MA 421: Linear Programming and Optimization Techniques
Fall 2025, Midterm

Instructor: Yip

- This test booklet has FIVE QUESTIONS, totaling 100 points for the whole test. You have 75 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is **closed book, closed note, with no electronic devices**.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.
- **As a rule of thumb, you should give explicit and useful answers.** No points will be given for just writing down some generically true statements. In other words, your answers should try to make use of all the information given in the question.
- **As a rule of thumb, you should only use those methods that have been covered in class.** If you use some other methods "for the sake of convenience", at our discretion, we might not give you any credit. You have the right to contest. In that event, **you will be asked to explain your answer using only what has been covered in class up to the point of time of this exam.**

Name: Answer Key (Major:)

Question	Score
1.(10 pts)	
2.(20 pts)	
3.(20 pts)	
4.(20 pts)	
5.(30 pts)	
Total (100 pts)	

Formula sheet.

Matrix form of simplex method.

Given the following linear program problem in its standard form:

$$\begin{aligned} \text{maximize} \quad & c^T X \\ \text{subject to} \quad & AX \le b; \\ & X \ge 0. \end{aligned}$$

During simplex iterations, the above can be transformed into the following matrix form:

$$\begin{aligned} \zeta &= c_{\mathcal{B}}^T(B^{-1}b) - \big((B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}}\big)^T X_{\mathcal{N}} \\ X_{\mathcal{B}} &= B^{-1}b - \big(B^{-1}N\big)X_{\mathcal{N}} \end{aligned}$$

with the following dual form:

$$\begin{aligned} -\xi &= -c_{\mathcal{B}}^T(B^{-1}b) - \big(B^{-1}b\big)^T Z_{\mathcal{B}} \\ Z_{\mathcal{N}} &= \big((B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}}\big) + \big(B^{-1}N\big)^T Z_{\mathcal{B}} \end{aligned}$$

where in the above

1. $\mathcal{B}$ and $\mathcal{N}$ are the basic and non-basic variable indices;
2. $c_{\mathcal{B}}$ and $c_{\mathcal{N}}$ are the coefficients in the objective function corresponding to the basic and non-basic variables;
3. B and N are matrices formed by the collecting the columns from the augmented matrix $[A \;\; I]$ corresponding to the basic and non-basic variables.
4. $X_{\mathcal{B}}$ and $X_{\mathcal{N}}$ are the basic and non-basic primal variables, and $Z_{\mathcal{B}}$ and $Z_{\mathcal{N}}$ are the basic and non-basic dual variables.

1. Find the dual of the following linear programming problem:

$$\begin{aligned} \text{maximize} \quad & c^T X \\ \text{subject to} \quad & a \le AX \le b; \\ & u \le X \le l. \end{aligned}$$

In the above, $X \in \mathbb{R}^n$ is the unknown vector, a, b, c, u and l are some given (column) vectors, and A is a matrix with appropriate dimensions. For two vectors X and Y, the notation $X \le Y$ is interpreted componentwise.

$$\begin{aligned} & p^T\,(AX \leq b) \qquad && p^T \geq 0 \\ & q^T\,(-AX \leq -a) && q^T \geq 0 \\ & r^T\,(X \leq l) && r^T \geq 0 \\ + \quad & s^T\,(-X \leq -u) && s^T \geq 0 \end{aligned}$$

$$c^T X \leq (p^T A - q^T A + r^T - s^T) X \leq p^T b - q^T a + r^T l - s^T u$$

Dual :

$$\boxed{\begin{aligned} \min \quad & p^T b - q^T a + r^T l - s^T u \\ \text{s.t.} \quad & p^T A - q^T A + r^T - s^T = c^T \\ & p, q, r, s \geq 0 \end{aligned}}$$

X has no sign constraint

Or equivalently,

$$\boxed{\begin{aligned} \min \quad & b^T p - a^T q + l^T r - u^T s \\ \text{s.t.} \quad & A^T (p - q) + (r - s) = c \\ & p, q, r, s \geq 0 \end{aligned}}$$

2. For each of the following linear programming problems, write down an initial feasible dictionary. Then perform one step of simplex method by just indicating which variable enters as a basic variable and which one leaves. Do not use Phase I unless it is *absolutely necessary*. If you do, just answer the question for Phase I is sufficient.

