MA 421 Fall 2024 (Aaron N. K. Yip)
Homework 7, due on Thursday, Nov 13th, 11:59pm, in Gradescope
(Abstract of your project is due on Friday, Nov 14th, 11:59pm, in Brightspace)

1. [V] p.207, Exercises: 12.1 (L^2-), 12.2 (L^1-), *and* L^∞-regression line of the data points, i.e., given the data points $\{(x_i, y_i) : i = 1, 2 \ldots\}$, find m and c so as to *minimize* the following L^2-, L^1- and L^∞-errors:

$$\sum_i (y_i - mx_i - c)^2, \ \sum_i |y_i - mx_i - c|, \ \max_i |y_i - mx_i - c|.$$

   Use any of your favorite computer program/software to plot the points and the regression lines you have found, *all in one plot*. Compare and contrast your results.

2. Consider the following labelled points (all lying on the x-axis):

$$\mathrm{P} = \{1, 2, 4, 5\}, \quad \mathrm{N} = \{-4, -3, -2\}.$$

   Rigorously find the binary classification point, i.e. *with proof*, find x_* that solve the following problem:

$$\max_x \left\{ \min_{i:x_i \in \mathrm{P}} \{x_i - x\}, \min_{i:x_i \in \mathrm{N}} \{x - x_i\} \right\}$$

   Do it again but with the following new set of "corrupted" points:

$$\mathrm{P} = \{-2.5, -1, 0, 1, 2, 4, 5\}, \quad \mathrm{N} = \{-4, -3, -2, 0.5, 1.5\}.$$

3. This question is for you to implement Caratheodory Theorem [V] Theorem 10.3. Consider the following convex combination of points:

$$\begin{pmatrix} 1.2 \\ 0.5 \\ 0.3 \end{pmatrix} = 0.1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 0.2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + 0.2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 0.2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + 0.1 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + 0.1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + 0.1 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

   Use LP to express the left-hand-side as a convex combination of $4(= 3 + 1)$ points out of the 7 points from the right-hand-side. (Well, for such a small size problem, you can certain try brute force search: you will have ${}_7C_4 = 35$ so many choices. The "advantage" of the brute force search is that you can indeed find all the possible solutions.)

   Note. The proof of the Caratheodory Theorem is "constructive" in the sense that it gives an algorithm to find the solution. With this said, the initial dictionary for this problem is not feasible. Probably the easiest way is to introduce three *auxiliary* variables (not slack variables) $w_1, w_2, w_3 \geq 0$ and use simplex to maximize $-w_1 - w_2 - w_3$. The end of this phase automatically gives you a solution. There is no Phase II for this problem because all you need is simply a feasible dictionary.

4. (a) Consider the following two polyhedrons:

$$\begin{aligned} P_1 &= \{x_1 \geq 0;\ x_2 \geq \frac{x_1}{2} + 5;\ x_1 + x_2 \geq 10\}; \\ P_2 &= \{x_2 \geq 0;\ x_2 \leq x_1;\ x_2 \leq -3x_1 + 30\}. \end{aligned}$$

Use Farkas' Lemma to find two disjoint half spaces H_1 and H_2 such that $P_1 \subseteq H_1$ and $P_2 \subseteq H_2$. After you have found your answer, use any of your favorite computer program/software to plot P_1, P_2, H_1 and H_2 in a single x_1x_2-plane.

(b) Consider the following two polyhedrons:

$$P_1 = \left\{ \begin{aligned} 2x_1 + 3x_2 + x_3 &\leq 5; \\ 3x_1 + 4x_2 + 2x_3 &\leq 8; \\ x_1,\ x_2,\ x_3 &\geq 0 \end{aligned} \right. \quad \text{and} \quad P_2 = \left\{ \begin{aligned} 5x_1 + 4x_2 + 3x_3 &\geq 14; \\ 4x_1 + x_2 + 2x_3 &\leq 11; \\ x_1,\ x_2,\ x_3 &\geq 0 \end{aligned} \right.$$

Use Farkas' Lemma to find two disjoint half spaces H_1 and H_2 such that $P_1 \subseteq H_1$ and $P_2 \subseteq H_2$.

5. Find the biggest circle that can fit inside the following set:

$$x_2 \leq \frac{x_1}{2} + 2,\ \ x_1 + x_2 \leq 5,\ \ x_1 \geq 0,\ \ x_2 \geq 0.$$

use any of your favorite computer program/software to draw the above set and also the circle you have found.

(Note: the above problem can be formulated as an LP – see [MG] p.23, Section 2.6.)

