MA 421: Linear Programming and Optimization Techniques
Fall 2024, Final Exam

Instructor: Yip

- This test booklet has SIX QUESTIONS, totaling 100 points for the whole test. You have 120 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is **closed book, closed note, with no electronic devices**.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.
- **As a rule of thumb, you should give explicit and useful answers.** No points will be given for just writing down some generically true statements. In other words, your answers should try to make use of all the information given in the question.
- **As a rule of thumb, you should only use those methods that have been covered in class.** If you use some other methods "for the sake of convenience", at our discretion, we might not give you any credit. You have the right to contest. In that event, **you will be asked to explain your answer using only what has been covered in class up to the point of time of this exam.**

Name: Answer Key (Major:)

Question	Score
1.(10 pts)	
2.(10 pts)	
3.(10 pts)	
4.(20 pts)	
5.(20 pts)	
6.(30 pts)	
Total (100 pts)	

Formula sheet

1. Matrix form of simplex method [V, p.91, 92].

   Given the following linear program problem and its dual in their standard forms:

$$
(\mathrm{P}):\left\{\begin{array}{lrcl}\text{maximize} & \zeta(x) & = & c^T x\\ \text{subject to} & Ax & \leq & b;\\ & x & \geq & 0.\end{array}\right. \qquad (\mathrm{D}):\left\{\begin{array}{lrcl}\text{minimize} & \xi(y) & = & b^T y\\ \text{subject to} & A^T y & \geq & c;\\ & y & \geq & 0.\end{array}\right.
$$

   During simplex iterations, the above can be transformed into the following matrix form:

$$
\begin{aligned}
\zeta &= c_{\mathcal{B}}^T(B^{-1}b) - \left((B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}}\right)^T X_{\mathcal{N}}\\
X_{\mathcal{B}} &= B^{-1}b - \left(B^{-1}N\right) X_{\mathcal{N}}
\end{aligned}
$$

   with the following dual form:

$$
\begin{aligned}
-\xi &= -c_{\mathcal{B}}^T(B^{-1}b) - \left(B^{-1}b\right)^T Z_{\mathcal{B}}\\
Z_{\mathcal{N}} &= \left((B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}}\right) + \left(B^{-1}N\right)^T Z_{\mathcal{B}}
\end{aligned}
$$

   where in the above

   (a) $\mathcal{B}$ and $\mathcal{N}$ are the basic and non-basic variable indices;

   (b) $c_{\mathcal{B}}$ and $c_{\mathcal{N}}$ are the coefficients in the objective function corresponding to the basic and non-basic variables;

   (c) B and N are matrices formed by the collecting the columns from the augmented matrix $[A\ \ I]$ corresponding to the basic and non-basic variables.

   (d) $X_{\mathcal{B}}$ and $X_{\mathcal{N}}$ are the basic and non-basic primal variables, and $Z_{\mathcal{B}}$ and $Z_{\mathcal{N}}$ are the basic and non-basic dual variables.

2. Infinitesimal change of matrix inverse. Let M be an invertible matrix. Then

$$
\delta(M^{-1}) = -M^{-1}(\delta M)M^{-1}
$$

   (where δ refers to infinitesimal change in a quantity).

3. Caratheodory Theorem [V, p.162].

   THEOREM 10.3. *The convex hull* conv(S) *of a set* S *in* $\mathbb{R}^m$ *consists of all convex combinations of* $m+1$ *points from* S*:*

$$
\operatorname{conv}(S) = \left\{ z = \sum_{j=1}^{m+1} t_j z_j : z_j \in S \text{ and } t_j \geq 0 \text{ for all } j, \text{ and } \sum_j t_j = 1 \right\}.
$$

4. Farkas Lemma [V, p. 165].

Lemma 10.5. *The system $Ax \le b$ has no solutions if and only if there is a y such that*

$$\begin{aligned} A^T y &= 0 \\ y &\ge 0 \\ b^T y &< 0. \end{aligned} \tag{10.8}$$

5. Separation Theorem [V, p.163].

Theorem 10.4. *Let P and $\tilde{P}$ be two disjoint nonempty polyhedra in $\mathbb{R}^n$. Then there exist disjoint half-spaces H and $\tilde{H}$ such that $P \subset H$ and $\tilde{P} \subset \tilde{H}$.*

If $P = \{x : Ax \le \tilde{b}\}$ and $\tilde{P} = \{x : \tilde{A}x \le \tilde{b}\}$, then

$$H = \{x : y^T Ax \le y^T b\}, \quad \text{and} \quad \tilde{H} = \{x : \tilde{y}^T \tilde{A}x \le \tilde{y}^T \tilde{b}\},$$

where y and $\tilde{y}$ are found by Farkas Lemma:

$$\begin{bmatrix} A^T & \tilde{A}^T \end{bmatrix} \begin{bmatrix} y \\ \tilde{y} \end{bmatrix} = A^T y + \tilde{A}^T \tilde{y} = 0 \tag{10.5}$$

$$\begin{bmatrix} y \\ \tilde{y} \end{bmatrix} \ge 0 \tag{10.6}$$

$$\begin{bmatrix} b^T & \tilde{b}^T \end{bmatrix} \begin{bmatrix} y \\ \tilde{y} \end{bmatrix} = b^T y + \tilde{b}^T \tilde{y} < 0. \tag{10.7}$$

(Note that the form of $\tilde{H}$ given here is equivalent to the one given in [V, p.164] but it looks nicer and is easier to remember.)

6. Distance formula. The distance from a point (x_0, y_0) to the straightline $ax + by + c = 0$ is given by

$$\left| \frac{ax_0 + by_0 + c}{\sqrt{a^2 + b^2}} \right|.$$

7. Variables of network flows. Given a network $(\mathcal{N}, \mathcal{A})$ (where $\mathcal{N}$ and $\mathcal{A}$ denote the nodes and arcs), and a spanning tree $\mathcal{T}$ of the network, we have the following set of variables [V, p.231]:

   (a) Flow variables: x_{ij} (found by balance equations)

   (b) Dual variables y_j and dual slacks z_{ij} which satisfy:

$$y_j - y_i + z_{ij} = c_{ij}, \quad \text{for } (i, j) \in \mathcal{A}.$$

Note that for $(i,j) \in \mathcal{T}$, we have (by complementary slackness) that $z_{ij} = 0$. The usual way to find y_i and z_{ij} is to *first* solve for y_i using $y_j - y_i = c_{ij}$ for $(i,j) \in \mathcal{T}$ and *then* use $z_{ij} = c_{ij} - y_j + y_i$ for $(i,j) \notin \mathcal{T}$.

