MA 421: Linear Programming and Optimization Techniques
Fall 2026, Midterm

Instructor: Yip

- This test booklet has FIVE QUESTIONS, totaling 100 points for the whole test. You have 75 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is **closed book, closed note, with no electronic devices**.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.
- **As a rule of thumb, you should give explicit and useful answers.** No points will be given for just writing down some generically true statements. In other words, your answers should try to make use of all the information given in the question.
- **As a rule of thumb, you should only use those methods that have been covered in class.** If you use some other methods "for the sake of convenience", at our discretion, we might not give you any credit. You have the right to contest. In that event, **you will be asked to explain your answer using only what has been covered in class up to the point of time of this exam.**

Name: (Major:)

Question	Score
1.(10 pts)	
2.(20 pts)	
3.(20 pts)	
4.(20 pts)	
5.(30 pts)	
Total (100 pts)	

Formula sheet.

Matrix form of simplex method.

Given the following linear program problem in its standard form:

$$\begin{array}{ll} \text{maximize} & c^T X \\ \text{subject to} & AX \leq b; \\ & X \geq 0. \end{array}$$

During simplex iterations, the above can be transformed into the following matrix form:

$$\begin{aligned} \zeta &= c_{\mathcal{B}}^T (B^{-1} b) - \left((B^{-1} N)^T c_{\mathcal{B}} - c_{\mathcal{N}} \right)^T X_{\mathcal{N}} \\ X_{\mathcal{B}} &= B^{-1} b - \left(B^{-1} N \right) X_{\mathcal{N}} \end{aligned}$$

with the following dual form:

$$\begin{aligned} -\xi &= -c_{\mathcal{B}}^T (B^{-1} b) - \left(B^{-1} b \right)^T Z_{\mathcal{B}} \\ Z_{\mathcal{N}} &= \left((B^{-1} N)^T c_{\mathcal{B}} - c_{\mathcal{N}} \right) + \left(B^{-1} N \right)^T Z_{\mathcal{B}} \end{aligned}$$

where in the above

1. $\mathcal{B}$ and $\mathcal{N}$ are the basic and non-basic variable indices;
2. $c_{\mathcal{B}}$ and $c_{\mathcal{N}}$ are the coefficients in the objective function corresponding to the basic and non-basic variables;
3. B and N are matrices formed by the collecting the columns from the augmented matrix $[A \;\; I]$ corresponding to the basic and non-basic variables.
4. $X_{\mathcal{B}}$ and $X_{\mathcal{N}}$ are the basic and non-basic primal variables, and $Z_{\mathcal{B}}$ and $Z_{\mathcal{N}}$ are the basic and non-basic dual variables.

1. Consider the following linear programming problem:

$$\begin{array}{lrll} \max & \zeta & = & x_1 + 2x_2 - x_3 + x_4 \\ \text{s.t.} & & & 2x_1 - x_2 + x_3 \leq -1 \\ & & & x_1 + x_2 + x_3 + x_4 \leq 3 \\ & & & x_1 - x_2 + 2x_3 + 3x_4 = -2 \\ & & & x_1,\, x_2,\, x_3,\, x_4 \geq 0 \end{array}$$

Write down an initial feasible dictionary to initate the simplex method. Indicate which variables to enter and leave in the first step.

M1

$$\begin{aligned} 2x_1 - x_2 + x_3 + w_1 - w_2 &= -1 \\ x_1 + x_2 + x_3 + x_4 + w_3 &= 3 \\ x_1 - x_2 + 2x_3 + 3x_4 - w_4 &= -2 \end{aligned}$$

w_1, w_3 are slacks, w_2, w_4 are auxiliary vars

$$\begin{aligned} w_2 &= 1 + 2x_1 - x_2 + x_3 + w_1 \\ w_3 &= 3 - x_1 - x_2 - x_3 - x_4 \\ w_4 &= 2 + x_1 - x_2 + 2x_3 + 3x_4 \end{aligned}$$

$$\begin{aligned} \max \zeta &= -w_2 - w_4 \quad = 0\ ? \quad \leftarrow \text{Phase I} \\ &= -1 - 2x_1 + x_2 - x_3 \\ &\quad -2 - x_1 + x_2 - 2x_3 - 3x_4 \\ &= -3 - 3x_1 + 2x_2 - 3x_3 - 3x_4 \end{aligned}$$

