MA 421: Linear Programming and Optimization Techniques

Fall 2024, Final Exam

Instructor: Yip

- This test booklet has SEVEN QUESTIONS, totaling 100 points for the whole test. You have 120 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is **closed book, closed note, with no electronic devices**.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.
- **As a rule of thumb, you should give explicit and useful answers.** No points will be given for just writing down some generically true statements. In other words, your answers should try to make use of all the information given in the question.
- **As a rule of thumb, you should only use those methods that have been covered in class.** If you use some other methods "for the sake of convenience", at our discretion, we might not give you any credit. You have the right to contest. In that event, **you will be asked to explain your answer using only what has been covered in class up to the point of time of this exam.**

Name: Answer Key (Major:)

Question	Score
1.(10 pts)	
2.(10 pts)	
3.(10 pts)	
4.(10 pts)	
5.(20 pts)	
6.(20 pts)	
7.(20 pts)	
Total (100 pts)	

Formula sheet

1. Matrix form of simplex method [V, p.91, 92].

   Given the following linear program problem and its dual in their standard forms:

$$\text{(P)}: \begin{cases} \text{maximize} & \zeta(x) & = & c^T x \\ \text{subject to} & Ax & \leq & b; \\ & x & \geq & 0. \end{cases} \qquad \text{(D)}: \begin{cases} \text{minimize} & \xi(y) & = & b^T y \\ \text{subject to} & A^T y & \geq & c; \\ & y & \geq & 0. \end{cases}$$

   During simplex iterations, the above can be transformed into the following matrix form:

$$\begin{aligned} \zeta &= c_{\mathcal{B}}^T(B^{-1}b) - \left((B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}}\right)^T X_{\mathcal{N}} \\ X_{\mathcal{B}} &= B^{-1}b - \left(B^{-1}N\right) X_{\mathcal{N}} \end{aligned}$$

   with the following dual form:

$$\begin{aligned} -\xi &= -c_{\mathcal{B}}^T(B^{-1}b) - \left(B^{-1}b\right)^T Z_{\mathcal{B}} \\ Z_{\mathcal{N}} &= \left((B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}}\right) + \left(B^{-1}N\right)^T Z_{\mathcal{B}} \end{aligned}$$

   where in the above

   (a) $\mathcal{B}$ and $\mathcal{N}$ are the basic and non-basic variable indices;

   (b) $c_{\mathcal{B}}$ and $c_{\mathcal{N}}$ are the coefficients in the objective function corresponding to the basic and non-basic variables;

   (c) B and N are matrices formed by the collecting the columns from the augmented matrix $[A \;\; I]$ corresponding to the basic and non-basic variables.

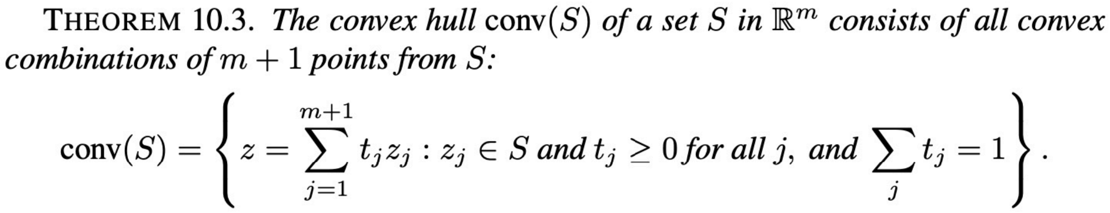
   (d) $X_{\mathcal{B}}$ and $X_{\mathcal{N}}$ are the basic and non-basic primal variables, and $Z_{\mathcal{B}}$ and $Z_{\mathcal{N}}$ are the basic and non-basic dual variables.

2. Change of matrix inverse. Let M be an invertible matrix. Then

$$\delta(M^{-1}) = -M^{-1}(\delta M)M^{-1}$$

   (where δ refers to infinitesimal change in a quantity).

3. Caratheodory Theorem [V, p.162].



THEOREM 10.3. *The convex hull* conv(S) *of a set* S *in* $\mathbb{R}^m$ *consists of all convex combinations of* $m+1$ *points from* S*:*

$$\text{conv}(S) = \left\{ z = \sum_{j=1}^{m+1} t_j z_j : z_j \in S \text{ and } t_j \geq 0 \text{ for all } j, \text{ and } \sum_j t_j = 1 \right\}.$$

4. Farkas Lemma [V, p. 165].

   LEMMA 10.5. *The system $Ax \leq b$ has no solutions if and only if there is a y such that*

$$\begin{aligned} A^T y &= 0 \\ y &\geq 0 \\ b^T y &< 0. \end{aligned} \tag{10.8}$$

5. Separation Theorem [V, p.163].

   THEOREM 10.4. *Let P and $\tilde{P}$ be two disjoint nonempty polyhedra in $\mathbb{R}^n$. Then there exist disjoint half-spaces H and $\tilde{H}$ such that $P \subset H$ and $\tilde{P} \subset \tilde{H}$.*

   If $P = \{x : Ax \leq \tilde{b}\}$ and $\tilde{P} = \{x : \tilde{A}x \leq \tilde{b}\}$, then

$$H = \{x : y^T A x \leq y^T b\}, \quad \text{and} \quad \tilde{H} = \{x : \tilde{y}^T \tilde{A} x \leq \tilde{y}^T \tilde{b}\},$$

   where y and $\tilde{y}$ are found by Farkas Lemma:

$$\begin{bmatrix} A^T & \tilde{A}^T \end{bmatrix} \begin{bmatrix} y \\ \tilde{y} \end{bmatrix} = A^T y + \tilde{A}^T \tilde{y} = 0 \tag{10.5}$$

$$\begin{bmatrix} y \\ \tilde{y} \end{bmatrix} \geq 0 \tag{10.6}$$

$$\begin{bmatrix} b^T & \tilde{b}^T \end{bmatrix} \begin{bmatrix} y \\ \tilde{y} \end{bmatrix} = b^T y + \tilde{b}^T \tilde{y} < 0. \tag{10.7}$$

   (Note that the form of $\tilde{H}$ given here is equivalent to the one given in [V, p.164] but it looks nicer and is easier to remember.)