8. Bellman's equation for the label (or value function) for finding the shortest distance/path from (any) node i to a root node (r) is given by [V, p.261, 262]

$$v_i = \min_j \{c_{ij} + v_j : \ (i,j) \in \mathcal{A}\}, \quad \text{for } i \neq r \quad v_r = 0.$$

The function v can be found by the following iteration procedure:

3.2.1. *Method of Successive Approximation.* Bellman's equation is an implicit system of equations for the values v_i, $i \in \mathcal{N}$. Implicit equations such as this arise frequently and beg to be solved by starting with a guess at the solution, using this guess in the right-hand side, and computing a new guess by evaluating the right-hand side. This approach is called the *method of successive approximations*. To apply it to the shortest-path problem, we initialize the labels as follows:

$$v_i^{(0)} = \begin{cases} 0 & i = r \\ \infty & i \neq r. \end{cases}$$

Then the updates are computed using Bellman's equation:

$$v_i^{(k+1)} = \begin{cases} 0 & i = r \\ \min\{c_{ij} + v_j^{(k)} : (i,j) \in \mathcal{A}\} & i \neq r. \end{cases}$$

9. Dijkstra algorithm for finding the shortest path (with non-negative costs c_{ij}) [V, p.263].

Initialize:

    $\mathcal{F} = \emptyset$

$$v_j = \begin{cases} 0 & j = r, \\ \infty & j \neq r. \end{cases}$$

while ($|\mathcal{F}^c| > 0$){

    $j = \text{argmin}\{v_k : k \notin \mathcal{F}\}$

    $\mathcal{F} \leftarrow \mathcal{F} \cup \{j\}$

    for each i for which $(i,j) \in \mathcal{A}$ and $i \notin \mathcal{F}$ {

        if $(c_{ij} + v_j < v_i)$ {

            $v_i = c_{ij} + v_j$

            $h_i = j$

        }

    }

}

10. Greedy algorithm for finding minimum spanning tree [Berksimas-Tsitsiklis, Intro. Lin. Opt., p.344].

Greedy algorithm for the minimum spanning tree problem

1. The input to the algorithm is a connected undirected graph $G = (\mathcal{N}, \mathcal{E})$ and a coefficient c_e for each edge $e \in \mathcal{E}$. The algorithm is initialized with a tree $(\mathcal{N}_1, \mathcal{E}_1)$ that has a single node and no edges ($\mathcal{E}_1$ is empty).

2. Once $(\mathcal{N}_k, \mathcal{E}_k)$ is available, and if $k < n$, we consider all edges $\{i, j\} \in \mathcal{E}$ such that $i \in \mathcal{N}_k$ and $j \notin \mathcal{N}_k$. Choose an edge $e^* = \{i, j\}$ of this type whose cost is smallest. Let

$$\mathcal{N}_{k+1} = \mathcal{N}_k \cup \{j\}, \qquad \mathcal{E}_{k+1} = \mathcal{E}_k \cup \{e^*\}.$$

11. Karush-Kuhn-Tucker (KKT) condition [Chong-Zak]

Theorem 21.1 ***Karush-Kuhn-Tucker (KKT) Theorem.*** *Let $f, \boldsymbol{h}, \boldsymbol{g} \in \mathcal{C}^1$. Let $\boldsymbol{x}^*$ be a regular point and a local minimizer for the problem of minimizing f subject to $\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}$, $\boldsymbol{g}(\boldsymbol{x}) \le \boldsymbol{0}$. Then, there exist $\boldsymbol{\lambda}^* \in \mathbb{R}^m$ and $\boldsymbol{\mu}^* \in \mathbb{R}^p$ such that:*

1. $\boldsymbol{\mu}^* \ge \boldsymbol{0}$.
2. $Df(\boldsymbol{x}^*) + \boldsymbol{\lambda}^{*\top} D\boldsymbol{h}(\boldsymbol{x}^*) + \boldsymbol{\mu}^{*\top} D\boldsymbol{g}(\boldsymbol{x}^*) = \boldsymbol{0}^\top$.
3. $\boldsymbol{\mu}^{*\top} \boldsymbol{g}(\boldsymbol{x}^*) = 0$.
4. $\boldsymbol{h}(\boldsymbol{x}^*) = \boldsymbol{0}$.
5. $\boldsymbol{g}(\boldsymbol{x}^*) \le \boldsymbol{0}$.

1. Consider the following (linear programming) problem:

$$\begin{array}{lrcrcrcl} \text{maximize} & 5x_1 & + & 5x_2 & + & 3x_3 & & \\ \text{subject to} & x_1 & + & 3x_2 & + & x_3 & \leq & 3 \\ & -x_1 & & & + & 3x_3 & \leq & 2 \\ & 2x_1 & - & x_2 & + & 2x_3 & \leq & 4 \\ & 2x_1 & + & 3x_2 & - & x_3 & \leq & 2 \\ & & & & & x_1, x_2, x_3 & \geq & 0. \end{array}$$

$2 < 3 \Rightarrow w_1 > 0$

$\frac{2}{3} < 2 \Rightarrow w_2 > 0$

$4 = 4 \Rightarrow w_3 = 0$

$2 = 2 \Rightarrow w_4 = 0$

Someone claims that $x_1 = \frac{4}{3},\ x_2 = 0,\ x_3 = \frac{2}{3}$ is an optimal solution. Prove or disprove this claim.

$x_1 > 0 \Rightarrow z_1 = 0$

$x_3 > 0 \Rightarrow z_3 = 0$

$w_1 > 0 \Rightarrow y_1 = 0$

$w_2 > 0 \Rightarrow y_2 = 0$

Dual Problem:

$$y_1 - y_2 + 2y_3 + 2y_4 \cancel{\geq 5} = 5 \quad z_1 = 0$$

$$3y_1 - y_3 + 3y_4 \geq 5$$

$$y_1 + 3y_2 + 2y_3 - y_4 \cancel{\geq 3} = 3 \quad z_3 = 0$$

$$\Rightarrow \quad 2y_3 + 2y_4 = 5$$

$$2y_3 - y_4 = 3$$

$$\Rightarrow \quad y_4 = \frac{2}{3}, \quad y_3 = \frac{1}{2}\left(3 + \frac{2}{3}\right) = \frac{11}{6}$$

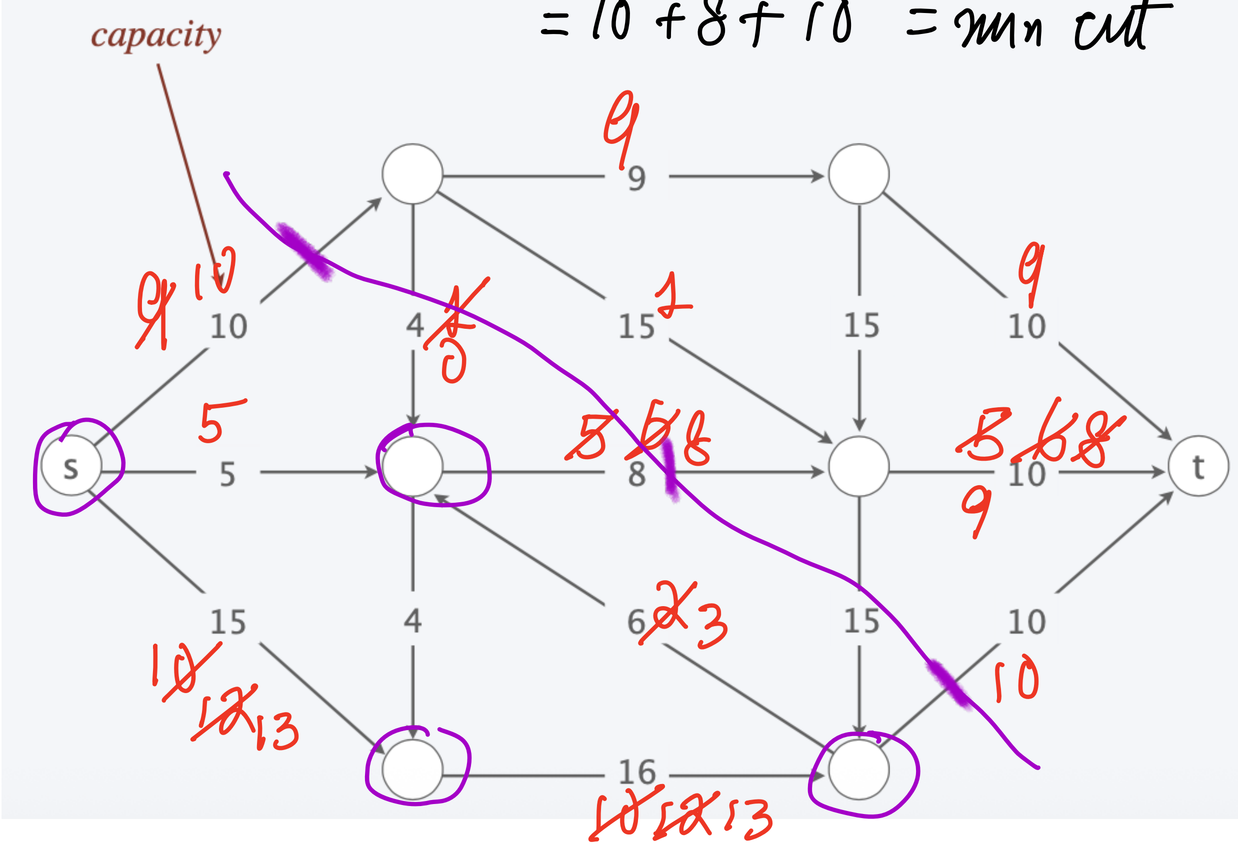
$$-y_3 + 3y_4 = -\frac{11}{6} + 2 \underline{< 5} \ !!$$