(a)

$$\begin{array}{lrrcr} \text{maximize} & -x_1 & -2x_2 & & \\ \text{subject to} & -3x_1 & +x_2 & \leq & -1 \\ & x_1 & -x_2 & \leq & 1 \\ & -2x_1 & +7x_2 & \leq & 1 \\ & 9x_1 & -4x_2 & \leq & -3 \\ & x_1, & x_2 & \geq & 0 \end{array}$$

(b)

$$\begin{array}{lrrrrcr} \text{maximize} & x_1 & -2x_2 & & & & \\ \text{subject to} & x_1 & +2x_2 & -x_3 & +x_4 & \geq & 0 \\ & 4x_1 & +3x_2 & +4x_3 & -2x_4 & \leq & 3 \\ & -x_1 & -2x_2 & +x_3 & +x_4 & = & 1 \\ & x_1, & x_2, & x_3, & x_4 & \geq & 0 \end{array}$$

(a) Consider dual as $-x_1 - x_2$ has neg. coeff.

$$(P): \max\ c^T x \quad \text{s.t.}\ Ax \leq b,\ x \geq 0 \iff \max\ -b^T y \quad \text{s.t.}\ -A^T y \leq -c,\ y \geq 0$$

(neg. transpose)

$$z = -c + A^T y$$

$$\max\ -\xi = y_1 - y_2 - y_3 + 3y_4$$

$$\text{s.t.}\quad z_1 = 1 - 3y_1 + y_2 - 2y_3 + 9y_4$$

$$z_2 = 2 + y_1 - y_2 + 7y_3 - 4y_4$$

$$y_i,\ z_i \geq 0$$

$\Rightarrow$ $\boxed{y_4 \text{ enters},\ z_2 \text{ leaves}}$

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(b) Constraints $\Rightarrow$

$$\begin{aligned}
-x_1 - 2x_2 + x_3 + x_4 + w_1 \qquad\qquad &= 0\\
4x_1 + 3x_2 + 4x_3 - 2x_4 \qquad + w_2 \qquad &= 0\\
-x_1 - 2x_2 + x_3 + x_4 \qquad\qquad + w_3 &= 1
\end{aligned}$$

Phase I: $\max \; -w_3$ (You want to see if $w_3 = 0$)

$$\begin{aligned}
\text{s.t.}\quad w_1 &= \quad x_1 + 2x_2 - x_3 - x_4\\
w_2 &= \quad -4x_1 - 3x_2 - 4x_3 + 2x_4\\
w_3 &= 1 + x_1 + 2x_2 - x_3 - x_4
\end{aligned}$$

$$\Longleftrightarrow \quad \left.\begin{aligned}
\max \quad \zeta &= -1 - x_1 - 2x_2 + x_3 + x_4\\
\text{s.t.}\quad w_1 &= \quad x_1 + 2x_2 - x_3 - x_4\\
w_2 &= \quad -4x_1 - 3x_2 - 4x_3 + 2x_4\\
w_3 &= 1 + x_1 + 2x_2 - x_3 - x_4
\end{aligned}\right\} \quad \begin{aligned}&x_3 \text{ enters,}\\ &w_2 \text{ leaves}\end{aligned}$$

3. Consider the following linear programming problem:

$$\begin{array}{lll} \text{maximize} & 6x_1 + x_2 - x_3 - x_4 & \\ \text{subject to} & x_1 + 2x_2 + x_3 + x_4 & \leq 5 \\ & 3x_1 + x_2 - x_3 & \leq 8 \\ & x_2 + x_3 + x_4 & = 1 \\ & x_1,\ x_2,\ x_3,\ x_4 & \geq 0 \end{array}$$

(a) Write down the dual problem. (Beware: the primal is not quite in standard form.)

(b) Someone claims that the optimal point is $(x_1^* = 3, x_2^* = 0, x_3^* = 1, x_4^* = 0)$. Prove or disprove the claim by *not solving the problem explicitly*. If the proposed solution is optimal, give also the solution of the dual problem.

(a)

$$\begin{array}{rlll} & y_1(x_1 + 2x_2 + x_3 + x_4) & \leq 5y_1 & y_1 \geq 0 \\ + & y_2(3x_1 + x_2 - x_3) & \leq 8y_2 & y_2 \geq 0 \\ + & y_3(x_2 + x_3 + x_4) & = y_3 & \end{array}$$

$$(y_1 + 3y_2)x_1 + (2y_1 + y_2 - y_3)x_2 + (y_1 - y_2 + y_3)x_3 + (y_1 + y_3)x_4$$

$$\Downarrow$$

$$6x_1 + x_2 - x_3 - x_4 \leq 5y_1 + 8y_2 + y_3$$

Dual:

$$\boxed{\begin{array}{lll} \min & \xi = 5y_1 + 8y_2 + y_3 & \\ \text{s.t.} & y_1 + 3y_2 & \geq 6 \\ & 2y_1 + y_2 - y_3 & \geq 1 \\ & y_1 - y_2 + y_3 & \geq -1 \\ & y_1 + y_3 & \geq -1 \\ & y_1, y_2 \geq 0 & \end{array}}$$