x_2 enters, w_2 leaves

M2

$$
\begin{array}{lrll}
\max & \zeta & = & x_1 + 2x_2 - x_3 + x_4 \\
\text{s.t.} & & & 2x_1 - x_2 + x_3 \leq -1 \\
& & & x_1 + x_2 + x_3 + x_4 \leq 3 \\
& & & x_1 - x_2 + 2x_3 + 3x_4 = -2 \\
& & & x_1,\ x_2,\ x_3,\ x_4 \geq 0
\end{array}
$$

≤ -2 / ≥ -2

① $2x_1 - x_2 + x_3 + w_1 - x_0 = -1$

② $x_1 + x_2 + x_3 + x_4 + w_2 - x_0 = 3$

③ $x_1 - x_2 + 2x_3 + 3x_4 + w_3 - x_0 = -2$ ← most infeasible

④ $-x_1 + x_2 - 2x_3 - 3x_4 + w_4 - x_0 = 2$

③ ⇒ $\boxed{x_0 = 2 + x_1 - x_2 + 2x_3 + 3x_4 + w_3}$ ←

① ⇒ $2x_1 - x_2 + x_3 + w_1 - 2 - x_1 + x_2 - 2x_3 - 3x_4 - w_3 = -1$

$\boxed{w_1 = 1 - x_1 + x_3 + 3x_4 + w_3}$

② ⇒ $x_1 + x_2 + x_3 + x_4 + w_2 - 2 - x_1 + x_2 - 2x_3 - 3x_4 - w_3 = 3$

$\boxed{w_2 = 5 - 2x_2 + x_3 + 2x_4 + w_3}$

④ ⇒ $-x_1 + x_2 - 2x_3 - 3x_4 + w_4 - 2 - x_1 + x_2 - 2x_3 - 3x_4 - w_3 = 2$

$\boxed{w_4 = 4 + 2x_1 - 2x_2 + 4x_3 + 6x_4 + w_3}$

$\boxed{\max \zeta = -x_0 = -2 - x_1 + x_2 - 2x_3 - 3x_4 - w_3}$

0 ?

x_2 enters, x_0 or w_4 leaves

2. Consider the following linear programming problem:

$$\begin{array}{lrll} \max & \zeta & = & x_1 + 2x_2 - x_3 + x_4 \\ \text{s.t.} & & & 2x_1 - x_2 + x_3 - 2x_4 \leq 1 \\ & & & x_1 + x_2 + x_3 + x_4 \leq 3 \\ & & & x_1,\, x_2,\, x_3,\, x_4 \geq 0 \end{array}$$

$w_1 > 0$
$w_2 = 0$

You are told that $x_1 = x_3 = x_4 = 0,\ x_2 = 3$ is an optimal solution. Check that if this is true. If so, find the optimal dictionary.

Dual:

$2y_1 + y_2 \geq 1$ ✓ z_1

$-y_1 + y_2 \geq 2$ ✓ z_2

$y_1 + y_2 \geq -1$ ✓ z_3

$-2y_2 + y_2 \geq 1$ ✓ z_4

$x_2 = 3 > 0 \Rightarrow z_2 = 0 \Rightarrow -y_1 + y_2 = 2$

$w_1 = 4 > 0 \Rightarrow y_1 = 0$

Hence $y_2 = 2$

Hence opt. [∵ There is a feasible dual variables satisfying C.S.)

x_2, w_1 are basic variables

(1) $w_1 = 1 - 2x_1 + x_2 - x_3 + 2x_4$

(2) $w_2 = 3 - x_1 - x_2 - x_3 - x_4$

Solve for x_2 and w_1

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(2) $\Rightarrow$ $x_2 = 3 - x_1 - x_3 - x_4 - w_2$

(1) $\Rightarrow$ $w_1 = 1 - 2x_1 + (3 - x_1 - x_3 - x_4 - w_2)$
$- x_3 + 2x_4$

$w_1 = 4 - 3x_1 - 2x_3 + x_4 - w_2$

$\zeta = x_1 + 2x_2 - x_3 + x_4$

$= x_1 + 2(3 - x_1 - x_3 - x_4 - w_2) - x_3 + x_4$

$\zeta = 6 - x_1 - 3x_3 - x_4 - 2w_2$

Opt. (with negative coeff's.)