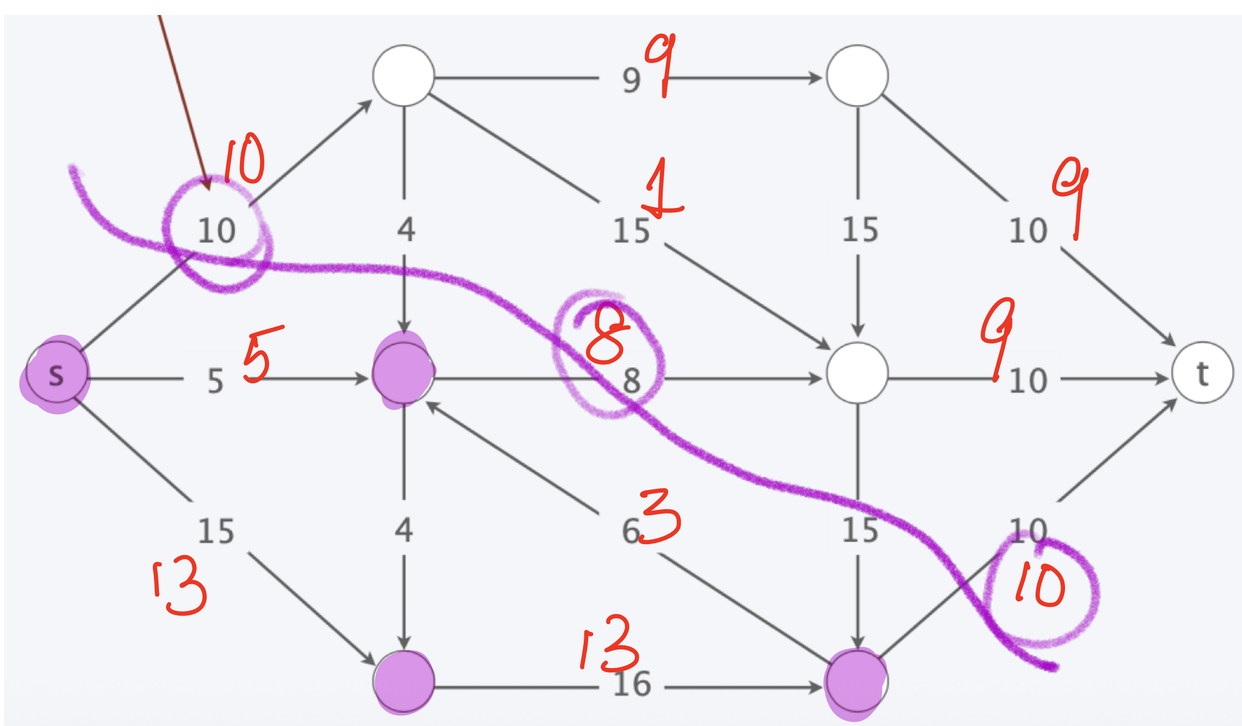
Not opt.!

2. Find the maximum flow and minumum cut for the following network. (The node s is the source and t is the sink. The extra graph copies in the next pages can be used for scrap work.)

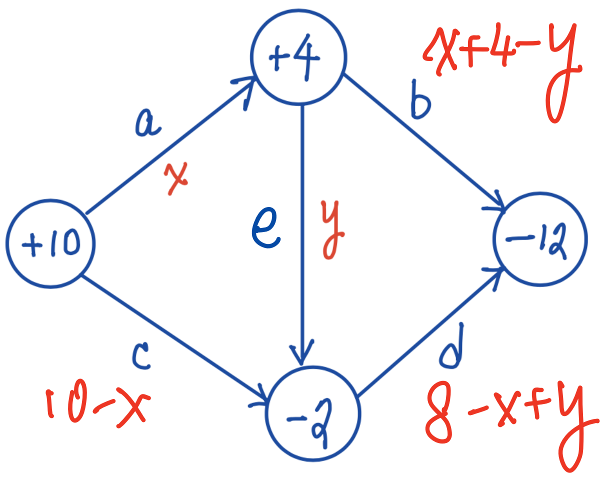
Max flow $= 10 + 5 + 13 = 28$ $(= 9 + 9 + 10)$

$= 10 + 8 + 10 =$ min cut





3. Consider the following network:



where the numbers inside each node is the production level (supply if positive/demand if negative) and $a, b, c, d \geq 0$ are the unit transportation costs between the nodes. Find the optimal transshipment scheme for any values of a, b, c, d.

