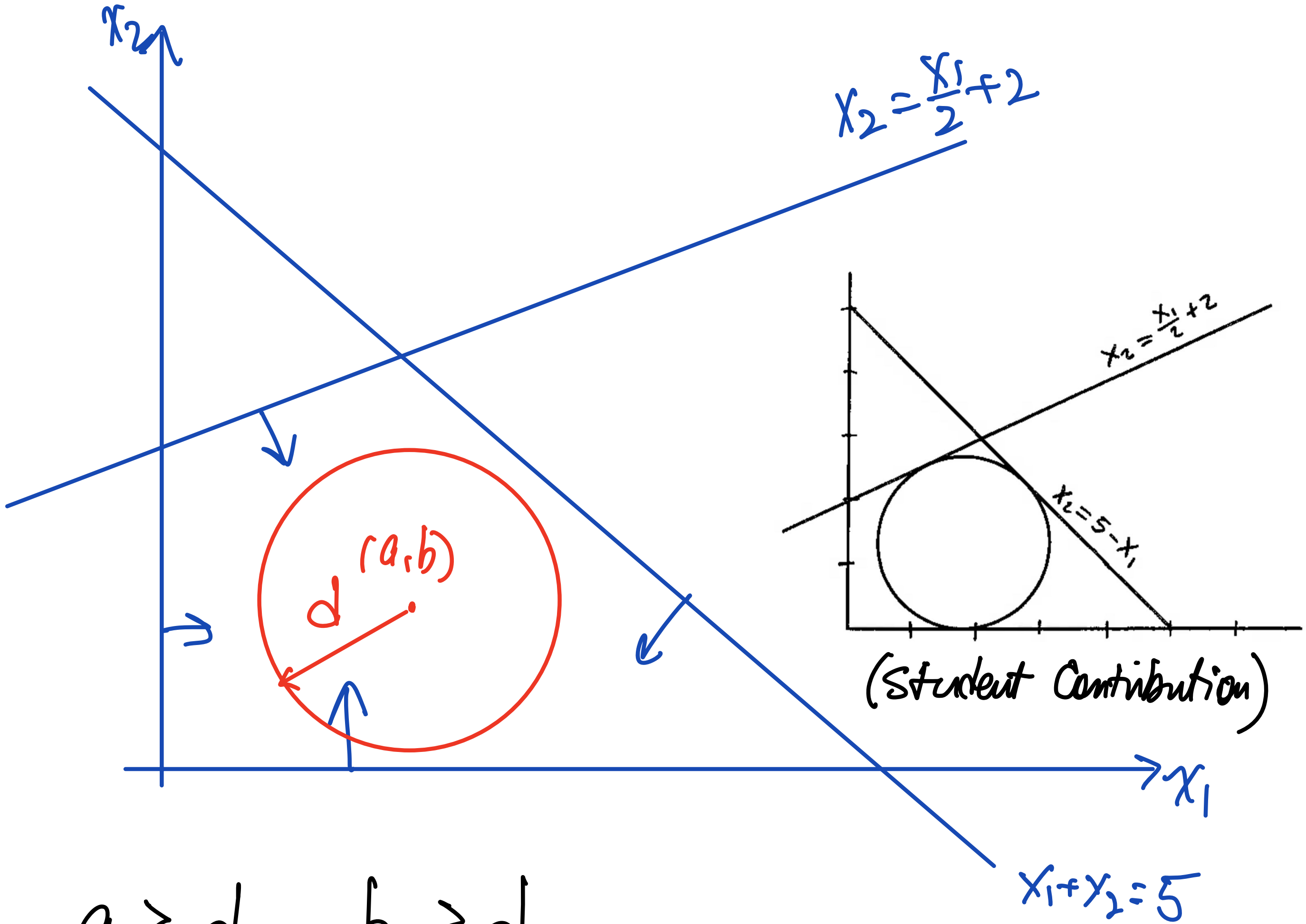
$= p-\frac{1}{2}$, (at $u_8 = 0$)

If $p < \frac{1}{2}$ $\max \left(p-\frac{1}{2}\right)\left(1-2u_8\right)$

$= -\left(p-\frac{1}{2}\right)$, (at $u_8 = 1$)

$$\boxed{\text{Hence } \max \left(p-\frac{1}{2}\right)\left(1-2u_8\right) = \left|p-\frac{1}{2}\right|}$$

#3



$a \geq d, \quad b \geq d,$

$$\left| \frac{a+b-5}{\sqrt{2}} \right| \geq d \implies \frac{5-a-b}{\sqrt{2}} \geq d$$