6. Distance formula. The distance from a point (x_0, y_0) to the straightline $ax + by + c = 0$ is given by

$$\left| \frac{ax_0 + by_0 + c}{\sqrt{a^2 + b^2}} \right|.$$

7. Variables of network flows. Given a network $(\mathcal{N}, \mathcal{A})$ (where $\mathcal{N}$ and $\mathcal{A}$ denote the nodes and arcs), and a spanning tree $\mathcal{T}$ of the network, we have the following set of variables [V, p.231]:

   (a) Flow variables: x_{ij} (found by balance equations)

   (b) Dual variables y_j and dual slacks z_{ij} which satisfy:

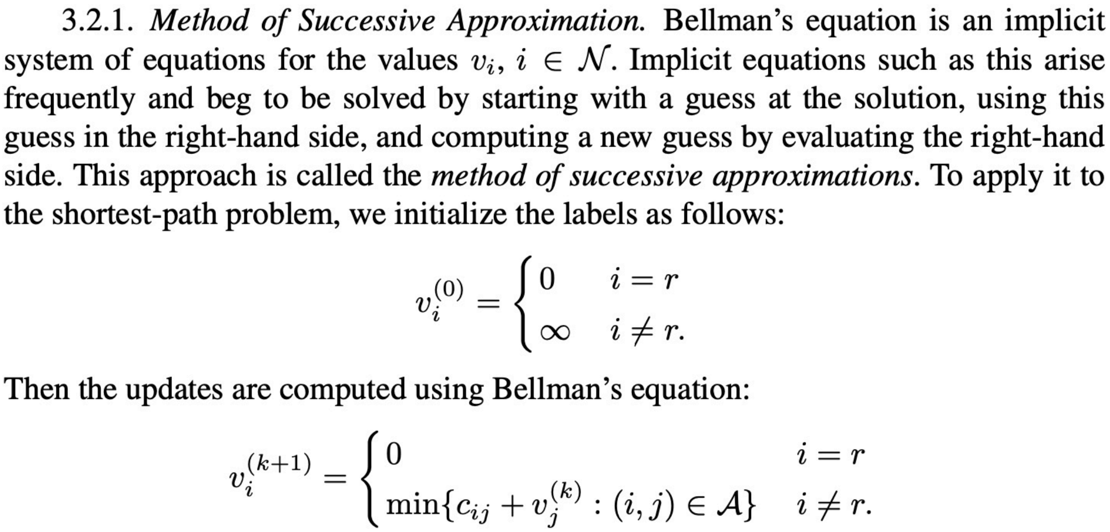
$$y_j - y_i + z_{ij} = c_{ij}, \quad \text{for } (i, j) \in \mathcal{A}.$$

Note that for $(i,j) \in \mathcal{T}$, we have (by complementary slackness) that $z_{ij} = 0$. The usual way to find y_i and z_{ij} is to *first* solve for y_i using $y_j - y_i = c_{ij}$ for $(i,j) \in \mathcal{T}$ and *then* use $z_{ij} = c_{ij} - y_j + y_i$ for $(i,j) \notin \mathcal{T}$.

8. Bellman's equation for the label (or value function) for finding the shortest distance/path from (any) node i to a root node (r) is given by [V, p.261, 262]

$$v_i = \min_j \{c_{ij} + v_j : \ (i,j) \in \mathcal{A}\}, \quad \text{for } i \neq r \quad v_r = 0.$$

The function v can be found by the following iteration procedure:



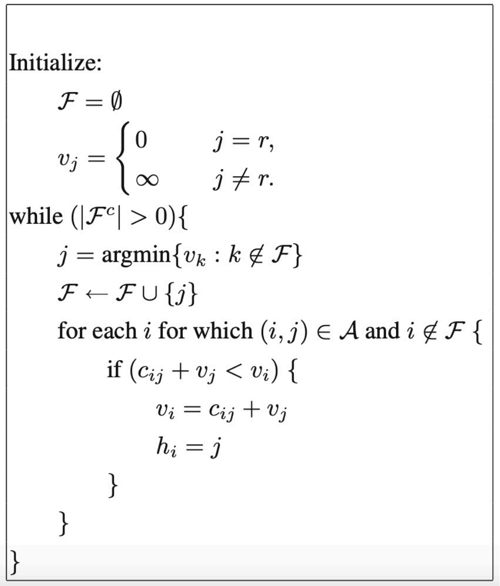
3.2.1. *Method of Successive Approximation.* Bellman's equation is an implicit system of equations for the values v_i, $i \in \mathcal{N}$. Implicit equations such as this arise frequently and beg to be solved by starting with a guess at the solution, using this guess in the right-hand side, and computing a new guess by evaluating the right-hand side. This approach is called the *method of successive approximations*. To apply it to the shortest-path problem, we initialize the labels as follows:

$$v_i^{(0)} = \begin{cases} 0 & i = r \\ \infty & i \neq r. \end{cases}$$

Then the updates are computed using Bellman's equation:

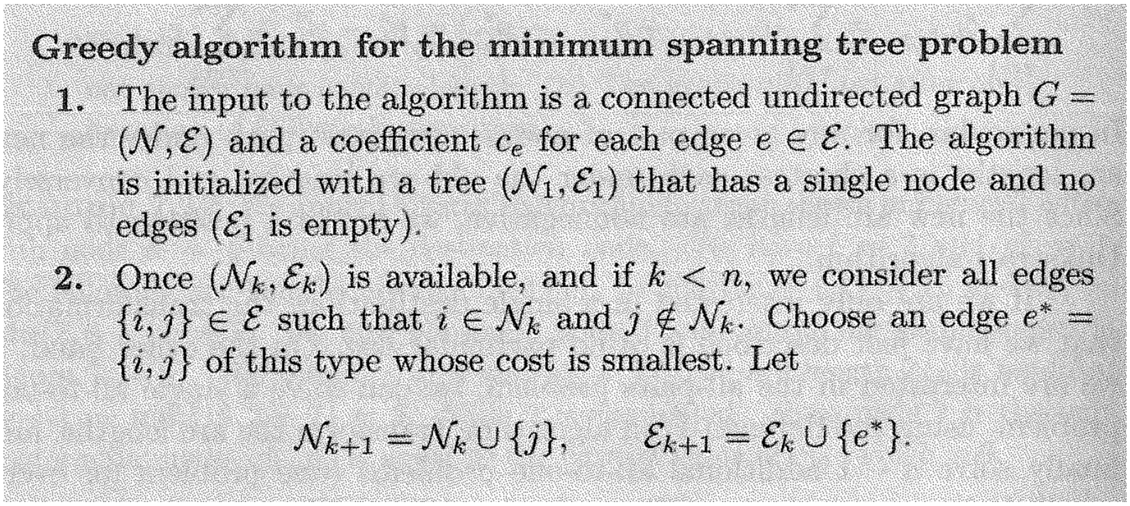
$$v_i^{(k+1)} = \begin{cases} 0 & i = r \\ \min\{c_{ij} + v_j^{(k)} : (i,j) \in \mathcal{A}\} & i \neq r. \end{cases}$$

9. Dijkstra algorithm for finding the shortest path (with non-negative costs c_{ij}) [V, p.263].



```
Initialize:
    $\mathcal{F} = \emptyset$
    $v_j = \begin{cases} 0 & j = r, \\ \infty & j \neq r. \end{cases}$
while ($|\mathcal{F}^c| > 0$){
    $j = \text{argmin}\{v_k : k \notin \mathcal{F}\}$
    $\mathcal{F} \leftarrow \mathcal{F} \cup \{j\}$
    for each $i$ for which $(i,j) \in \mathcal{A}$ and $i \notin \mathcal{F}$ {
        if ($c_{ij} + v_j < v_i$) {
            $v_i = c_{ij} + v_j$
            $h_i = j$
        }
    }
}
```