(b) $(x_1 = 3,\ x_2 = 0,\ x_3 = 1,\ x_4 = 0)$

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$$6x_1 + x_2 - x_3 - x_4$$

$$\overset{①}{\leq} (y_1 + 3y_2)x_1 + (2y_1 + y_2 - y_3)x_2 + (y_1 - y_2 + y_3)x_3 + (y_1 + y_3)x_4$$

$$\overset{②}{\leq} 5y_1 + 8y_2 + y_3$$

$= ?$

For ① to be $\overset{?}{=}$:

$$y_1 + 3y_2 = 6 \quad (x_1 > 0)$$
$$2y_1 + y_2 - y_3 \geq 1 \quad (x_2 = 0)$$
$$y_1 - y_2 + y_3 = -1 \quad (x_3 > 0)$$
$$y_1 + y_3 \geq -1 \quad (x_4 = 0)$$

For ② to be $\overset{?}{=}$:

$$\begin{aligned}
x_1 + 2x_2 + x_3 + x_4 &\leq 5 \\
3x_1 + x_2 - x_3 &\leq 8 \\
x_2 + x_3 + x_4 &= 1 \\
x_1,\ x_2,\ x_3,\ x_4 &\geq 0
\end{aligned}$$

$3 + 1 < 5 \Rightarrow y_1 = 0$

$3(3) - 1 = 8 \Rightarrow y_2 \geq 0$

$1 = 1$

$y_1 + 3y_2 = 6 \Rightarrow y_2 = 2$

$y_1 - y_2 + y_3 = -1 \Rightarrow y_3 = 1$

Check other conditions for y_i's: $2(0) + 2 - 1 \geq 1$ ✓

$0 - 2 + 1 \geq -1$ ✓

(Check (but not necessary)

$6(3) + 1(0) - 1 - 0$

$= 17 = 5(0) + 8(2) + 1$)

Hence opt. !

4. Using the method of Lagrange multiplier, find the point in $\mathbb{R}^4$ that satisfies the following system of linear equations and is closest to the origin:

$$\begin{aligned} x_1 + 2x_2 - x_3 - x_4 &= 1 \\ 4x_1 + x_2 + 3x_3 - x_4 &= 0 \end{aligned}$$

Give also the value(s) of the Lagrange multiplier(s).

$$L(x_1, x_2, x_3, x_4, \lambda_1, \lambda_2)$$
$$= x_1^2 + x_2^2 + x_3^2 + x_4^2 + \lambda_1(x_1 + 2x_2 - x_3 - x_4 - 1) + \lambda_2(4x_1 + x_2 + 3x_3 - x_4)$$

$$\partial_{x_1} L = 0 \Rightarrow \begin{aligned} 2x_1 + \lambda_1 + 4\lambda_2 &= 0 \\ 2x_2 + 2\lambda_1 + \lambda_2 &= 0 \\ 2x_3 - \lambda_1 + 3\lambda_2 &= 0 \\ 2x_4 - \lambda_1 - \lambda_2 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 4 \\ 2 & 1 \\ -1 & 3 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Constraint:

$$\begin{bmatrix} 1 & 2 & -1 & -1 \\ 4 & 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 - x_3 - x_4 &= 1 \\ 4x_1 + x_2 + 3x_3 - x_4 &= 0 \end{aligned}$$

Hence

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$$\left(-\frac{1}{2}\right)\begin{bmatrix} 1 & 2 & -1 & -1 \\ 4 & 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 1 \\ -1 & 3 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 4 \\ 4 & 27 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 4 & 27 \end{bmatrix}^{-1}\begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$= \frac{1}{173}\begin{bmatrix} 27 & -4 \\ -4 & 7 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \end{bmatrix} = \frac{1}{173}\begin{bmatrix} -54 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 1 & 4 \\ 2 & 1 \\ -1 & 3 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 1 & 4 \\ 2 & 1 \\ -1 & 3 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -54 \\ 8 \end{bmatrix}\frac{1}{173}$$

$$= \frac{1}{173}\begin{bmatrix} 11 \\ 50 \\ -39 \\ -23 \end{bmatrix}$$

5. As a new data scientist, your very first assignment was to solve the following problem:

$$
\begin{array}{lrl}
\text{maximize} & 5x_1 + 4x_2 + 3x_3 & \\
\text{subject to} & 2x_1 + 3x_2 + x_3 & \leq 5 \\
 & 4x_1 + x_2 + 2x_3 & \leq 11 \\
 & 3x_1 + 4x_2 + 2x_3 & \leq 8 \\
 & x_1, x_2, x_3 & \geq 0
\end{array}
$$