1. [V] p.207, Exercises: 12.1 (L^2-), 12.2 (L^1-), *and* L^∞-regression line of the data points, i.e., given the data points $\{(x_i, y_i) : i = 1, 2 \ldots\}$, find m and c so as to *minimize* the following L^2-, L^1- and L^∞-errors:

$$\sum_i (y_i - mx_i - c)^2, \quad \sum_i |y_i - mx_i - c|, \quad \max_i |y_i - mx_i - c|.$$

L^2 least square:

$$AX = b$$

$$\Rightarrow A^TA\hat{X} = A^Tb \quad \text{normal eqn}$$

$$\Rightarrow \hat{X} = (A^TA)^{-1}A^Tb$$

(least square solution)

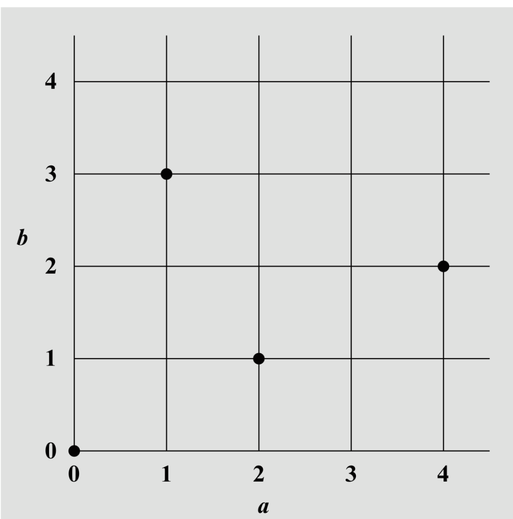


FIGURE 12.8. Four data points for a linear regression.

$$mx_i + c = y_i \qquad i = 1, 2, 3, 4$$

$(0,0)$: $0m + c = 0$

$(1,3)$: $1m + c = 3$

$(2,1)$: $2m + c = 1$

$(4,2)$: $4m + c = 2$

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 4 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ c \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 0 \\ 3 \\ 1 \\ 2 \end{bmatrix}}_{b}$$

L^1 – regression

$$\min \underbrace{|0-c|}_{t_1}+\underbrace{|3-m-c|}_{t_2}+\underbrace{|1-2m-c|}_{t_3}+\underbrace{|2-4m-c|}_{t_4}$$

$$\min \quad t_1+t_2+t_3+t_4$$

$$\text{s.t.} \quad \begin{cases} |0-c| \le t_1 \\ |3-m-c| \le t_2 \\ |1-2m-c| \le t_3 \\ |2-4m-c| \le t_4 \end{cases}$$

$$\begin{cases} -t_1 \le 0-c \le t_1 \\ -t_2 \le 3-m-c \le t_2 \\ -t_3 \le 1-2m-c \le t_3 \\ -t_4 \le 2-4m-c \le t_4 \end{cases}$$

$\max \quad -t_1-t_2-t_3-t_4$ use dual

(t_i's automatically positive)

L^∞-regression

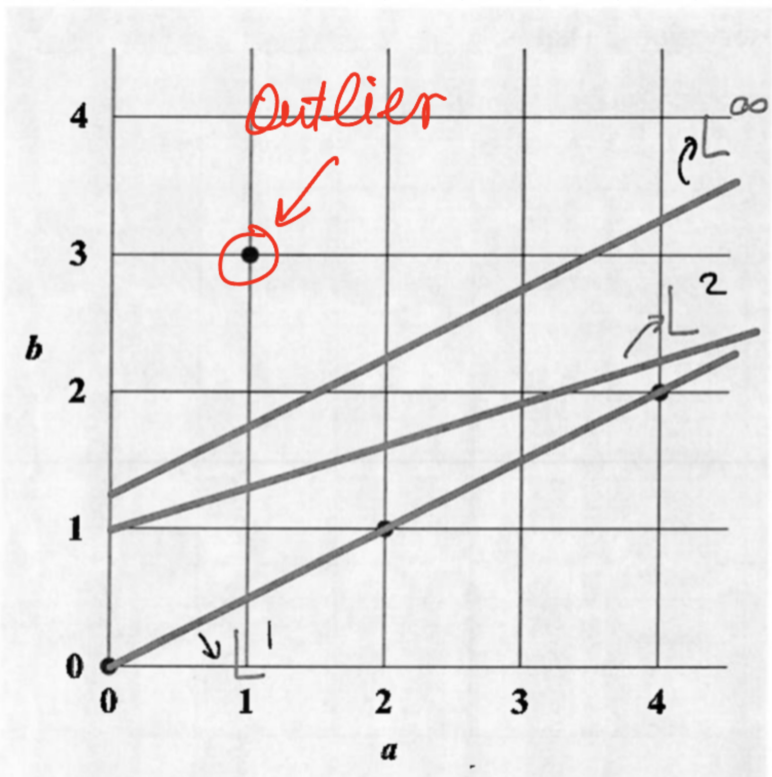
$$\min_{m,c}\left\{\underbrace{\max\left\{|0-c|,\ |3-m-c|,\ |1-2m-c|,\ |2-4m-c|\right\}}_{\delta}\right\}$$

$$\min_{m,c,\delta}\ \delta \qquad \max\ -\delta$$

use dual:

$$\text{s.t.}\quad \begin{cases} |0-c| \le \delta \\ |3-m-c| \le \delta \\ |1-2m-c| \le \delta \\ |2-4m-c| \le \delta \end{cases}$$

$$\begin{cases} -\delta \le 0-c \le \delta \\ -\delta \le 3-m-c \le \delta \\ -\delta \le 1-2m-c \le \delta \\ -\delta \le 2-4m-c \le \delta \end{cases}$$