10. Greedy algorithm for finding minimum spanning tree [Berksimas-Tsitsiklis, Intro. Lin. Opt., p.344].



Greedy algorithm for the minimum spanning tree problem

1. The input to the algorithm is a connected undirected graph $G = (\mathcal{N}, \mathcal{E})$ and a coefficient c_e for each edge $e \in \mathcal{E}$. The algorithm is initialized with a tree $(\mathcal{N}_1, \mathcal{E}_1)$ that has a single node and no edges ($\mathcal{E}_1$ is empty).
2. Once $(\mathcal{N}_k, \mathcal{E}_k)$ is available, and if $k < n$, we consider all edges $\{i, j\} \in \mathcal{E}$ such that $i \in \mathcal{N}_k$ and $j \notin \mathcal{N}_k$. Choose an edge $e^* = \{i, j\}$ of this type whose cost is smallest. Let

$$\mathcal{N}_{k+1} = \mathcal{N}_k \cup \{j\}, \qquad \mathcal{E}_{k+1} = \mathcal{E}_k \cup \{e^*\}.$$

1. Find the dual of the following optimization problem:

$$
\begin{array}{lrcl}
\text{maximize} & & 5x_1 + 4x_2 + 3x_3 & \\
\text{subject to} & 1 \leq & 2x_1 + 3x_2 + x_3 & \leq 5 \\
& -2 \leq & 4x_1 + x_2 + 2x_3 & \leq 11 \\
& 4 \leq & 3x_1 + 4x_2 + 2x_3 & \leq 8 \\
& & -1 \leq x_1 \leq 4 & \\
& & 3 \leq x_2 \leq 5 & \\
& & -2 \leq x_3 \leq 9 &
\end{array}
$$

$$
\begin{array}{l}
y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \\ y_{10} \\ y_{11} \\ y_{12}
\end{array}
\left(
\begin{array}{l}
2x_1 + 3x_2 + x_3 \leq 5 \\
-2x_1 - 3x_2 - x_3 \leq -1 \\
4x_1 + x_2 + 2x_3 \leq 11 \\
-4x_1 - x_2 - 2x_3 \leq 2 \\
3x_1 + 4x_2 + 2x_3 \leq 8 \\
-3x_1 - 4x_2 - 2x_3 \leq -4 \\
x_1 \leq 4 \\
-x_1 \leq 1 \\
x_2 \leq 5 \\
-x_2 \leq -3 \\
x_3 \leq 9 \\
-x_3 \leq 2
\end{array}
\right)
\qquad y_i \geq 0
$$

You can use this blank page.

$$\min \xi(y) = 5y_1 - y_2 + 11y_3 + 2y_4 + 8y_5 - 4y_6 + 4y_7 + y_8 + 5y_9 - 3y_{10} + 9y_{11} + 2y_{12}$$

$$\text{s.t.} \cdot 2(y_1 - y_2) + 4(y_3 - y_4) + 3(y_5 - y_6) + (y_7 - y_8) = 5$$

$$3(y_1 - y_2) + 1(y_3 - y_4) + 4(y_5 - y_6) + (y_9 - y_{10}) = 4$$

$$1(y_1 - y_2) + 2(y_3 - y_4) + 2(y_5 - y_6) + (y_{11} - y_{12}) = 3$$

$$y_1, y_2, \ldots y_{12} \geq 0$$

2. Let A be an $m \times n$ matrix, b and c be an m and n vector. Simplify the formulation of the following max-min problem:

$$\max_{x\geq 0}\min_{y\geq 0}\left(c^T x - y^T Ax + y^T b\right),$$

by carrying out the inner optimization problem.

Do the same but for the following min-max problem:

$$\min_{y\geq 0}\max_{x\geq 0}\left(c^T x - y^T Ax + y^T b\right).$$

(a)

$$\max_{x\geq 0} \underbrace{\min_{y\geq 0}\left(c^T x - y^T Ax + y^T b\right)}$$

$$= \underbrace{\min_{y\geq 0}\left(c^T x - y^T(Ax-b)\right)}$$

$$= \begin{cases} -\infty & \text{if } Ax > b \quad (y \to +\infty) \\ c^T x & \text{if } Ax = b \\ c^T x & \text{if } Ax < b \quad (y = 0) \end{cases}$$

$$\Leftrightarrow \quad \max_{x\geq 0} \; c^T x$$

$$\text{s.t.} \quad Ax \leq b$$

(Otherwise, if $Ax > b$. then $\max_{x\geq 0} -\infty \Rightarrow -\infty$)

You can use this blank page.

(b)

$$\min_{y\geqslant 0}\ \underbrace{\max_{x\geqslant 0}\left(c^Tx-y^TAx+y^Tb\right)}$$

$$=\max_{x\geqslant 0}\left[(c^T-y^TA)x+y^Tb\right]$$

$$=\begin{cases}+\infty & \text{if } c^T > y^TA \quad (x\to+\infty)\\ y^Tb & \text{if } c^T = y^TA\\ y^Tb & \text{if } c^T < y^TA \quad (x=0)\end{cases}$$

$$\Longleftrightarrow \quad \min_{y\geqslant 0}\ y^Tb$$

$$y^TA \geqslant c^T$$

(Otherwise if $y^TA < c^T$, then $\min_{y\geqslant 0} +\infty = +\infty$)

3. Consider the following optimization problem:

$$\begin{array}{lrcl} \text{maximize} & \zeta(x) & = & c^T x \\ \text{subject to} & Ax & \leq & b; \\ & x & \geq & 0. \end{array}$$

(a) Let c be fixed. Let $\zeta^*(b)$ be the maximum value for the objective function written as a function of b. Show that ζ^* is a *concave* function of b, i.e.

$$\zeta^*(\lambda b_1 + (1-\lambda)b_2) \geq \lambda\zeta^*(b_1) + (1-\lambda)\zeta^*(b_2), \quad \text{for } 0 \leq \lambda \leq 1.$$

(b) Now let b be fixed. Let $\zeta^*(c)$ be the maximum value for the objective function written as a function of c. Show that ζ^* is a *convex* function of c, i.e.

$$\zeta^*(\lambda c_1 + (1-\lambda)c_2) \leq \lambda\zeta^*(c_1) + (1-\lambda)\zeta^*(c_2), \quad \text{for } 0 \leq \lambda \leq 1.$$

(a) Let X_1^*, X_2^* be the solution for

$AX \le b_1$ and $AX \le b_2$.