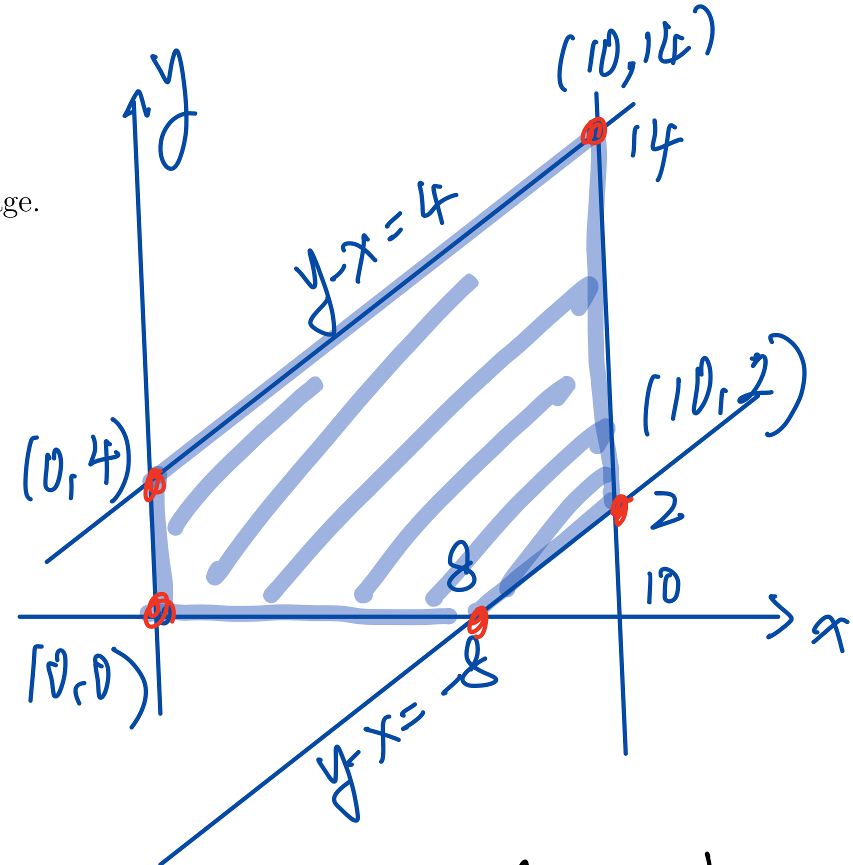
(Hint: you can set up the problem using the x and y variables which represent the flows between the respective nodes. Make sure you identify the feasible region for x and y. The use of graphical method is permissible.)

Cost fct : $f(x,y) = ax + b(x+4-y) + c(10-x)$
$$+ d(8-x+y) + ey$$
$$= (a+b-c-d)x + (-b+d+e)y$$
$$+ 4b + 10c + 8d$$

Constraints: $\begin{cases} x \geq 0, \ y \geq 0 \\ 10-x \geq 0, \ 4+x-y \geq 0, \ 8-x+y \geq 0 \end{cases}$

$$\Longleftrightarrow \begin{cases} 0 \leq x \leq 10 \\ -8 \leq y-x \leq 4 \end{cases}$$

You can use this blank page.