(need $a+b \leq 5$)

$$\left| \frac{\frac{a}{2} - b + 2}{\sqrt{\frac{1}{2^2} + 1^2}} \right| \geq d \implies \frac{a - 2b + 4}{\sqrt{5}} \geq d$$

(need $\frac{a}{2} - b + 2 \geq 0$)

(LP)

$$\boxed{\begin{aligned} \max \quad & d \\ \text{s.t.} \quad & a \geq d, \quad b \geq d, \\ & \frac{5-a-b}{\sqrt{2}} \geq d; \quad \frac{a-2b+4}{\sqrt{5}} \geq d \end{aligned}}$$

#4 (a)

$$P_1 = \left\{ \begin{array}{l} -x_1 \le 0 \\ \frac{x_1}{2} - x_2 \le -5 \\ -x_1 - x_2 \le -10 \end{array} \right\} = \underbrace{\begin{bmatrix} -1 & 0 \\ \frac{1}{2} & -1 \\ -1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \underbrace{\begin{bmatrix} 0 \\ -5 \\ -10 \end{bmatrix}}_{b}$$

$$P_2 = \left\{ \begin{array}{l} -x_2 \le 0 \\ -x_1 + x_2 \le 0 \\ 3x_1 + x_2 \le 30 \end{array} \right\} = \underbrace{\begin{bmatrix} 0 & -1 \\ -1 & 1 \\ 3 & 1 \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \underbrace{\begin{bmatrix} 0 \\ 0 \\ 30 \end{bmatrix}}_{\tilde{b}}$$

$$P_1 \cap P_2 = \left\{ \begin{array}{l} Ax \le b \\ \tilde{A}x \le \tilde{b} \end{array} \right. = \begin{bmatrix} A \\ \tilde{A} \end{bmatrix} x \le \begin{bmatrix} b \\ \tilde{b} \end{bmatrix}$$

Apply FL to this system.

ie.

$$\boxed{\begin{array}{ll} \min & \begin{pmatrix} b^T & \tilde{b}^T \end{pmatrix} \begin{pmatrix} y \\ \tilde{y} \end{pmatrix} \\ \text{s.t.} & \begin{pmatrix} A^T & \tilde{A}^T \end{pmatrix} \begin{pmatrix} y \\ \tilde{y} \end{pmatrix} = 0 \\ & y, \tilde{y} \ge 0 \end{array}}$$

Using simplex to find $y, \tilde{y}$ s.t.

$$\boxed{\begin{aligned} &\begin{pmatrix} A^T & \tilde{A}^T \end{pmatrix} \begin{pmatrix} y \\ \tilde{y} \end{pmatrix} = 0, \quad y, \tilde{y} \geq 0 \\ &\text{and} \quad \begin{pmatrix} b^T & \tilde{b}^T \end{pmatrix} \begin{pmatrix} y \\ \tilde{y} \end{pmatrix} < 0 \end{aligned}}$$

$$\underline{A^T y + \tilde{A}^T \tilde{y} = 0} \qquad b^T y + \tilde{b}^T \tilde{y} < 0$$

Then

$$H_1 = \{ x : (y^T A) x \leq y^T b \} \quad (= b^T y)$$

$$H_2 = \{ x : (\tilde{y}^T \tilde{A}) x \leq \tilde{y}^T \tilde{b} \} \quad = \tilde{b}^T \tilde{y}$$

Note: H_1: $\quad y^T A x \leq y^T b \qquad y^T b < -\tilde{y}^T \tilde{b}$

$y^T A = -\tilde{y}^T \tilde{A}$

$$-\tilde{y}^T \tilde{A} x < -\tilde{y}^T \tilde{b}$$

i.e. $\quad \tilde{y}^T \tilde{b} < \tilde{y}^T \tilde{A} x \quad \leftarrow$ not in H_2