Then for $b = \lambda b_1 + (1-\lambda) b_2$,

Note that for $X = \lambda X_1^* + (1-\lambda) X_2^*$,

$$\begin{aligned} AX &= A(\lambda X_1^* + (1-\lambda) A X_2^*) \\ &= \lambda A X_1^* + (1-\lambda) A X_2^* \\ &\le \lambda b_1 + (1-\lambda) b_2 \end{aligned}$$

and

$$\begin{aligned} \zeta(X) &= c^T(\lambda X_1^* + (1-\lambda) X_2^*) \\ &= \lambda(c^T X_1^*) + (1-\lambda) c^T X_2^* \end{aligned}$$

a candidate

$$\boxed{= \underbrace{\lambda \zeta^*(b_1) + (1-\lambda)\zeta^*(b_2)}_{\text{a possible value}} \le \underbrace{\zeta^*(\lambda b_1 + (1-\lambda) b_2^*)}_{\text{true opt. (max.)}}}$$

You can use this blank page.

(b) Let x_1^*, x_2^* be the solution for c_1 & c_2.

Then for $\lambda c_1^T + (1-\lambda) c_2^T$, we have

$$\begin{aligned}\zeta(x) &= (\lambda c_1^T + (1-\lambda) c_2^T) x \\ &= \lambda c_1^T x + (1-\lambda) c_2^T x \\ &\leq \lambda c_1^T x_1^* + (1-\lambda) c_2^T x_2^* \\ &= \lambda \zeta^*(c_1) + (1-\lambda) \zeta^*(c_2)\end{aligned}$$

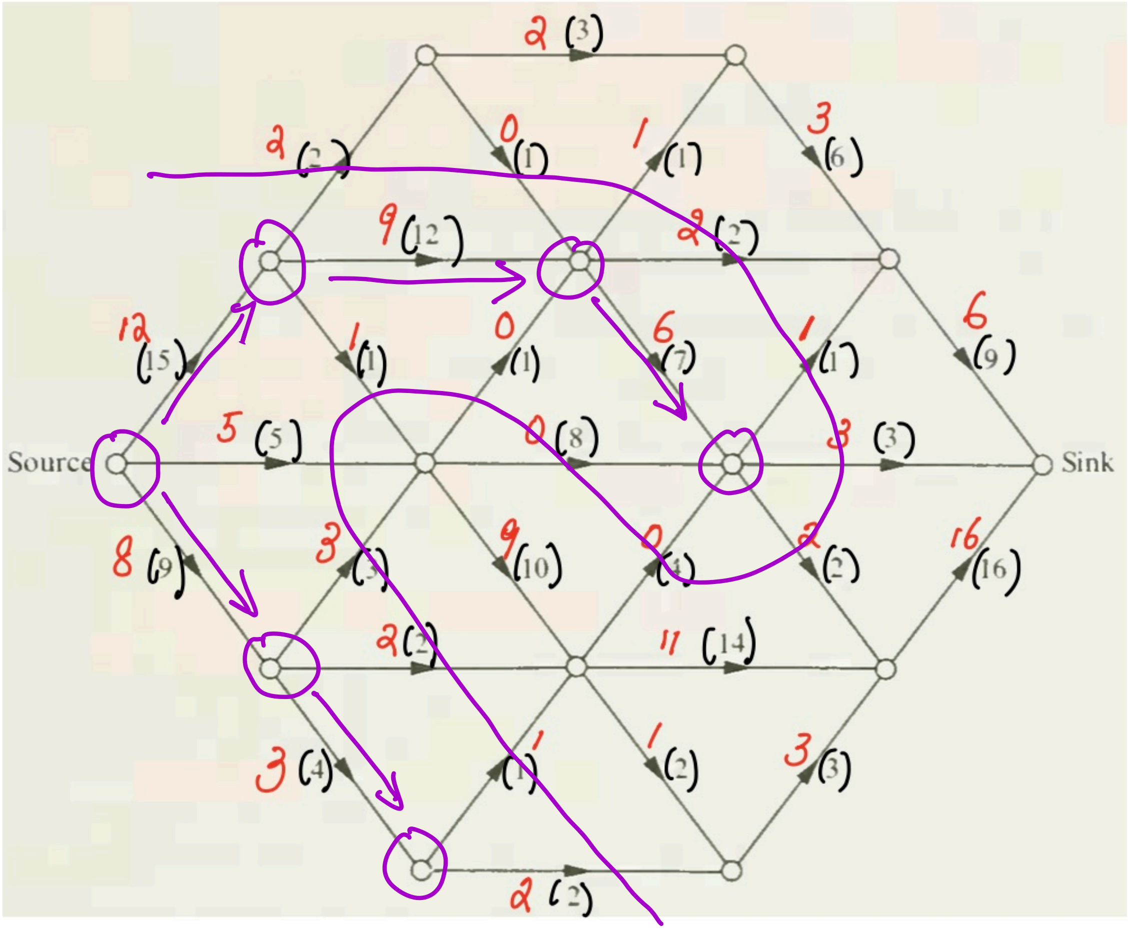
this is true for any x s.t. $Ax \leq b$

i.e. $\underset{\uparrow}{\zeta(x)} \leq \lambda \zeta^*(c_1) + (1-\lambda) \zeta^*(c_2)$

max over x s.t. $Ax \leq b$

$$\Longrightarrow \boxed{\zeta^*(\lambda c_1 + (1-\lambda) c_2) \leq \lambda \zeta^*(c_1) + (1-\lambda) \zeta^*(c_2)}$$

4. Consider the following flow network from Source to Sink. The numbers with paranthesis refer to the arc capacities and those without refer to the actual flows.



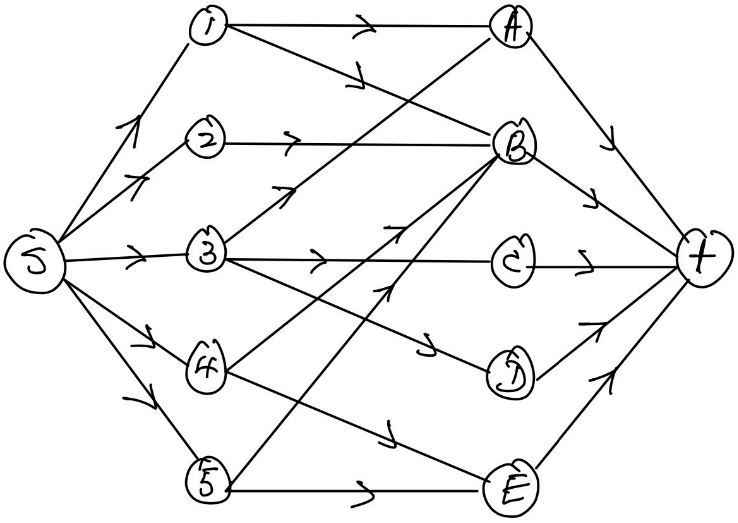
Does the flow pattern give a maximum flow from Source to Sink? If so, find the cut with minimum cut capacity. If not, find one augmented path and determine how much more flow can be pushed to the Sink.

Look for all the nodes which have an augmented path from the source (s.t. more flows can be pushed further).