This was such a difficult problem that you needed to use one week to solve it. You submitted the following solution to your boss:

$$
\begin{array}{lrcl}
\textbf{maximize} & \zeta & = & \quad + 5\,x_1 + 4\,x_2 + 3\,x_3 \\
\textbf{subject to} & w_1 & = & 5 - 2\,x_1 - 3\,x_2 - \,x_3 \\
 & w_2 & = & 11 - 4\,x_1 - \,x_2 - 2\,x_3 \\
 & w_3 & = & 8 - 3\,x_1 - 4\,x_2 - 2\,x_3 \\
 & & & x_1,\ x_2,\ x_3,\ w_1,\ w_2,\ w_3 \geq 0.
\end{array}
\qquad
\begin{array}{rcl}
\zeta & = & 13 - 1\,w_1 - 3\,x_2 - 1\,w_3 \\
\hline
x_1 & = & 2 - 2\,w_1 - 2\,x_2 + \,w_3 \\
w_2 & = & 1 + 2\,w_1 + 5\,x_2 \\
x_3 & = & 1 + 3\,w_1 + \,x_2 - 2\,w_3 \,.
\end{array}
$$

You boss said, "That's great. Hey, listen, can you modify the above problem according to the following two separate scenarios?"

(a) Delete the first constraint, $2x_1 + 3x_2 + x_3 \leq 5$.

You can try replacing 5 by a "really huge" number.

(b) Add a new variable x_4 to the problem:

$$
\begin{array}{lrl}
\text{maximize} & 5x_1 + 4x_2 + 3x_3 + 10x_4 & \\
\text{subject to} & 2x_1 + 3x_2 + x_3 + x_4 & \leq 5 \\
 & 4x_1 + x_2 + 2x_3 + x_4 & \leq 11 \\
 & 3x_1 + 4x_2 + 2x_3 + x_4 & \leq 8 \\
 & x_1, x_2, x_3, x_4 & \geq 0
\end{array}
$$

You can try modifying the matrix notation during the simplex algorithm.

You boss continued, "I know this is such a hard and time consuming problem. *I don't want you to start from scratch and spend another week* on this. So for each of the above scenarios, all I need from you is just a feasible dictionary to start with and tell me which variable to enter and leave. Then I can go from there. I am expecting a response from you by 1:15pm today. To tell you the truth, being in a managerial position gets me kind of rusty in these LP stuff. But I am happy to brush up on my technical skill again. *So make sure you explain to me how you get your answers from your previous answers.*"[1]

[1]You said to yourself, "My boss' position sounds 'soooooooooo exciting'!".

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(a) $\mathcal{B} = \{x_1, w_2, x_3\}$, $\mathcal{N} = \{w_1, x_2, w_3\}$

Replace $2x_1 + 3x_2 + x_3 \leq 5$ by $2x_1 + 3x_2 + x_3 \leq \underbrace{M \text{ (very large)}}$ ∼ no constraint.

$$b = \begin{pmatrix} 5 \\ 11 \\ 8 \end{pmatrix} \longrightarrow b_{new} = \begin{pmatrix} M \\ 11 \\ 8 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 0 & 1 \\ 4 & 1 & 2 \\ 3 & 0 & 2 \end{pmatrix} \qquad B^{-1}N = \begin{pmatrix} 2 & 2 & -1 \\ -2 & -5 & 0 \\ -3 & -1 & 2 \end{pmatrix}$$

Find B^{-1}: $\left(\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 4 & 1 & 2 & 0 & 1 & 0 \\ 3 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1/2 & 0 & 0 \\ 4 & 1 & 2 & 0 & 1 & 0 \\ 3 & 0 & 2 & 0 & 0 & 1 \end{array}\right)$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1/2 & -3/2 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1/2 & -3/2 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 2 \end{array}\right) \qquad B^{-1} = \begin{pmatrix} 2 & 0 & -1 \\ -2 & 1 & 0 \\ -3 & 0 & 2 \end{pmatrix}$$

$$B^{-1} b_{new} = \begin{pmatrix} 2 & 0 & -1 \\ -2 & 1 & 0 \\ -3 & 0 & 2 \end{pmatrix} \begin{pmatrix} M \\ 11 \\ 8 \end{pmatrix} = \begin{pmatrix} 2M - 8 \\ -2M + 11 \\ -3M + 16 \end{pmatrix}$$ ← not feasible ←