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3. Find the solution of the following system of linear equations:

$$\begin{aligned} x + 2y - z &= 1 \\ x + y + z &= 0 \end{aligned}$$

that is closest to the point $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

(M1)

$$\min \quad (x-1)^2 + (y-1)^2 + z^2$$

s.t.

$$L(x, y, z, \lambda_1, \lambda_2) = (x-1)^2 + (y-1)^2 + z^2 + \lambda_1 (x + 2y - z - 1) + \lambda_2 (x + y + z)$$

$$\begin{aligned} \partial_x L = 0 \Rightarrow \quad & 2(x-1) + \lambda_1 + \lambda_2 = 0 \\ \partial_y L = 0 \Rightarrow \quad & 2(y-1) + 2\lambda_1 + \lambda_2 = 0 \\ \partial_z L = 0 \Rightarrow \quad & 2z - \lambda_1 + \lambda_2 = 0 \end{aligned}$$

$$\begin{aligned} x &= 1 - \tfrac{1}{2}(\lambda_1 + \lambda_2) \\ y &= 1 - \tfrac{1}{2}(2\lambda_1 + \lambda_2) \\ z &= 0 - \tfrac{1}{2}(-\lambda_1 + \lambda_2) \end{aligned}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \right)$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$= \frac{\begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix}}{14} = \frac{1}{7} \begin{pmatrix} 2 \\ 8 \end{pmatrix}$$

$$x = 1 - \frac{1}{2}(\lambda_1 + \lambda_2) = 1 - \frac{10}{14} = \frac{2}{7}$$

$$y = 1 - \frac{1}{2}(2\lambda_1 + \lambda_2) = 1 - \frac{12}{14} = \frac{1}{7}$$

$$z = 0 - \frac{1}{2}(-\lambda_1 + \lambda_2) = -\frac{6}{14} = -\frac{3}{7}$$

M2

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$$\begin{cases} x+2y-z=1 \\ x+y+z=0 \end{cases}$$

$$\Rightarrow \begin{cases} x+2y = 1+z \\ x+y = -z \end{cases}$$

$$\Rightarrow \begin{cases} y = 1+2z, \\ x = -y-z \\ x = -1-3z \end{cases}$$

$$(x-1)^2+(y-1)^2+z^2$$

$$= (2+3z)^2+(2z)^2+z^2$$

$$\frac{d}{dz} = 2\left[(2+3z)3+2z(2)+z\right] = 0$$

$$6+14z=0 \Rightarrow \boxed{\begin{array}{l} z=-\frac{3}{7} \\ y=\frac{1}{7} \\ x=\frac{2}{7} \end{array}}$$

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$\frac{pq}{p+q} = 1$

4. Given two numbers $p, q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, or equivalently, $pq = p + q$. Consider the following optimization problem:

$$\begin{array}{ll} \min_{x,y} & f(x,y) = \frac{x^p}{p} + \frac{y^q}{q} \\ \text{s.t.} & xy = c, \quad a, b \geq 0. \end{array}$$

where in the above, c is some positive number.

(a) Solve the above problem by means of Lagrange Multiplier method. Find the optimal, i.e. minimum value of f, the Lagrange multiplier and the optimal choice of x and y.

(b) Note that for simplicity, you can just forget about the positivity constraint for x and y. Explain why.

(c) For a fixed c, let $\zeta(c)$ denote the optimal value of f you have found in (a). Compute $\zeta'(c)$. What do you observe?