Cut capacity

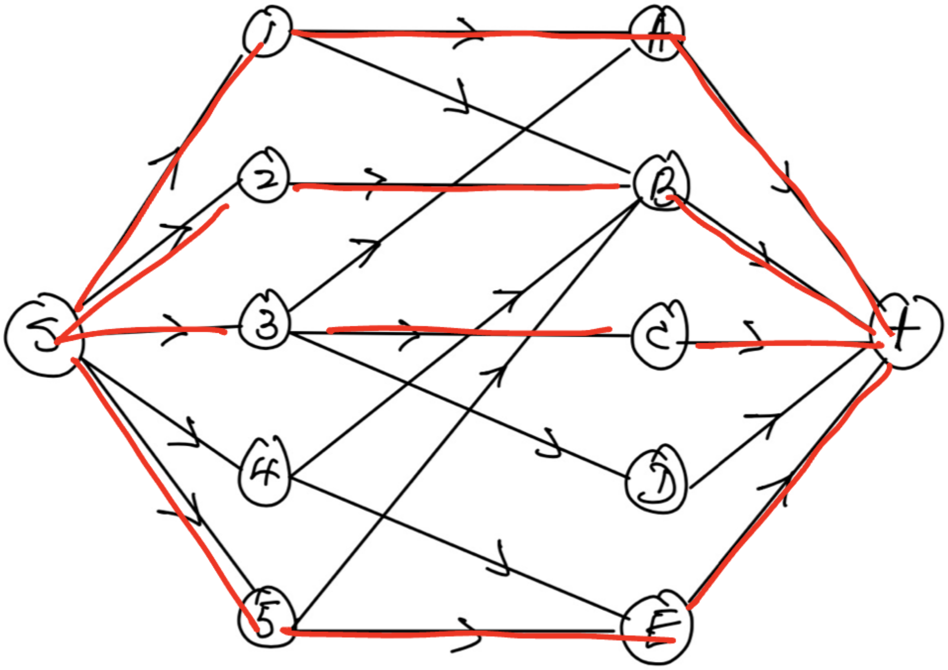
$= 2 + 0 + 1 + 2 + 1 + 3 + 2 + 1 + 5 + 3$
$+ 2 + 1 + 2 = 25$

5. Consider the following flow network from s to t. The capacity of each arc is 1. Find the maximum flow and the cut with minimum cut capacity.



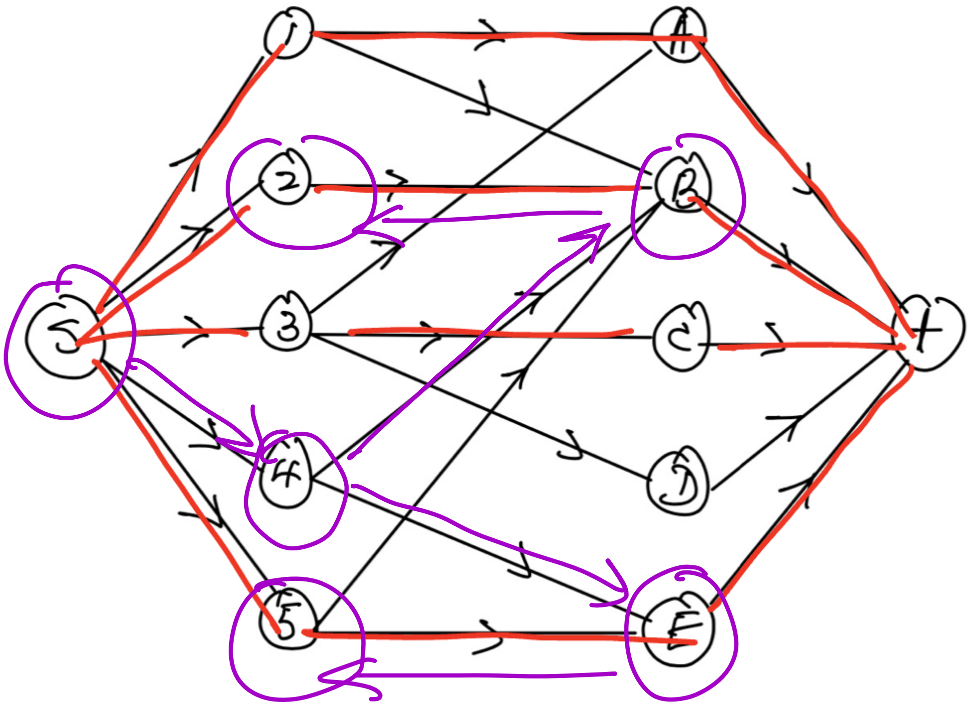
(The solution of the above problem in fact provides a solution of finding the maximum assignment or matching between vertices $1, 2, \ldots 5$ (e.g. employers) and vertices $A, B, \ldots E$ (e.g. job seekers).)

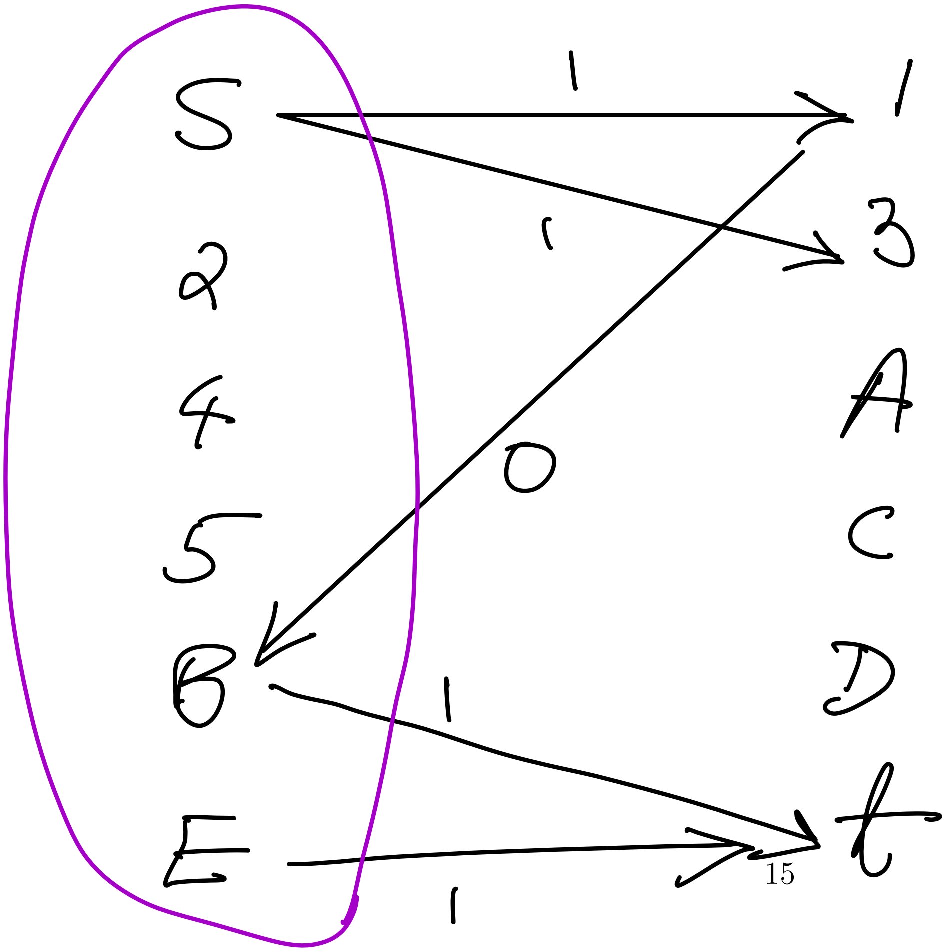
Step 1



You can use this blank page.