(Solution from student)

Note that

L^1 is least sensitive to outlier.

L^∞ is most sensitive to outlier

2. Consider the following labelled points (all lying on the x-axis):

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Rigorously find the binary classification point, i.e. *with proof*, find x_* that solve the following problem:

$$\max_{x} \left\{ \min_{i:x_i \in \mathrm{P}} \{x_i - x\}, \min_{i:x_i \in \mathrm{N}} \{x - x_i\} \right\}$$

Do it again but with the following new set of "corrupted" points:

$$\mathrm{P} = \{-2.5, -1, 0, 1, 2, 4, 5\}, \quad \mathrm{N} = \{-4, -3, -2, 0.5, 1.5\}.$$

$$\max_{x} \left\{ \underline{\min} \left\{ \underbrace{\min_{x_i \in P} \{x_i - x\}}_{f(x)}, \underbrace{\min_{x_i \in N} \{x - x_i\}}_{g(x)} \right\} \right\}$$

$$\max_{x} \left\{ \min \{ f(x), g(x) \} \right\}$$

$$f(x) = \min_{x_i \in P} \{ x_i - x \}$$

$$= \min \{ 1-x, 2-x, 4-x, 5-x \} = 1-x$$

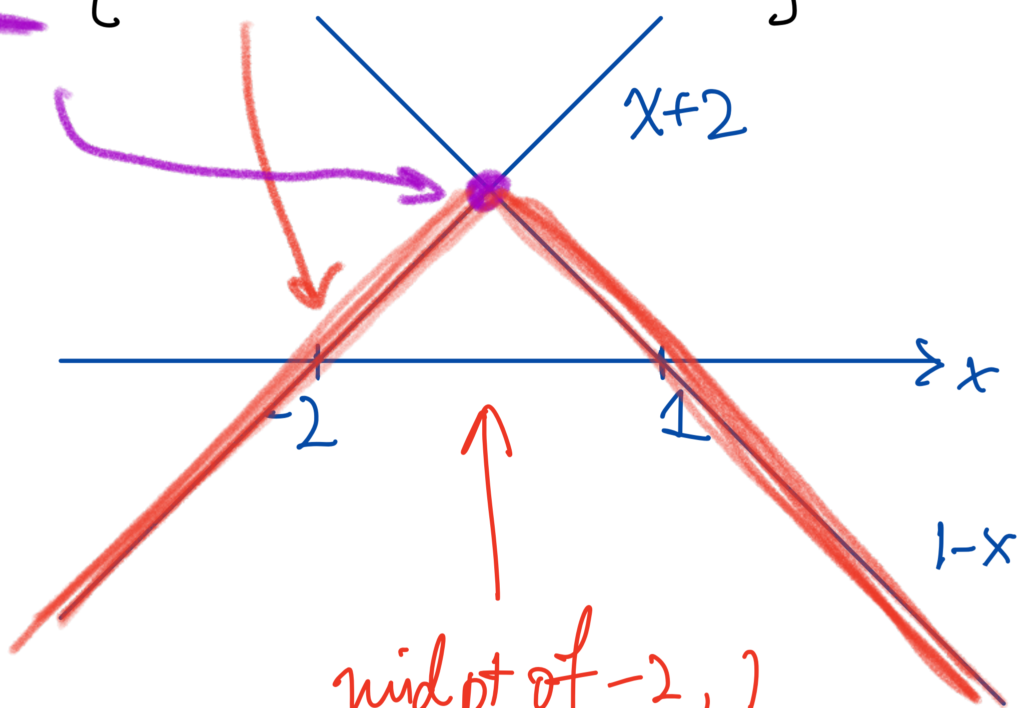
$$g(x) = \min_{x_i \in N} \{ x - x_i \}$$

$$= \min\{x-(-4),\ x-(-3),\ x-(-2)\}$$

$$= \min\{x+4,\ x+3,\ x+2\}$$

$$= x+2$$

$$\max_x \left\{ \min\{1-x,\ x+2\} \right\}$$



midpt of $-2, 1$

$$= \frac{-2+1}{2} = -1/2$$

3. This question is for you to implement Caratheodory Theorem [V] Theorem 10.3. Consider the following convex combination of points:

$$\begin{pmatrix} 1.2 \\ 0.5 \\ 0.3 \end{pmatrix} = 0.1 \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_{x_1} + 0.2 \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{x_2} + 0.2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 0.2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + 0.1 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + 0.1 \underbrace{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}_{x_7} + 0.1 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