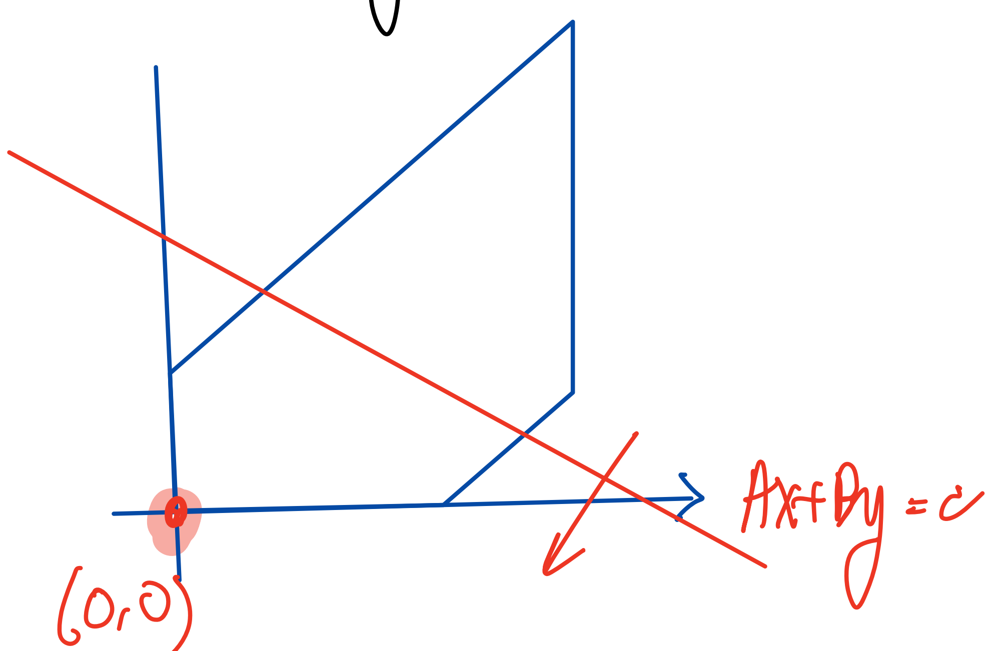
$$\min f(x,y) = Ax + By + C$$

$A = a+b-c-d$

$B = -b+d+e$

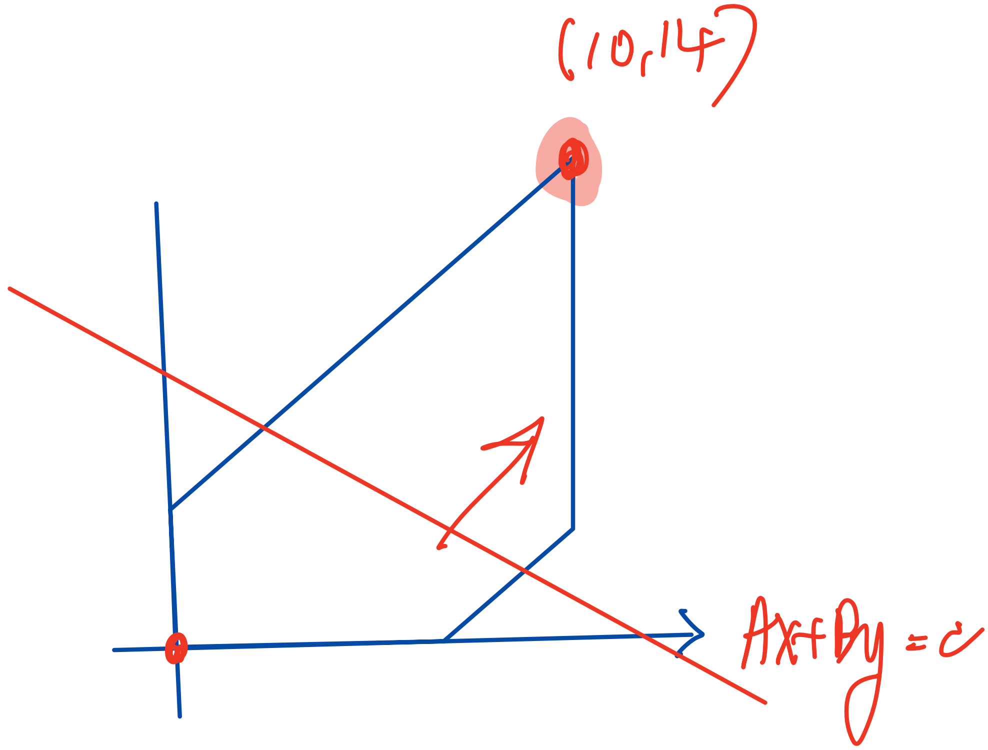
$C = 4b+10c+8d$

① $A, B \geq 0 \Rightarrow x^* = 0,\ y^* = 0$



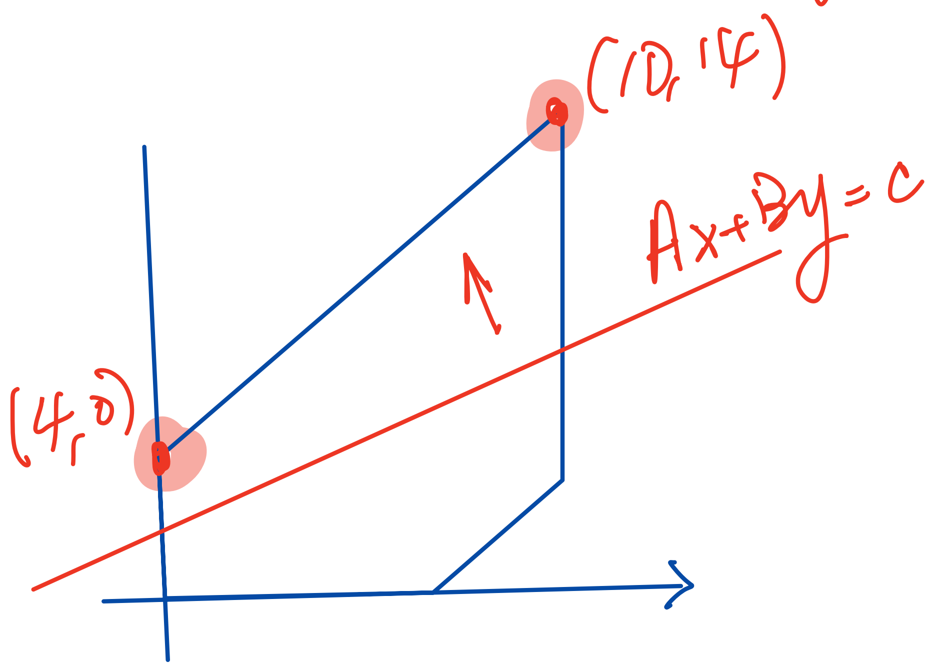
(2) $A, B < 0 \Rightarrow x^* = 10, \; y^* = 14$

You can use this blank page.



(3) $A \geq 0, \; B < 0$

(eg $x - y = c$)



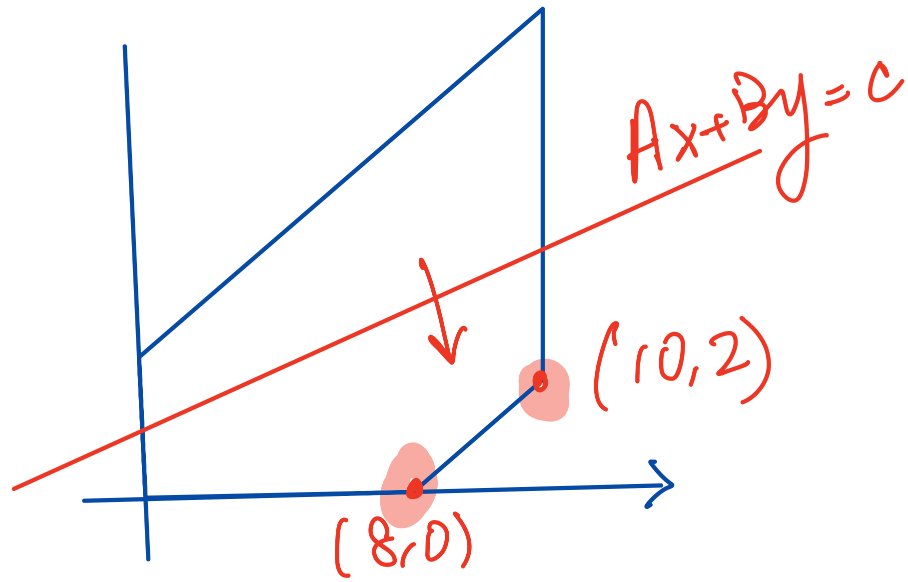
If $\left|\frac{A}{B}\right| < 1 \Rightarrow x^* = 10, \; y^* = 14$

If $\left|\frac{A}{B}\right| > 1 \Rightarrow x^* = 0, \; y^* = 4$

You can use this blank page.

4 $A<0,\ B\geq 0$

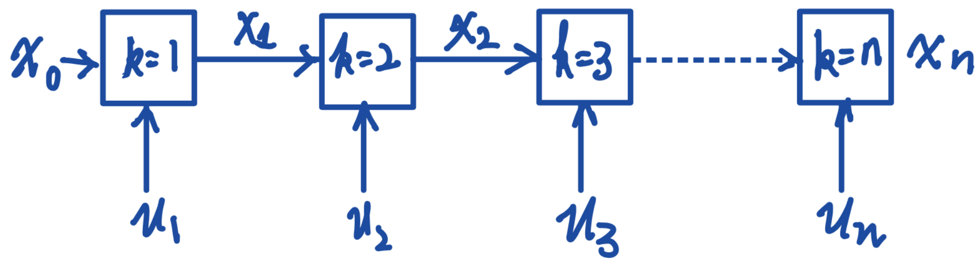
(eg $-x+y=c$)



If $\left|\frac{A}{B}\right|<1 \Rightarrow x^*=8,\ y^*=0$

If $\left|\frac{A}{B}\right|>1 \Rightarrow x^*=10,\ y^*=2$

4. Consider the following input-output-control system:



where $x_0, x_1, x_2 \dots$ are the initial and subsequent state variables and $u_1, u_2 \dots$ are the control variables. The x_k's are determined by the following rule:

$$x_{k+1} = ax_k + bu_{k+1}, \quad \text{for } 0 \le k \le n-1 \text{ (and } x_0 \text{ is given/prescribed).} \tag{1}$$

The objective is to find the controls so as to

$$\min_{u_1, u_2, \dots u_n} x_n^2 + \sum_{k=1}^{n} u_k^2, \tag{2}$$

i.e. you want the end state and the controls to be as small as possible.

(a) Solve (1)–(2) for $x_0 = 1,\ a = -1,\ b = 2,\ n = 2.$

(b) Now suppose (2) is changed to

$$\min_{u_1, u_2, \dots u_n} |x_n| + \sum_{k=1}^{n} |u_k|. \tag{3}$$

Formulate the problem as a linear programming problem for the same parameters: $x_0 = 1,\ a = -1,\ b = 2,\ n = 2.$

Write your formulation in *standard form* and produce an *initial feasible dictionary.*

(a)
$$\begin{cases} x_1 = -x_0 + 2u_1 \\ x_2 = -x_1 + 2u_2 \end{cases} \iff \begin{aligned} x_1 - 2u_1 &= -1 \\ x_2 + x_1 - 2u_2 &= 0 \end{aligned}$$

$$\boxed{\begin{aligned} \min \quad & x_2^2 + u_1^2 + u_2^2 \\ \text{s.t.} \quad & x_1 - 2u_1 = -1 \\ & x_2 + x_1 - 2u_2 = 0 \end{aligned}}$$

M1

$L(x_1, x_2, u_1, u_2, \lambda_1, \lambda_2)$

You can use this blank page.