Hence $\quad H_1 \cap H_2 = \phi$

$$\min \begin{pmatrix} b^T & \tilde{b}^T \end{pmatrix} \begin{pmatrix} y \\ \tilde{y} \end{pmatrix}$$

$$\text{s.t.} \quad \begin{pmatrix} A^T & \tilde{A}^T \end{pmatrix} \begin{pmatrix} y \\ \tilde{y} \end{pmatrix} = 0$$

$$y, \tilde{y} \geq 0$$

maximize	ζ	=			0	x_1	+	5	x_2	+	10	x_3	+	0	x_4	+	0	x_5	+	-30	x_6
subject to:	w_1	=	0	-	-1	x_1	-	1/2	x_2	-	-1	x_3	-	0	x_4	-	-1	x_5	-	3	x_6
	w_2	=	0	-	0	x_1	-	-1	x_2	-	-1	x_3	-	-1	x_4	-	1	x_5	-	1	x_6
	w_3	=	0	-	1	x_1	-	-1/2	x_2	-	1	x_3	-	0	x_4	-	1	x_5	-	-3	x_6
	w_4	=	0	-	0	x_1	-	1	x_2	-	1	x_3	-	1	x_4	-	-1	x_5	-	-1	x_6

$x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6 \quad w_1 \ w_2 \ w_3 \ w_4 \geq 0$

maximize	ζ	=			-10	x_1	+	10	x_2	+	-10	w_3	+	0	x_4	+	-10	x_5	+	0	x_6
subject to:	w_1	=	0	-	0	x_1	-	0	x_2	-	1	w_3	-	0	x_4	-	0	x_5	-	0	x_6
	w_2	=	0	-	1	x_1	-	-3/2	x_2	-	1	w_3	-	-1	x_4	-	2	x_5	-	-2	x_6
	x_3	=	0	-	1	x_1	-	-1/2	x_2	-	1	w_3	-	0	x_4	-	1	x_5	-	-3	x_6
	w_4	=	0	-	-1	x_1	-	3/2	x_2	-	-1	w_3	-	1	x_4	-	-2	x_5	-	2	x_6

$x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6 \quad w_1 \ w_2 \ w_3 \ w_4 \geq 0$

maximize	ζ	=			-10/3	x_1	+	-20/3	w_4	+	-10/3	w_3	+	-20/3	x_4	+	10/3	x_5	+	-40/3	x_6
subject to:	w_1	=	0	-	0	x_1	-	0	w_4	-	1	w_3	-	0	x_4	-	0	x_5	-	0	x_6
	w_2	=	0	-	0	x_1	-	1	w_4	-	0	w_3	-	0	x_4	-	0	x_5	-	0	x_6
	x_3	=	0	-	2/3	x_1	-	1/3	w_4	-	2/3	w_3	-	1/3	x_4	-	1/3	x_5	-	-7/3	x_6
	x_2	=	0	-	-2/3	x_1	-	2/3	w_4	-	-2/3	w_3	-	2/3	x_4	-	-4/3	x_5	-	4/3	x_6

$x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6 \quad w_1 \ w_2 \ w_3 \ w_4 \geq 0$

maximize	ζ	=			-10	x_1	+	-10	w_4	+	-10	w_3	+	-10	x_4	+	-10	x_3	+	10	x_6
subject to:	w_1	=	0	-	0	x_1	-	0	w_4	-	1	w_3	-	0	x_4	-	0	x_3	-	0	x_6
	w_2	=	0	-	0	x_1	-	1	w_4	-	0	w_3	-	0	x_4	-	0	x_3	-	0	x_6
	x_5	=	0	-	2	x_1	-	1	w_4	-	2	w_3	-	1	x_4	-	3	x_3	-	-7	x_6
	x_2	=	0	-	2	x_1	-	2	w_4	-	2	w_3	-	2	x_4	-	4	x_3	-	-8	x_6

$x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6 \quad w_1 \ w_2 \ w_3 \ w_4 \geq 0$

$(y_1 = 0, \ y_2 = 8, \ y_3 = 0), \quad (\tilde{y}_1 = 0, \ \tilde{y}_2 = 7, \ \tilde{y}_3 = 1)$

$b = (0, \ -5, \ -10), \quad \tilde{b} = (0, \ 0, \ 30)$

$$\boxed{b^T y + \tilde{b}^T \tilde{y} = -40 + 30 = -10 < 0}$$

$$H = \{x : y^T A x \leq y^T b\}$$

$$\begin{pmatrix} 0 & 8 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ \frac{1}{2} & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 0 & 8 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -5 \\ -10 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq -40$$

$$\boxed{x_1 - 2x_2 \leq -10} \quad H$$

$$\tilde{H} = \{x : \tilde{y}^T \tilde{A} x \leq \tilde{y}^T \tilde{b}\}$$

$$\begin{pmatrix} 0 & 7 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 0 & 7 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 30 \end{pmatrix}$$

$$\begin{pmatrix} -4 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq 30$$

$$\boxed{x_1 - 2x_2 \geq -7.5} \quad \tilde{H}$$

(b)