Step2





Cut capacity

= 1 + 1 + 0 + 1 + 1

= 4

Flow = 4

Flow = Cut Cap.

\\ //

4

6. Consider the following optimization problem:

$$\begin{array}{ll} \textbf{maximize} & 5\,x_1 + 4\,x_2 + 3\,x_3 \\ \\ \textbf{subject to} & 2\,x_1 + 3\,x_2 + \,x_3 \le 5 \\ & 4\,x_1 + \,x_2 + 2\,x_3 \le 11 \\ & 3\,x_1 + 4\,x_2 + 2\,x_3 \le 8 \\ \\ & \quad x_1,\ x_2,\ x_3 \ \ \ge \ \ 0. \end{array}$$

The initial and final dictionaries are as follows:

$$\begin{array}{lcccl} \textbf{maximize} & \zeta & = & & +\,5\,x_1 + 4\,x_2 + 3\,x_3 \\ \textbf{subject to} & w_1 & = & 5 & -\,2\,x_1 - 3\,x_2 - \,x_3 \\ & w_2 & = & 11 & -\,4\,x_1 - \,x_2 - 2\,x_3 \\ & w_3 & = & 8 & -\,3\,x_1 - 4\,x_2 - 2\,x_3 \end{array} \qquad \begin{array}{ccl} \zeta & = & 13 - 1\,w_1 - 3\,x_2 - 1\,w_3 \\ \hline x_1 & = & 2 - 2\,w_1 - 2\,x_2 + \,w_3 \\ w_2 & = & 1 + 2\,w_1 + 5\,x_2 \\ x_3 & = & 1 + 3\,w_1 + \,x_2 - 2\,w_3\,. \end{array}$$

(a) Suppose the objective function is changed to $\zeta(x) = 5x_1 + px_2 + 3x_3$. Find the maximum range of p such that the above final dictionary remains optimal. Within this range, how do the optimal solution x_1^*, x_2^*, x_3^* and optimal objective value ζ^* change?

(b) Suppose the second constraint is changed to $4x_1+x_2+2x_3 \le q$. Find the maximum range of q such that the above final dictionary remains optimal. Within this range, how do the optimal solution x_1^*, x_2^*, x_3^* and optimal objective value ζ^* change?

(c) Suppose the second constraint is changed to $4x_1 + (1+\epsilon)x_2 + 2x_3 \le 11$ where ϵ is some small number, say 10^{-6}. How do the optimal solution x_1^*, x_2^*, x_3^* and optimal objective value ζ^* change? (You can assume ϵ is small enough that the above final dictionary remains optimal.)

$$\zeta = C_B^T B^{-1} b - ((B^{-1}N)^T C_B - C_N)^T X_N$$

$$X_B = B^{-1} b - (B^{-1}N)\, X_N$$

$$A = \left[\begin{array}{ccc} 2 & 3 & 1 \\ 4 & 1 & 2 \\ 3 & 4 & 2 \\ x_1 & x_2 & x_3 \end{array}\right], \qquad N = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ w_1 & w_2 & w_3 \end{array}\right]$$

$$C = (5, 4, 3, 0, 0, 0), \qquad b = \begin{bmatrix} 5 \\ 11 \\ 8 \end{bmatrix}$$

$\mathcal{B} = \{x_1, w_2, x_3\}, \quad \mathcal{N} = \{w_1, x_2, w_3\}$

You can use this blank page.

$$B = \begin{bmatrix} 2 & 0 & 1 \\ 4 & 1 & 2 \\ 3 & 0 & 2 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

$$c_B^T = (5, 0, 3) \quad c_N^T = (0, 4, 0)$$

$$(B^{-1}N) = \begin{bmatrix} 2 & 2 & -1 \\ -2 & -5 & 0 \\ -3 & -1 & 2 \end{bmatrix} \quad \text{(read from dictionary)}$$

(a) We need $(B^{-1}N)^T c_B - c_N \geq 0$

$$\begin{bmatrix} 2 & -2 & -3 \\ 2 & -5 & -1 \\ -1 & 0 & 2 \end{bmatrix} \overset{c_B}{\begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix}} - \overset{c_N}{\begin{bmatrix} 0 \\ p \\ 0 \end{bmatrix}} \geq 0$$

$$\begin{bmatrix} 1 \\ 7 \\ 1 \end{bmatrix} \geq \begin{bmatrix} 0 \\ p \\ 0 \end{bmatrix} \Longrightarrow \boxed{7 > p}$$

You can use this blank page.

$x_N^* = 0, \quad x_B^* = B^{-1}b$, no change

$$\zeta^* = \underbrace{c_B^T B^{-1} b}_{} - \left(\text{———} \right) \underset{0}{\cancel{x_N}}$$

no change as $c_B = \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix}$ (c_2 is not in c_B)

(b) We need $B^{-1}b \geq 0$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 4 & 1 & 2 & 0 & 1 & 0 \\ 3 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 0 & 1 \\ 4 & 1 & 2 & 0 & 1 & 0 \\ 3 & 0 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & -2 & 4 & 1 & -4 \\ 0 & 0 & -1 & 3 & 0 & -2 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 2 \end{array}\right)$$

$$\underbrace{\begin{matrix} 2 & 0 & -1 \\ -2 & 1 & 0 \\ -3 & 0 & 2 \end{matrix}}_{B^{-1}}$$

$$B^{-1}b = \begin{bmatrix} 2 & 0 & -1 \\ -2 & 1 & 0 \\ -3 & 0 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

You can use this blank page.