$$= x_2^2 + u_1^2 + u_2^2 + \lambda_1(x_1 - 2u_1 - 1) + \lambda_2(x_2 + x_1 - 2u_2)$$

$$\left.\begin{aligned} \partial_{x_1} = 0 &\Rightarrow \lambda_1 + \lambda_2 = 0 \\ \partial_{x_2} = 0 &\Rightarrow 2x_2 + \lambda_2 = 0 \\ \partial_{u_1} = 0 &\Rightarrow 2u_1 - 2\lambda_1 = 0 \\ \partial_{u_2} = 0 &\Rightarrow 2u_2 - 2\lambda_2 = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} &u_1 = \lambda_1,\ u_2 = \lambda_2 = -\lambda_1 \\ &x_2 = -\frac{\lambda_2}{2} = \frac{\lambda_1}{2}, \end{aligned}$$

$$\begin{aligned} x_1 - 2u_1 &= -1 \\ x_2 + x_1 - 2u_2 &= 0 \end{aligned} \Longrightarrow \begin{aligned} &x_1 = -1 + 2u_1 = -1 + 2\lambda_1 \\ &\frac{\lambda_1}{2} - 1 + 2\lambda_1 + 2\lambda_1 = 0 \\ &\lambda_1 = \frac{2}{9} \\ &\lambda_2 = -\frac{2}{9} \end{aligned}$$

$$\boxed{\begin{aligned} &x_1 = -1 + \frac{4}{9} = -\frac{5}{9}, \quad u_1 = \frac{2}{9}, \qquad \min = \left(\frac{1}{9}\right)^2 + \left(\frac{2}{9}\right)^2 \times 2 \\ &x_2 = \frac{1}{9}, \quad u_2 = -\frac{2}{9}, \qquad = \frac{1}{9} \end{aligned}}$$

M2

$$\min \; x_2^2 + u_1^2 + u_2^2$$
$$\text{s.t.} \quad x_1 - 2u_1 = -1$$
$$x_2 + x_1 - 2u_2 = 0$$

$$x_2 = -x_1 + 2u_2 = -2u_1 + 1 + 2u_2$$

$$\min \; (1 - 2u_1 + 2u_2)^2 + u_1^2 + u_2^2$$

$$\partial u_1 = 0 \Rightarrow (1 - 2u_1 + 2u_2)(-2) + u_1 = 0$$
$$5u_1 - 4u_2 = 2$$

$$\partial u_2 = 0 \Rightarrow (1 - 2u_1 + 2u_2)(2) + u_2 = 0$$
$$-4u_1 + 5u_2 = -2$$

$$u_1 = \frac{\begin{vmatrix} 2 & -4 \\ -2 & 5 \end{vmatrix}}{\begin{vmatrix} 5 & -4 \\ -4 & 5 \end{vmatrix}} = \frac{2}{9}, \quad u_2 = \frac{\begin{vmatrix} 5 & 2 \\ -4 & -2 \end{vmatrix}}{\begin{vmatrix} 5 & -4 \\ -4 & 5 \end{vmatrix}} = -\frac{2}{9}$$

$$x_1 = -\frac{5}{9}, \quad x_2 = \frac{1}{9}$$

You can use this blank page.

(b) $$\min \ |x_2| + |u_1| + |u_2|$$

$$\text{s.t.} \quad x_1 - 2u_1 = -1$$

$$x_2 + x_1 - 2u_2 = 0$$

$$\min \ t_1 + t_2 + t_3$$

$$-t_1 \le x_2 \le t_1$$

$$-t_2 \le u_1 \le t_2$$

$$-t_3 \le u_2 \le t_3$$

$$\left.\begin{aligned} x_1 - 2u_1 &= -1 \\ x_1 + x_2 - 2u_2 &= 0 \end{aligned}\right\} \quad \begin{aligned} w_1 &= 1 + x_1 - 2u_1 \ge 0 \\ w_2 &= x_1 + x_2 - 2u_2 \ge 0 \end{aligned}$$

$$\min \ t_1 + t_2 + t_3 + w_1 + w_2$$

$$\left.\begin{aligned} \text{s.t.} \quad & x_2 - t_1 \le 0, \quad -x_2 - t_1 \le 0 \\ & u_1 - t_2 \le 0, \quad -u_1 - t_2 \le 0 \\ & u_2 - t_3 \le 0, \quad -u_2 - t_3 \le 0 \\ & w_1 = 1 + x_1 - 2u_1 \\ & w_2 = x_1 + x_2 - 2u_2 \\ & t_1, t_2, t_3, w_1, w_2 \ge 0 \end{aligned}\right\}$$

feasible !

$$\max \quad -t_1-t_2-t_3-w_1-w_2$$

You can use this blank page.

$$= -t_1-t_2-t_3-1-x_1+2u_1-x_1-x_2+2u_2$$

$$= -1-t_1-t_2-t_3-2x_1-x_2+2u_1+2u_2$$

$$\text{LP:} \quad \max \ \zeta = -1-t_1-t_2-t_3-2x_1-x_2+2u_1+2u_2$$

$$\begin{aligned}
\text{s.t.} \quad w_1 &= 1+x_1-2u_1 \\
w_2 &= x_1+x_2-2u_2 \\
w_3 &= t_1-x_2 \\
w_4 &= t_1+x_2 \\
w_5 &= t_2-u_1 \\
w_6 &= t_2+u_1 \\
w_7 &= t_3-u_2 \\
w_8 &= t_3+u_2
\end{aligned}$$

$$t_1, t_2, t_3, w_1, \ldots, w_8 \geq 0$$

$u_1 \to \frac{1}{2}$ 1^{st} iteration

exchange u_1 & w_1

5. Consider the following quadratic programming problem:

$$\begin{aligned} \min_{x_1,x_2,x_3} \quad & f(x_1,x_2,x_3) = x_1^2 + x_2^2 + x_3^2 - 7x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 1, \\ & 4x_1 + 5x_2 + 2x_3 = 3, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

(a) Relate the above to a linear programming problem and write down a corresponding feasible dictionary (plus any extra rule(s)).

(b) Solve the above minimization problem. You can use the linear programming formulation in (a) or any other method of your choice.

$$L(x_1, x_2, x_3, \lambda_1, \lambda_2, \mu_1, \mu_2, \mu_3)$$

$$= x_1^2 + x_2^2 + x_3^2 - 7x_3 + \lambda_1(x_1 + x_2 + x_3 - 1) + \lambda_2(4x_1 + 5x_2 + 2x_3 - 3) - \mu_1 x_1 - \mu_2 x_2 - \mu_3 x_3$$

$$\partial_{x_1} = 0 \Rightarrow 2x_1 + \lambda_1 + 4\lambda_2 - \mu_1 = 0$$

$$\partial_{x_2} = 0 \Rightarrow 2x_2 + \lambda_1 + 5\lambda_2 - \mu_2 = 0$$

$$\partial_{x_3} = 0 \Rightarrow 2x_3 - 7 + \lambda_1 + 2\lambda_2 - \mu_3 = 0$$

$$x_1 + x_2 + x_3 - 1 = 0$$

$$4x_1 + 5x_2 + 2x_3 - 3 = 0$$

$$x_1, x_2, x_3, \mu_1, \mu_2, \mu_3 \geq 0$$

$$x_1\mu_1 = 0, \quad x_2\mu_2 = 0, \quad x_3\mu_3 = 0$$

(a) LP Formulation: introduce auxiliary variables:

You can use this blank page.