$$P_1 = \begin{cases} 2x_1 + 3x_2 + x_3 \le 5 \\ 3x_1 + 4x_2 + 2x_3 \le 8 \\ -x_1 \le 0 \\ -x_2 \le 0 \\ -x_3 \le 0 \end{cases} = \underbrace{\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 2 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \le \underbrace{\begin{bmatrix} 5 \\ 8 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{b}$$

$$P_2 = \begin{cases} -5x_1 - 4x_2 - 3x_3 \le -14 \\ 4x_1 + x_2 + 2x_3 \le 11 \\ -x_1 \le 0 \\ -x_2 \le 0 \\ -x_3 \le 0 \end{cases} = \underbrace{\begin{bmatrix} -5 & -4 & -3 \\ 4 & 1 & 2 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \le \underbrace{\begin{bmatrix} -14 \\ 11 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\tilde{b}}$$

maximize ζ = −5 x_1 + −8 x_2 + 0 x_3 + 0 x_4 + 0 x_5 + 14 x_6 + −11 x_7 + 0 x_8 + 0 x_9 + 0 x_{10}

subject to:

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
w_1 =	0 −	2	3	−1	0	0	−5	4	−1	0	0
w_2 =	0 −	−2	−3	1	0	0	5	−4	1	0	0
w_3 =	0 −	3	4	0	−1	0	−4	1	0	−1	0
w_4 =	0 −	−3	−4	0	1	0	4	−1	0	1	0
w_5 =	0 −	1	2	0	0	−1	−3	2	0	0	−1
w_6 =	0 −	−1	−2	0	0	1	3	−2	0	0	1

$x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10}\quad w_1\ w_2\ w_3\ w_4\ w_5\ w_6 \ge 0$

maximize ζ = −1/3 x_1 + 4/3 x_2 + 0 x_3 + 0 x_4 + −14/3 x_5 + −14/3 w_6 + −5/3 x_7 + 0 x_8 + 0 x_9 + −14/3 x_{10}

subject to:

		x_1	x_2	x_3	x_4	x_5	w_6	x_7	x_8	x_9	x_{10}
w_1 =	0 −	1/3	−1/3	−1	0	5/3	5/3	2/3	−1	0	5/3
w_2 =	0 −	−1/3	1/3	1	0	−5/3	−5/3	−2/3	1	0	−5/3
w_3 =	0 −	5/3	4/3	0	−1	4/3	4/3	−5/3	0	−1	4/3
w_4 =	0 −	−5/3	−4/3	0	1	−4/3	−4/3	5/3	0	1	−4/3
w_5 =	0 −	0	0	0	0	0	1	0	0	0	0
x_6 =	0 −	−1/3	−2/3	0	0	1/3	1/3	−2/3	0	0	1/3

$x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10}\quad w_1\ w_2\ w_3\ w_4\ w_5\ w_6 \ge 0$

maximize ζ = 1 x_1 + −4 w_2 + −4 x_3 + 0 x_4 + 2 x_5 + 2 w_6 + 1 x_7 + −4 x_8 + 0 x_9 + 2 x_{10}

subject to:

		x_1	w_2	x_3	x_4	x_5	w_6	x_7	x_8	x_9	x_{10}
w_1 =	0 −	0	1	0	0	0	0	0	0	0	0
x_2 =	0 −	−1	3	3	0	−5	−5	−2	3	0	−5
w_3 =	0 −	3	−4	−4	−1	8	8	1	−4	−1	8
w_4 =	0 −	−3	4	4	1	−8	−8	−1	4	1	−8
w_5 =	0 −	0	0	0	0	0	1	0	0	0	0
x_6 =	0 −	−1	2	2	0	−3	−3	−2	2	0	−3

$x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10}\quad w_1\ w_2\ w_3\ w_4\ w_5\ w_6 \ge 0$