(a) $L(x,y,\lambda) = \frac{x^p}{p} + \frac{y^q}{q} + \lambda(xy - c)$

$\partial_x L = 0 \Rightarrow \quad x^{p-1} + \lambda y = 0$

$\partial_y L = 0 \Rightarrow \quad y^{q-1} + \lambda x = 0$

$\partial_\lambda L = 0 \Rightarrow \quad xy = c$

$\frac{x^{p-1}}{y} = -\lambda = \frac{y^{q-1}}{x}$

$\Rightarrow x^p = y^q \qquad y = x^{\frac{p}{q}}$ or $x = y^{\frac{q}{p}}$

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$$xy = C \implies x x^{\frac{p}{q}} = C$$

$$x^{\frac{p+q}{q}} = C$$

$$x = C^{\frac{q}{p+q}} = C^{\frac{pq}{p+q}\frac{1}{p}} = C^{\frac{1}{p}}$$

$$y = x^{\frac{p}{q}} = C^{\frac{p}{p+q}} = C^{\frac{pq}{p+q}\frac{1}{q}} = C^{\frac{1}{q}}$$

$$\lambda = -\frac{x^{p-1}}{y} = -\frac{\left(C^{\frac{q}{p+q}}\right)^{(p-1)}}{C^{\frac{p}{p+q}}}$$

$$= -C^{\frac{q(p-1)}{p+q} - \frac{p}{p+q}}$$

$$= -C^{\frac{qp-p-q}{p+q}}$$

$$= -C^{\frac{pq}{p+q}-1}$$

$$= -1$$

$$\frac{1}{p} + \frac{1}{q} = 1$$

$$\Updownarrow$$

$$pq = p+q$$

$$\Updownarrow$$

$$\frac{pq}{p+q} = 1$$

(b) $xy = c$, x, y cannot approach 0

You can use this blank page.

otherwise x or $y \to +\infty$ and

$f(x,y) = \frac{x^p}{p} + \frac{y^q}{q} \to +\infty$ ← not min.

(c) $f(x,y) = \frac{x^p}{p} + \frac{y^q}{q}$

$= \frac{1}{p} C^{\left(\frac{q}{p+q}\right)p} + \frac{1}{q} C^{\left(\frac{p}{p+q}\right)q}$

$= \frac{1}{p} C^{\frac{pq}{p+q}} + \frac{1}{q} C^{\frac{pq}{p+q}}$

$\left(\frac{pq}{p+q} = 1\right)$ $= \frac{1}{p} C^1 + \frac{1}{q} C^1$

$= \left(\frac{1}{p} + \frac{1}{q}\right) C = C = \zeta(c)$

$\zeta'(c) = 1 = -\lambda$

$(-1)\lambda$ is the rate of change, i.e. sensitivity of the optimal value w.r.t. c

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5. You are running a fancy furniture store, making tables and chairs. Suppose

   - each table, sold for \$500, requires 3 hours of labor and 9 feet of wood;
   - each chair, sold for \$300, requires 6 hours of labor and 3 feet of wood;
   - You have 120 labor hours and 90 feet of wood at your disposal, i.e., you have already paid for these resources.