$$\begin{cases} w_1 = -2x_1 - \lambda_1 - 4\lambda_2 + \mu_1 \\ w_2 = -2x_2 - \lambda_1 - 5\lambda_2 + \mu_2 \\ w_3 = 7 - 2x_3 - \lambda_1 - 2\lambda_2 + \mu_3 \\ w_4 = 1 - x_1 - x_2 - x_3 \\ w_5 = 3 - 4x_1 - 5x_2 - 2x_3 \end{cases}$$

$$w_1, \dots, w_5, \ x_1, x_2, x_3, \ \mu_1, \mu_2, \mu_3 \geq 0$$

$$x_1\mu_1 = 0, \ x_2\mu_2 = 0, \ x_3\mu_3 = 0$$

$$\min \ \underbrace{w_1 + w_2 + w_3 + w_4 + w_5}$$

$$= -7x_1 - 8x_2 - 5x_3 - 3\lambda_1 - 11\lambda_2 + \mu_1 + \mu_2 + \mu_3$$

$$\Updownarrow$$

$$\boxed{\max \ 7x_1 + 8x_2 + 5x_3 + 3\lambda_1 + 11\lambda_2 - \mu_1 - \mu_2 - \mu_3}$$

(b) Solve for

You can use this blank page.

$$\begin{cases} x_1 + x_2 + x_3 - 1 = 0 \\ 4x_1 + 5x_2 + 2x_3 - 3 = 0 \end{cases}$$

$$\begin{cases} x_1 + x_2 = 1 - x_3 \\ 4x_1 + 5x_2 = 3 - 2x_3 \end{cases}$$

$$x_1 = 2 - 5x_3 + 2x_3 = 2 - 3x_3$$

$$x_2 = 3 - 4 - 2x_3 + 4x_3 = -1 + 2x_3$$

$$\min \quad x_1^2 + x_2^2 + x_3^2 - 7x_3$$

$$= (2 - 3x_3)^2 + (-1 + 2x_3)^2 + x_3^2 - 7x_3$$

$$= 4 - \underline{12x_3} + \underline{9x_3^2} + 1 - \underline{4x_3} + \underline{4x_3^2} + \underline{x_3^2} - \underline{7x_3}$$

$$= 14x_3^2 - 23x_3 + 5$$

$$\left.\begin{array}{lll} x_1 > 0 & \Rightarrow & x_3 \leq \frac{2}{3} \\ x_2 > 0 & \Rightarrow & x_3 \geq \frac{1}{2} \\ x_3 > 0 & \Rightarrow & x_3 \geq 0 \end{array}\right\} \quad \frac{1}{2} \leq x_3 \leq \frac{2}{3}$$

You can use this blank page.

$$\min \quad f(x_3) = 14x_3^2 - 23x_3 + 5$$

$$\text{s.t.} \quad \frac{1}{2} \le x_3 \le \frac{2}{3}$$

Try interior pt.

$$f'(x_3) = 0 \Rightarrow 28x_3 = 23 \Rightarrow x_3 = \underbrace{\frac{23}{28}}_{> \frac{2}{3}\,!}$$

Try end pt:

$$x_3 = \frac{1}{2} \Rightarrow f\left(\frac{1}{2}\right) = \frac{14}{4} - \frac{23}{2} = -8$$

$$x_3 = \frac{2}{3} \Rightarrow f\left(\frac{2}{3}\right) = 14\left(\frac{4}{9}\right) - \frac{23(2)}{3} = -\frac{82}{9}$$

$(x_1 = 0,\ x_2 = \frac{1}{3})$ $\quad (56 - 138 = -82)$ $\quad$ min

6. Now, as a veteran data scienctist in a renowned company, you attempt an internal technical evaluation to seek for a promotion. You are presented the following problem and its solution:

$$
\begin{array}{ll}
\text{maximize} & 17x_1 + 12x_2 \\
\text{subject to} & 10x_1 + 7x_2 \leq 40 \\
& x_1 + x_2 \leq 5 \\
& x_1 - x_2 \leq 3 \\
& x_1, x_2, x_3 \geq 0
\end{array}
$$

maximize	ζ	=	0	+	17	x₁	+	12	x₂
subject to:	w₁	=	40	-	10	x₁	-	7	x₂
	w₂	=	5	-	1	x₁	-	1	x₂
	w₃	=	3	-	1	x₁	-	-1	x₂

x₁ x₂ w₁ w₂ w₃ ≥ 0

maximize	ζ	=	205/3	+	-1/3	w₂	+	-5/3	w₁
subject to:	x₂	=	10/3	-	10/3	w₂	-	-1/3	w₁
	w₃	=	14/3	-	17/3	w₂	-	-2/3	w₁
	x₁	=	5/3	-	-7/3	w₂	-	1/3	w₁

x₁ x₂ w₁ w₂ w₃ ≥ 0

You are asked to use *the final dictionary* as so to incorporate each of the following separate new scenarios of the original problem[1]

(a) The problem is changed to a "mixed interger" programming where variable x_1 is further constrained to be an integer. Create an initial feasible dictionary to demontrate how you would continue the problem.

(b) Suppose the right hand side of the constraints are changed to $40 + a, 5 + b, 3 + c$.

i. Assuming a, b, c are small enough so that the optimal dictionary remains optimal. Find the rate of change of the optimal objective functional value ξ^* with respect to a, b, c, i.e. find $\frac{\partial \xi^*}{\partial a}, \frac{\partial \xi^*}{\partial b}, \frac{\partial \xi^*}{\partial c}$.

ii. Find condition(s) on the constants a, b, c such that the optimal dictionary remains optimal.

(c) Suppose the coefficients in the objective function are changed to $17 + d, 12 + e$.

i. Assuming d, e are small enough so that the optimal dictionary remains optimal. Find the rate of change of the optimal objective functional value ξ^* with respect to d, e, i.e. find $\frac{\partial \xi^*}{\partial d}, \frac{\partial \xi^*}{\partial e}$.

ii. Find condition(s) on the constants d, e such that the optimal dictionary remains optimal.

[1] *You are told that your promotion will not go through if you only know how to start from scratch, i.e. the beginning dictionary.*

(g) (MI) Use Branch and Bound

You can use this blank page.

$x_1 = 5/3 \swarrow \searrow$

$x_1 = \frac{5}{3} \Rightarrow$ introduce $x_1 \leq 1$, $x_2 \geq 2$

$$\zeta = \frac{205}{3} - \frac{1}{3}w_2 - \frac{5}{3}w_1 \quad \leftarrow \text{try dual}$$

$$x_2 = \frac{10}{3} - \frac{10}{3}w_2 + \frac{1}{3}w_1 \quad \leftarrow y_1$$

$$w_3 = \frac{14}{3} - \frac{17}{3}w_2 + \frac{2}{3}w_1 \quad \leftarrow y_2$$

$$x_1 = \frac{5}{3} + \frac{7}{3}w_2 - \frac{1}{3}w_1 \quad \leftarrow y_3$$

$x_1 \leq 1$

$$\Rightarrow w_4 = 1 - x_1 = 1 - \frac{5}{3} - \frac{7w_2}{3} + \frac{w_1}{3}$$

$$w_4 = \underbrace{-\frac{2}{3}}_{<0} - \frac{7w_2}{3} + \frac{w_1}{3} \quad \leftarrow y_4$$

not feasible

$$\min \ \frac{10}{3}y_1 + \frac{14}{3}y_2 + \frac{5}{3}y_3 - \frac{2}{3}y_4$$

s.t.

$$\frac{10}{3}y_1 + \frac{17}{3}y_2 - \frac{7}{3}y_3 + \frac{7}{3}y_4 \geq -\frac{1}{3}$$

$$-\frac{y_1}{3} - \frac{2}{3}y_2 + \frac{1}{3}y_3 - \frac{1}{3}y_4 \geq -\frac{5}{3}$$

$$y_1, y_2, y_3, y_4 \geq 0$$

$$A = \begin{pmatrix} \frac{10}{3} & -\frac{1}{3} \\ \frac{17}{3} & -\frac{2}{3} \\ -\frac{7}{3} & \frac{1}{3} \\ \frac{7}{3} & -\frac{1}{3} \end{pmatrix}$$

M2 Use Gomory cut.