10. $$\begin{bmatrix}2\\-10+q\\1\end{bmatrix} \geq \begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \boxed{q>10}$$

$$\Delta x_B = B^{-1}\Delta b = \begin{bmatrix}2&0&-1\\-2&1&0\\-3&0&2\end{bmatrix}\begin{bmatrix}0\\q-11\\0\end{bmatrix} = \begin{bmatrix}0\\q-11\\0\end{bmatrix} \leftarrow \Delta w_2^*$$

x_1^*, x_3^* no change, $x_2^* = 0$

$\Delta w_2^* = q-11$, Δw_1^*, $\Delta w_3^* = 0$

So no change for (x_1^*, x_2^*, x_2^*)

$$\Delta y^* = c_B^T B^{-1} \Delta b$$

$$= (5 \quad 0 \quad 3)\begin{bmatrix}2&0&-1\\-2&1&0\\-3&0&2\end{bmatrix}\begin{pmatrix}0\\q-11\\0\end{pmatrix}$$

$$= (5 \quad 0 \quad 3)\begin{pmatrix}0\\q-11\\0\end{pmatrix} = 0$$ no change

(c) $\Delta x_B = \Delta(B^{-1})\, b$

$\Delta(B^{-1}) = -B^{-1} \Delta B\, B^{-1}$

But $\Delta B = 0$ as x_2 is non-basic and hence the column corresponding to x_2 is not in B

$\Rightarrow \quad \Delta(B^{-1}) = 0$

$\Rightarrow \quad \Delta x_B = 0 \quad \& \quad x_N = 0$

So all variables do not change

$$\zeta = c_B^T B^{-1} b - \left((B^{-1}N)^T c_B - c_N \right)^T \underset{0}{\cancel{x_N}}$$

$$\Delta\zeta = c_B^T \Delta(B^{-1}) b = 0$$

So ζ does not change

7. Consider you try to solve a feasible but underdetermined problem $Ax = b$ so that there are infinitely many solutions. In practice, one often tries to select the solution x that has minimum norm, i.e. you need to solve $\min_x \|x\|$ subject to $Ax = b$ where $\|\cdot\|$ is some given norm. (Depending on the choice of norm, x might demonstrate different properties.)

   Solve the following problems:

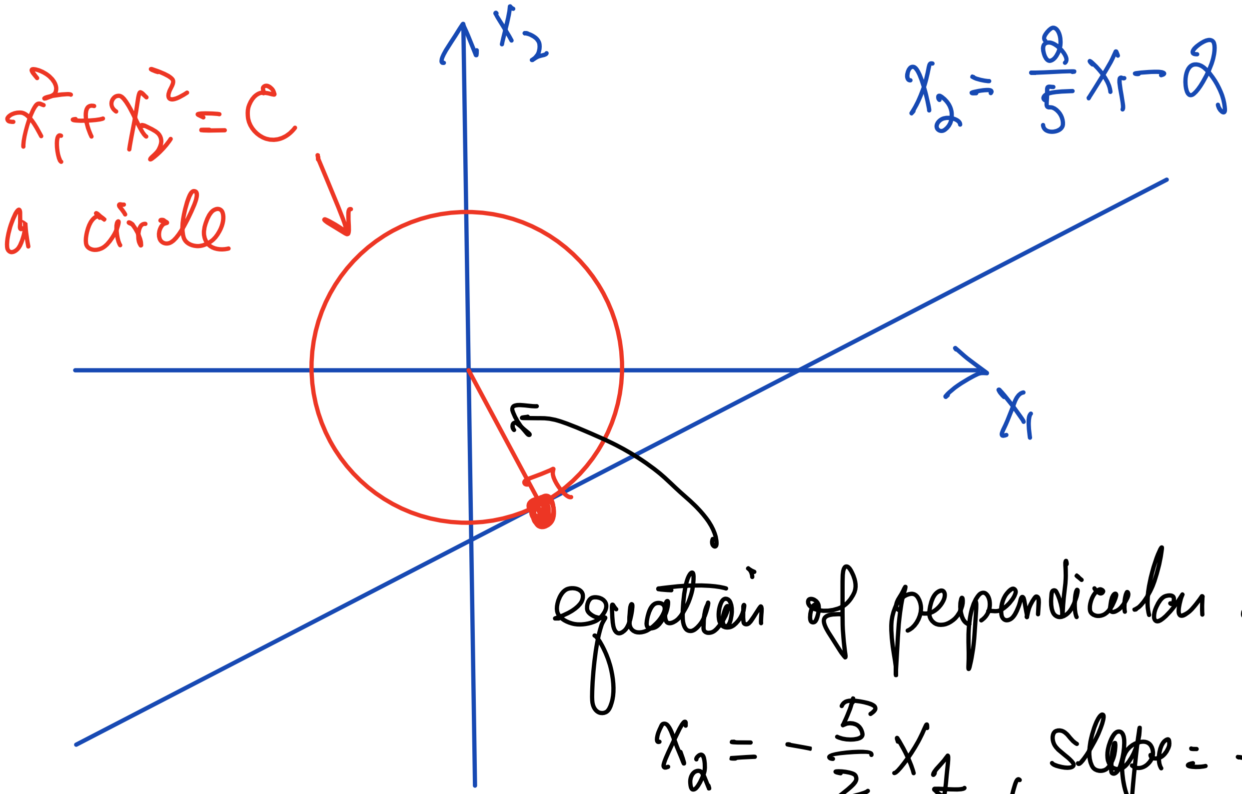
   (a) $\min_{x_1,x_2} \sqrt{x_1^2 + x_2^2}$, subject to $2x_1 - 5x_2 = 10$. $\rightarrow x_2 = \frac{2}{5}x_1 - 2$

   (b) $\min_{x_1,x_2} |x_1| + |x_2|$, subject to $2x_1 - 5x_2 = 10$.

   (c) $\min_{x_1,x_2} \max\{|x_1|, |x_2|\}$, subject to $2x_1 - 5x_2 = 10$.

   You can use any method, including graphical method. It is advantageous to have a visual understanding of $\|x\| =$ constant for the above norms.

(a)