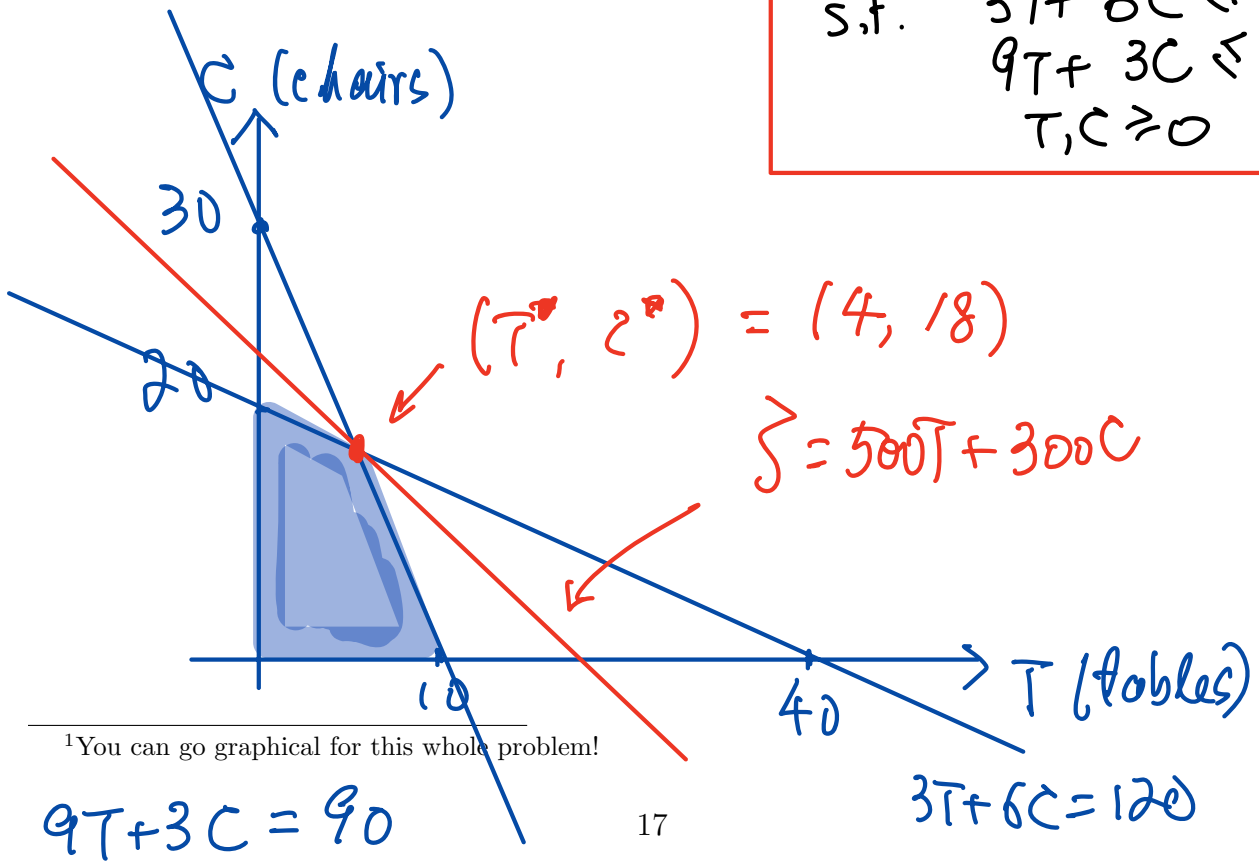
   (a) How many tables and chairs should you make in order to maximize your profit? (It is OK if your numbers turn out to be fractions as in the long run, the fractions will be averaged out. We are not performing integer programming here.)[1]

   (b) Suppose you plan to hire some extra labor to increase your production but *your wood supply remain fixed to be 90 feet*. Up to how many extra hours of labor would you hire and what is the maximum unit price (per hour) you are willing to pay for these *extra* hours?

   (c) Suppose you plan to purchase some extra wood to increase your production but *your labor supply remain fixed to be 120 hours*. Up to how many extra wood would you purchase and what is the maximum unit price (per feet) you are willing to pay for these *extra* wood?

(a)

$$\max \quad \pi = 500T + 300C$$
$$\text{s.t.} \quad 3T + 6C \le 120$$
$$9T + 3C \le 90$$
$$T, C \ge 0$$



[1] You can go graphical for this whole problem!

You can use this blank page.

$$\begin{cases} 3T+6C=120 \\ 9T+3C=90 \end{cases} \Rightarrow \begin{cases} T+2C=40 \\ 3T+C=30 \end{cases}$$