You can use this blank page.

$$x_1 = \frac{5}{3} + \frac{7}{3}w_2 - \frac{1}{3}w_1$$

$$x_1 - \frac{7}{3}w_1 + \frac{1}{3}w_2 = \frac{5}{3}$$

(int)

$$x_1 - 3w_1 + 0w_2 = \underbrace{\frac{5}{3} - \frac{2}{3}w_1 - \frac{1}{3}w_2 \leq \frac{5}{3} \leq 1}$$

$$w_4 = 1 - \left(\frac{5}{3} - \frac{2}{3}w_1 - \frac{1}{3}w_2\right)$$

$$= -\frac{2}{3} + \frac{2}{3}w_1 + \frac{w_2}{3}$$

$$\zeta = \frac{205}{3} - \frac{1}{3}w_2 - \frac{5}{3}w_1$$

$$x_2 = \frac{10}{3} - \frac{10}{3}w_2 + \frac{1}{3}w_1$$

$$w_3 = \frac{14}{3} - \frac{17}{3}w_2 + \frac{2}{3}w_1$$

$$x_1 = \frac{5}{3} + \frac{7}{3}w_2 - \frac{1}{3}w_1$$

$$w_4 = \underbrace{-\frac{2}{3}} + \frac{2}{3}w_1 + \frac{w_2}{3}$$

not feasible $\Longrightarrow$ try dual

$$\begin{aligned}
\min \quad & \frac{10}{3}y_1 + \frac{14}{3}y_2 + \frac{5}{3}y_3 - \frac{2}{3}y_4 \\
\text{s.t.} \quad & \frac{10}{3}y_1 + \frac{17}{3}y_2 - \frac{7}{3}y_3 - \frac{2}{3}y_4 \geq -\frac{1}{3} \\
& -\frac{y_1}{3} - \frac{2}{3}y_2 + \frac{1}{3}y_3 - \frac{1}{3}y_4 \geq -\frac{5}{3} \\
& y_1, y_2, y_3, y_4 \geq 0
\end{aligned}$$

You can use this blank page.

(b)
$$\begin{array}{ccccc} x_1 & x_2 & w_1 & w_2 & w_3 \end{array}$$
$$\begin{bmatrix} 10 & 7 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 40+a \\ 5+b \\ 3+c \end{bmatrix}$$

$\mathcal{B} = \{x_2, w_3, x_1\}$, $\mathcal{N} = \{w_2, w_1\}$

$$B = \begin{bmatrix} 7 & 0 & 10 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

B^{-1}:
$$\left[\begin{array}{ccc|ccc} 7 & 0 & 10 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 7 & 0 & 10 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 & -7 & 0 \\ 0 & 1 & 2 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{7}{3} & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{10}{3} & 0 \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{17}{3} & 1 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{7}{3} & 0 \end{array}\right]$$

B^{-1}

You can use this blank page

$$B^{-1}N = \begin{pmatrix} -\frac{1}{3} & \frac{10}{3} & 0 \\ -\frac{2}{3} & \frac{17}{3} & 1 \\ \frac{1}{3} & -\frac{7}{3} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{10}{3} & -\frac{1}{3} \\ \frac{17}{3} & -\frac{2}{3} \\ -\frac{7}{3} & \frac{1}{3} \end{pmatrix}}_{\text{the same as given in the opt. dict.}}$$

(1)

$$\zeta^* = c_B^T B^{-1} b - \left((B^{-1}N)^T c_B - c_N \right)^T x_N$$

$$\Delta \zeta^* = c_B^T B^{-1} \Delta b$$

$$= \underbrace{\begin{array}{ccc} x_2 & w_3 & x_1 \\ (12 & 0 & 17) \end{array}}_{c_B} \begin{pmatrix} -\frac{1}{3} & \frac{10}{3} & 0 \\ -\frac{2}{3} & \frac{17}{3} & 1 \\ \frac{1}{3} & -\frac{7}{3} & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$= \left(\frac{5}{3} \quad \frac{1}{3} \quad 0 \right) \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{5}{3} a + \frac{b}{3}$$

shadow price

$$\boxed{\frac{\partial \zeta^*}{\partial a} = \underbrace{\frac{5}{3}}_{= y_1^*}, \quad \frac{\partial \zeta^*}{\partial b} = \underbrace{\frac{1}{3}}_{= y_2^*}, \quad \frac{\partial \zeta^*}{\partial c} = 0}$$

(ii) We need $B^{-1}b \geqslant 0$

You can use this blank page.

$$\begin{pmatrix} \frac{1}{3} & \frac{10}{3} & 0 \\ -\frac{2}{3} & \frac{17}{3} & 1 \\ \frac{1}{3} & -\frac{7}{3} & 0 \end{pmatrix} \begin{pmatrix} 40+a \\ 5+b \\ 3+c \end{pmatrix} \geqslant \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} &\frac{40+a}{3} + \frac{10(5+b)}{3} \geqslant 0 \\ &\frac{-2(40+a)}{3} + \frac{17}{3}(5+b) + (3+c) \geqslant 0 \\ &\frac{1}{3}(40+a) - \frac{7}{3}(5+b) \geqslant 0 \end{aligned}\right\}$$

$$\Longleftrightarrow \begin{cases} a+10b \geqslant -90 \\ -2a+17b+3c \geqslant 80-85-9 = -14 \\ a-7b \geqslant 35-40 = -5 \end{cases}$$

$$\Longleftrightarrow \boxed{\begin{cases} a+10b \geqslant -90 \\ -2a+17b+3c \geqslant -14 \\ a-7b \geqslant -5 \end{cases}}$$

$$\zeta^{*} = c_B^T B^{-1} b - \left((B^{-1}N)^T c_B - c_N \right)^T \cancel{x_N} \quad 0$$

You can use this blank page.

$$\Delta \zeta^{*} = (\Delta c_B^T) B^{-1} b$$

$$= \begin{pmatrix} e & 0 & d \end{pmatrix} \begin{pmatrix} \frac{10}{3} \\ \frac{14}{3} \\ \frac{5}{3} \end{pmatrix} = \frac{10}{3} e + \frac{d5}{3}$$

$$\boxed{\frac{\partial \zeta^{*}}{\partial d} = \frac{5}{3} \;(= x_1^{*}), \qquad \frac{\partial \zeta^{*}}{\partial e} = \frac{10}{3} \;(= x_2^{*})}$$

shadow prices

We need $(B^{-1}N)^T c_B - c_N \geq 0$

$$\begin{pmatrix} \frac{10}{3} & \frac{17}{3} & -\frac{7}{3} \\ -\frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 12+e \\ 0 \\ 17+d \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} &120 + 10e - 119 - 7d \geq 0 \\ &-12 - e + 17 + d \geq 0 \end{aligned}\right\} \iff \boxed{\begin{aligned} &10e - 7d \geq -1 \\ &-e + d \geq -5 \end{aligned}}$$