maximize	ζ =		1/4 x_1 +	-3 w_2 +	-3 x_3 +	1/4 x_4 +	0 x_5 +	0 w_6 +	3/4 x_7 +	-3 x_8 +	1/4 x_9 +	-1/4 w_3
subject to:	w_1 =	0 -	0 x_1 -	1 w_2 -	0 x_3 -	0 x_4 -	0 x_5 -	0 w_6 -	0 x_7 -	0 x_8 -	0 x_9 -	0 w_3
	x_2 =	0 -	7/8 x_1 -	1/2 w_2 -	1/2 x_3 -	-5/8 x_4 -	0 x_5 -	0 w_6 -	-11/8 x_7 -	1/2 x_8 -	-5/8 x_9 -	5/8 w_3
	x_{10} =	0 -	3/8 x_1 -	-1/2 w_2 -	-1/2 x_3 -	-1/8 x_4 -	1 x_5 -	1 w_6 -	1/8 x_7 -	-1/2 x_8 -	-1/8 x_9 -	1/8 w_3
	w_4 =	0 -	0 x_1 -	0 w_2 -	0 x_3 -	0 x_4 -	0 x_5 -	0 w_6 -	0 x_7 -	0 x_8 -	0 x_9 -	1 w_3
	w_5 =	0 -	0 x_1 -	0 w_2 -	0 x_3 -	0 x_4 -	0 x_5 -	1 w_6 -	0 x_7 -	0 x_8 -	0 x_9 -	0 w_3
	x_6 =	0 -	1/8 x_1 -	1/2 w_2 -	1/2 x_3 -	-3/8 x_4 -	0 x_5 -	0 w_6 -	-13/8 x_7 -	1/2 x_8 -	-3/8 x_9 -	3/8 w_3

$x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10}\quad w_1\ w_2\ w_3\ w_4\ w_5\ w_6 \geq 0$

maximize	ζ =		-2 x_1 +	0 w_2 +	0 x_3 +	1 x_4 +	-6 x_5 +	-6 w_6 +	-6 x_{10} +	0 x_8 +	1 x_9 +	-1 w_3
subject to:	w_1 =	0 -	0 x_1 -	1 w_2 -	0 x_3 -	0 x_4 -	0 x_5 -	0 w_6 -	0 x_{10} -	0 x_8 -	0 x_9 -	0 w_3
	x_2 =	0 -	5 x_1 -	-5 w_2 -	-5 x_3 -	-2 x_4 -	11 x_5 -	11 w_6 -	11 x_{10} -	-5 x_8 -	-2 x_9 -	2 w_3
	x_7 =	0 -	3 x_1 -	-4 w_2 -	-4 x_3 -	-1 x_4 -	8 x_5 -	8 w_6 -	8 x_{10} -	-4 x_8 -	-1 x_9 -	1 w_3
	w_4 =	0 -	0 x_1 -	0 w_2 -	0 x_3 -	0 x_4 -	0 x_5 -	0 w_6 -	0 x_{10} -	0 x_8 -	0 x_9 -	1 w_3
	w_5 =	0 -	0 x_1 -	0 w_2 -	0 x_3 -	0 x_4 -	0 x_5 -	1 w_6 -	0 x_{10} -	0 x_8 -	0 x_9 -	0 w_3
	x_6 =	0 -	5 x_1 -	-6 w_2 -	-6 x_3 -	-2 x_4 -	13 x_5 -	13 w_6 -	13 x_{10} -	-6 x_8 -	-2 x_9 -	2 w_3

$x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10}\quad w_1\ w_2\ w_3\ w_4\ w_5\ w_6 \geq 0$

$$(y_1 = 0 \quad y_2 = 2 \quad y_3 = 0 \quad y_4 = 1 \quad y_5 = 0)$$

$$b = (5 \quad 8 \quad 0 \quad 0 \quad 0)$$

$$(\tilde{y}_1 = 2, \ \tilde{y}_2 = 1, \ \tilde{y}_3 = 0, \ \tilde{y}_4 = 0, \ \tilde{y}_5 = 0)$$

$$\tilde{b} = (-14, \quad 11, \quad 0, \quad 0, \quad 0)$$

$$\boxed{b^T y + \tilde{b}^T \tilde{y} = 16 - 28 + 11 = -1 < 0}$$

$$H = \{ x : y^T A x \le y^T b \}$$

$$\begin{pmatrix} 0 & 2 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 3 & 4 & 2 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\le \begin{pmatrix} 0 & 2 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 8 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 7 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \le 16$$

$$\boxed{6x_1 + 7x_2 + 4x_3 \le 16} \quad H$$

$$\tilde{H} = \{ x : \tilde{y}^T \tilde{A} x \le \tilde{y}^T b \}$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -5 & -4 & -3 \\ 4 & 1 & 2 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\le \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -14 \\ 11 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -6 & -7 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \le -17$$

$$\boxed{6x_1 + 7x_2 + 4x_3 \geq 17} \quad \tilde{H}$$

(#5 Look at note on duality, Nov 3, 2024)