$$\Rightarrow \boxed{\begin{cases} T=4 \\ C=18 \end{cases}}$$

$$\Sigma^* = 500(4)+300(18) = \#7400$$

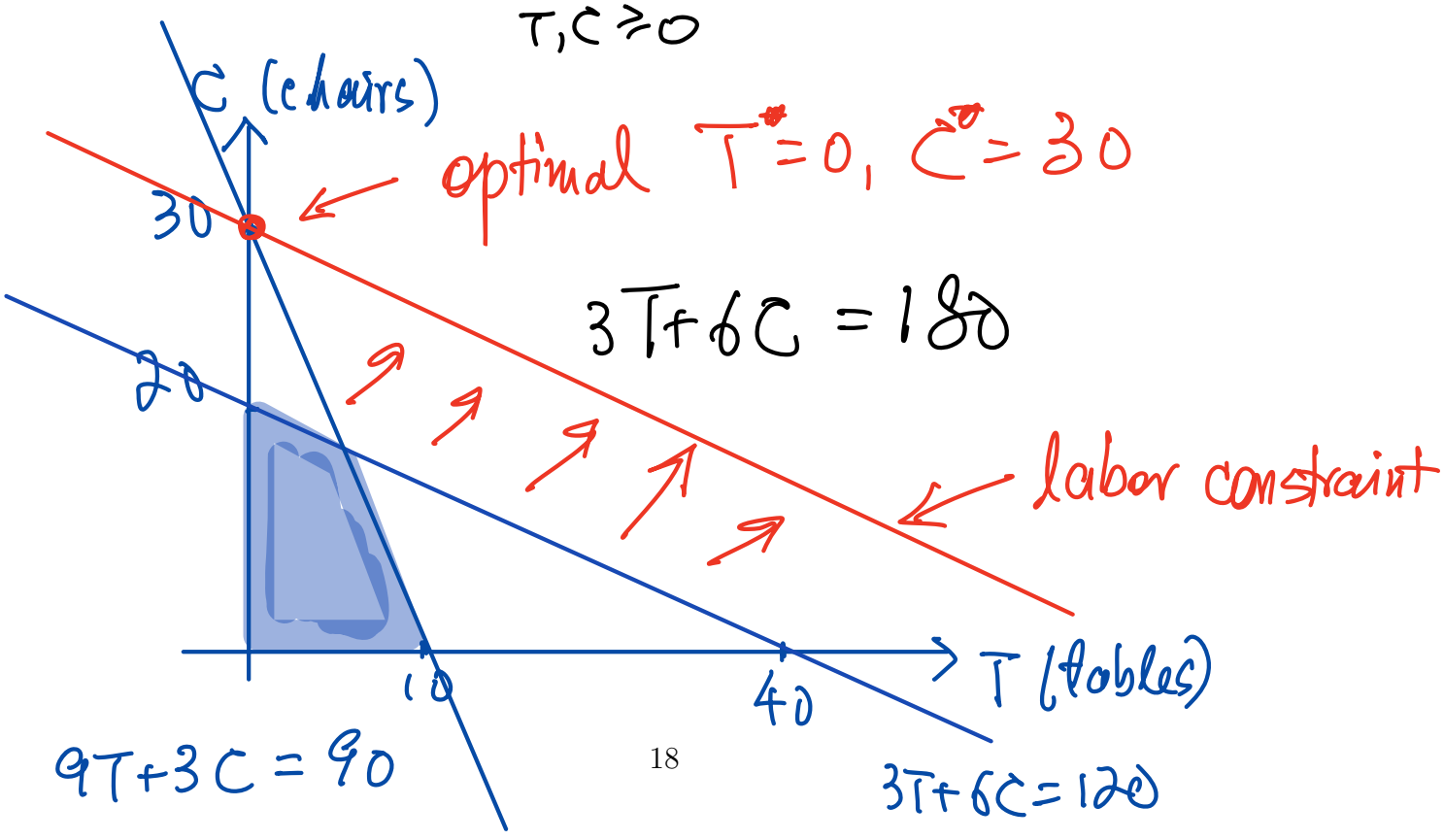
(b) Increase labor (only)

$$\max \ \Sigma = 500T+300C$$

$$\text{s.t.} \quad 3T+6C \le 120 \ \uparrow$$

$$9T+3C \le 90$$

$$T, C \ge 0$$



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Hire 60 hrs more (total 180 hrs)

[There is no reason to have more labor as there is not enough wood.)

new max profit

$$\zeta^* = 500(0) + 300(30) = \$9000$$

net increase in profit = $\$(9000 - 7400)$

$= \$1600$

$$\text{labor cost/hr} \leq \frac{\$1600}{60\text{ hr}} = \$\frac{80}{3} \approx \$26.7/\text{hr}$$

(c) Increase wood (only)