equation of perpendicular line

$x_2 = -\frac{5}{2}x_1$, slope $= -\frac{1}{\left(\frac{2}{5}\right)}$

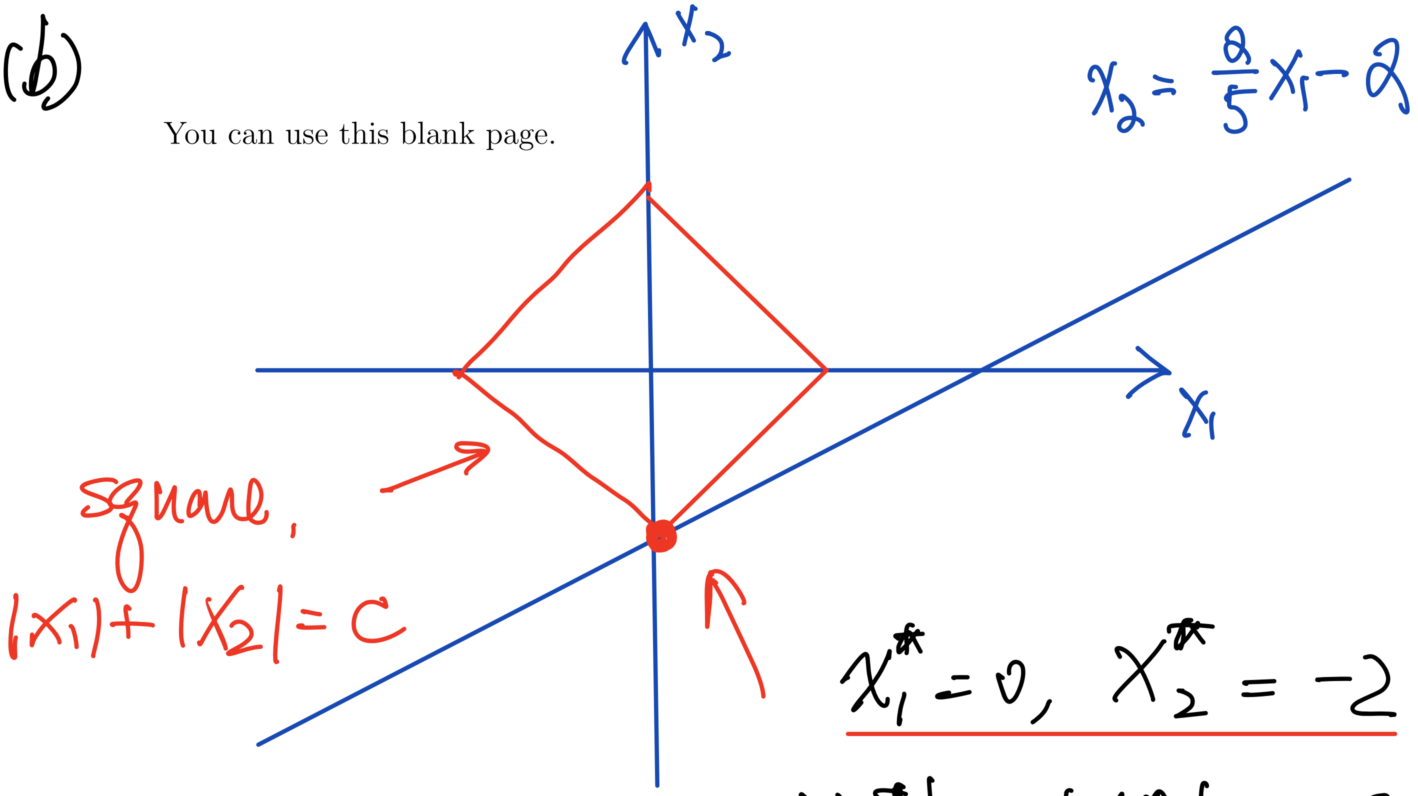
$$-\frac{5}{2}x_1 = \frac{2}{5}x_2 - 2$$

$$2 = \left(\frac{2}{5} + \frac{5}{2}\right)x_1 \Rightarrow x_1^* = \frac{20}{29}, \quad x_2^* = -\frac{50}{29}$$

$$\sqrt{(x_1^*)^2 + (x_2^*)^2} = \sqrt{\left(\frac{20}{29}\right)^2 + \left(\frac{50}{29}\right)^2} = \frac{10}{\sqrt{29}}$$

(b)

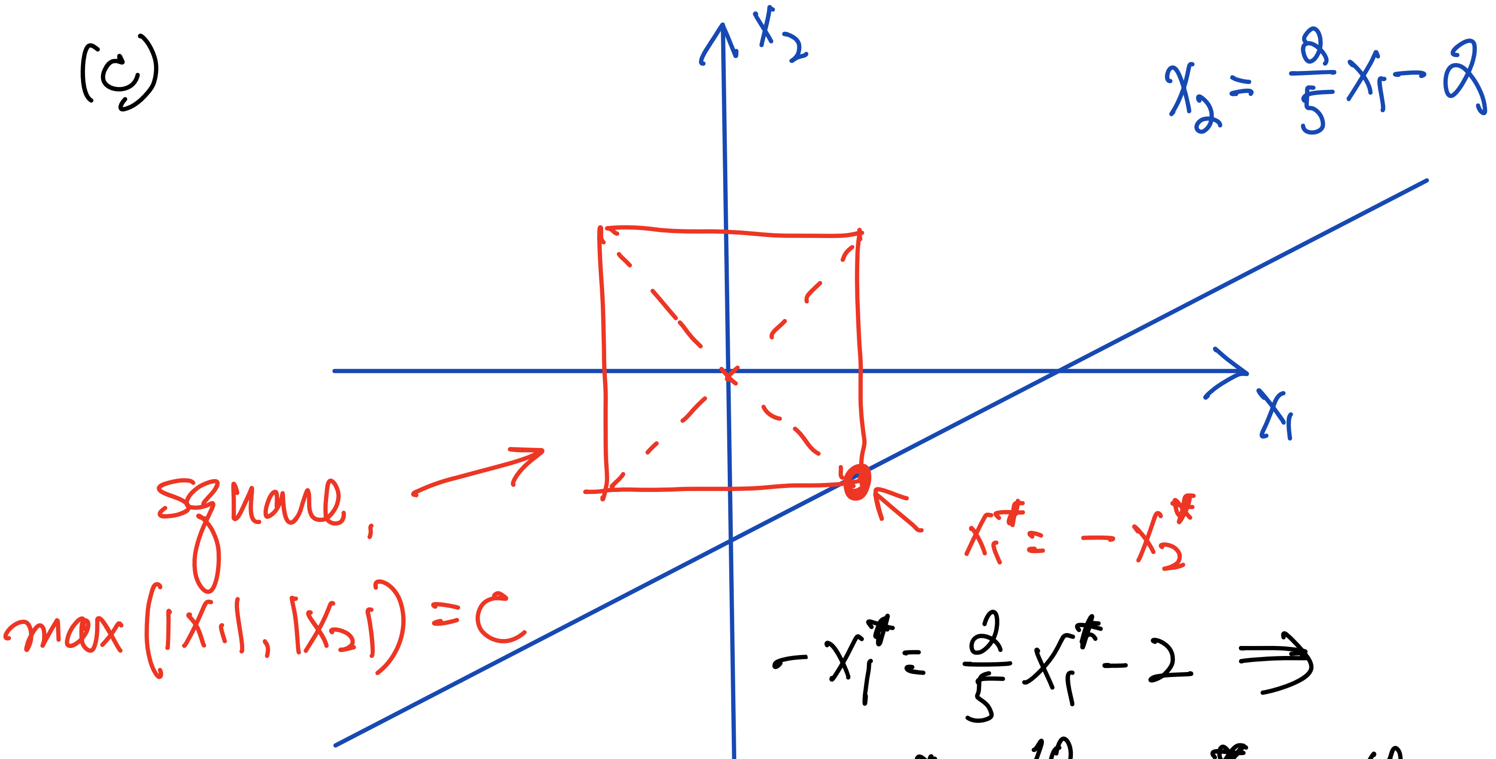
You can use this blank page.



$x_1^* = 0, \quad x_2^* = -2$

$|x_1^*| + |x_2^*| = 2$

(c)



$-x_1^* = \frac{2}{5}x_1^* - 2 \Rightarrow$

$\Rightarrow x_1^* = \frac{10}{7}, \quad x_2^* = -\frac{10}{7}$

$\max(|x_1^*|, |x_2^*|) = \frac{10}{7}$