Consider dual of new problem:

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		z_1	y_2	z_3
	-13	$-2M+8$	$2M-11$	$3M-16$
y_1	1	2	-2	-3
z_2	3	2	-5	-1
y_3	1	-1	0	2

z_3 enters & y_1 leaves

M1

(b) $B = \{x_1, w_2, x_3\}$, $N = \{w_1, x_2, w_3, \underbrace{x_4}_{\text{new}}\}$

$$C = (5 \quad 4 \quad 3 \quad \underbrace{10}_{} \quad 0 \quad 0 \quad 0)^T$$

$$C_B = (5 \quad 0 \quad 3)^T, \quad C_N = (0, 4, 0, \underbrace{10}_{\text{new}})^T$$

$$B^{-1}b = \text{same as before} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$N = \begin{pmatrix} 1 & 3 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 4 & 1 & 1 \end{pmatrix}$$

$$B^{-1}N = \begin{pmatrix} 2 & 0 & -1 \\ -2 & 1 & 0 \\ -3 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 4 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & -1 & 1 \\ -2 & -5 & 0 & -1 \\ -3 & -1 & 2 & -1 \end{pmatrix}$$

old new old new

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$$(B^{-1}N)^T C_B - C_N = \begin{pmatrix} 2 & -2 & -3 \\ 2 & -5 & -1 \\ -1 & 0 & 2 \\ \underline{1} & \underline{-2} & \underline{-1} \end{pmatrix} \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \\ 0 \\ \underline{10} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 3 \\ 1 \\ -8 \end{pmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{old} \\ \} \text{ new} \leftarrow \text{not opt.} \end{matrix}$$

New dictionary:

		w_1	x_2	w_3	x_4
	13	−1	−3	−1	8
x_1	2	−2	−2	1	−1
w_2	1	2	5	0	1
x_3	1	3	1	−2	1

x_4 enters
x_1 leaves

M2

$$
\begin{array}{lrl}
\text{maximize} & 5x_1 + 4x_2 + 3x_3 + 10x_4 & \\
\text{subject to} & 2x_1 + 3x_2 + x_3 + x_4 & \leq 5 \\
& 4x_1 + x_2 + 2x_3 + x_4 & \leq 11 \\
& 3x_1 + 4x_2 + 2x_3 + x_4 & \leq 8 \\
& x_1, x_2, x_3, x_4 & \geq 0
\end{array}
\iff
\begin{aligned}
w_1 &= 5 - 2x_1 - 3x_2 - x_3 - x_4 \\
w_2 &= 11 - 4x_1 - x_2 - 2x_3 - x_4 \\
w_3 &= 8 - 3x_1 - 4x_2 - 2x_3 - x_4
\end{aligned}
$$

$$
\begin{aligned}
w_1^{old} &= (w_1 + x_4) = 5 - 2x_1 - 3x_2 - x_3 \\
w_2^{old} &= (w_2 + x_4) = 11 - 4x_1 - x_2 - 2x_3 \\
w_3^{old} &= (w_3 + x_4) = 8 - 3x_1 - 4x_2 - 2x_3
\end{aligned}
$$

Hence new dictionary incorporating x_4 is:

$$
\begin{array}{rcrcrcrcr}
\zeta & = & 13 & - & 1\,w_1 & - & 3\,x_2 & - & 1\,w_3 \\
\hline
x_1 & = & 2 & - & 2\,w_1 & - & 2\,x_2 & + & w_3 \\
w_2 & = & 1 & + & 2\,w_1 & + & 5\,x_2 & & \\
x_3 & = & 1 & + & 3\,w_1 & + & x_2 & - & 2\,w_3 .
\end{array}
$$

$$
\begin{aligned}
\Rightarrow \quad \zeta &= 13 - (w_1 + x_4) - 3x_2 - (w_3 + x_4) + 10x_4 \\
&= \underline{13 - w_1 - 3x_2 - w_3 + 8x_4} \\
x_1 &= 2 - 2(w_1 + x_4) - 2x_2 + (w_3 + x_4) \\
&= \underline{2 - 2w_1 - 2x_2 + w_3 - x_4} \\
(w_2 + x_4) &= 1 + 2(w_1 + x_4) + 5x_2 \\
w_2 &= \underline{1 + 2w_1 + 5x_2 + x_4} \\
x_3 &= 1 + 3(w_1 + x_4) + x_2 - 2(w_3 + x_4) \\
&= \underline{1 + 3w_1 + x_2 - 2w_3 + x_4}
\end{aligned}
$$

(Same as M1)