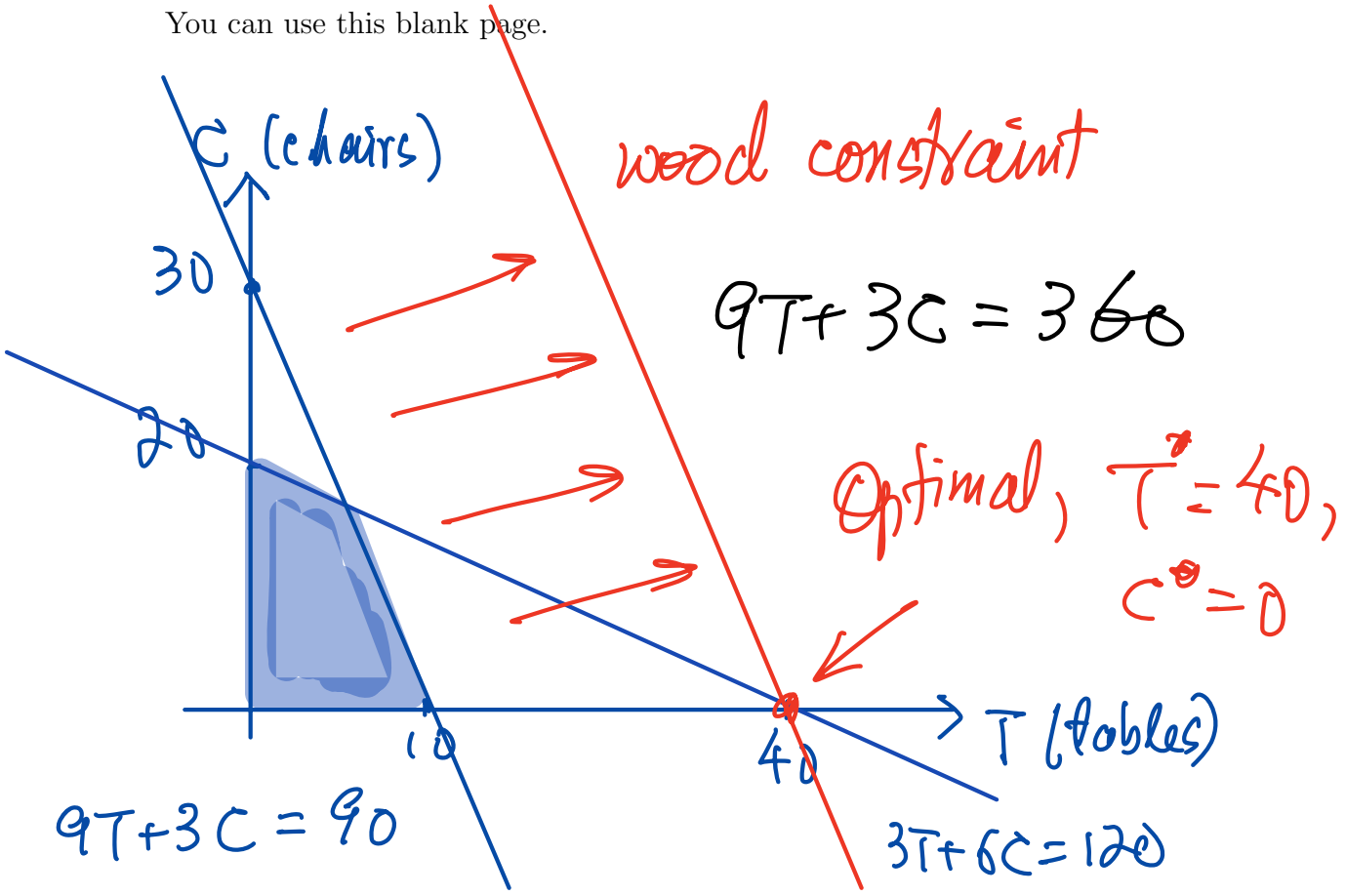
$$\max \zeta = 500T + 300C$$

$$\text{s.t. } 3T + 6C \leq 120$$

$$9T + 3C \leq 90 \uparrow$$

$$T, C \geq 0$$

You can use this blank page.



Purchase 270 ft of wood (total 360 ft)
(There is no reason to buy more as there is not enough labor hours.)

new max. profit

$\zeta^* = 500(40)+300(0) = \#20000$

net increase in profit

$= \#(20000-7400) = \#12600.$

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wood cost/ft $\leq \frac{\$12600}{270} = \frac{\$140}{3}$

$\approx \$46.7/\text{ft.}$

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