$x_1 \quad x_2 \quad \cdots\cdots \quad x_7$

$$\left.\begin{aligned} x_1 + x_3 + 3x_4 + x_5 + 2x_7 &= 1.2 \\ 2x_1 - x_3 + x_4 + x_5 + x_6 + x_7 &= 0.5 \\ 3x_1 + x_2 - x_5 - x_6 &= 0.3 \\ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 &= 1 \end{aligned}\right\} \text{4 eqn}$$

$$x_i \geq 0, \quad i = 1, 2, \cdots, 7$$

$$\min \quad w_1 + w_2 + w_3 + w_4$$

s.t.

$$\begin{aligned} w_1 &= 1.2 - (x_1 + x_3 + 3x_4 + x_5 + 2x_7) \\ w_2 &= 0.5 - (2x_1 - x_3 + x_4 + x_5 + x_6 + x_7) \\ w_3 &= 0.3 - (3x_1 + x_2 - x_5 - x_6) \\ w_4 &= 1 - (x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7) \end{aligned}$$

Note:

$$\min \quad w_1 + w_2 + w_3 + w_4 + \text{const}$$

$$= \min \ -(7x_1 + 2x_2 + x_3 + 5x_4 + 2x_5 + x_6 + 4x_7)$$

$$= \max \ 7x_1 + 2x_2 + x_3 + 5x_4 + 2x_5 + x_6 + 4x_7$$

LP :

$$\max \quad 7x_1 + 2x_2 + x_3 + 5x_4 + 2x_5 + x_6 + 4x_7$$

s.t.

$$
\begin{aligned}
w_1 &= 1.2 - (x_1 + x_3 + 3x_4 + x_5 + 2x_7) \\
w_2 &= 0.5 - (2x_1 - x_3 + x_4 + x_5 + x_6 + x_7) \\
w_3 &= 0.3 - (3x_1 + x_2 - x_5 - x_6) \\
w_4 &= 1 - (x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7)
\end{aligned}
$$

#4 (a)

$$P_1 = \left\{\begin{array}{l} -x_1 \le 0 \\ \frac{x_1}{2} - x_2 \le -5 \\ -x_1 - x_2 \le -10 \end{array}\right\} = \underbrace{\begin{bmatrix} -1 & 0 \\ \frac{1}{2} & -1 \\ -1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \underbrace{\begin{bmatrix} 0 \\ -5 \\ -10 \end{bmatrix}}_{b}$$

$$P_2 = \left\{\begin{array}{l} -x_2 \le 0 \\ -x_1 + x_2 \le 0 \\ 3x_1 + x_2 \le 30 \end{array}\right\} = \underbrace{\begin{bmatrix} 0 & -1 \\ -1 & 1 \\ 3 & 1 \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \underbrace{\begin{bmatrix} 0 \\ 0 \\ 30 \end{bmatrix}}_{\tilde{b}}$$

$$P_1 \cap P_2 = \left\{\begin{array}{l} Ax \le b \\ \tilde{A}x \le \tilde{b} \end{array}\right. = \boxed{\begin{bmatrix} A \\ \tilde{A} \end{bmatrix}} x \le \boxed{\begin{bmatrix} b \\ \tilde{b} \end{bmatrix}}$$

Apply FL to this system.

i.e.

$$\boxed{\begin{array}{ll} \min & \begin{pmatrix} b^T & \tilde{b}^T \end{pmatrix} \begin{pmatrix} y \\ \tilde{y} \end{pmatrix} \\ \text{s.t.} & \begin{pmatrix} A^T & \tilde{A}^T \end{pmatrix} \begin{pmatrix} y \\ \tilde{y} \end{pmatrix} = 0 \\ & y, \tilde{y} \ge 0 \end{array}}$$

Using simplex to find $y, \tilde{y}$ s.t.

$$\boxed{\begin{aligned} &\begin{pmatrix} A^T & \tilde{A}^T \end{pmatrix} \begin{pmatrix} y \\ \tilde{y} \end{pmatrix} = 0, \quad y, \tilde{y} \geq 0 \\ &\text{and} \quad \begin{pmatrix} b^T & \tilde{b}^T \end{pmatrix} \begin{pmatrix} y \\ \tilde{y} \end{pmatrix} < 0 \end{aligned}}$$

$$\underline{A^T y + \tilde{A}^T \tilde{y} = 0} \qquad b^T y + \tilde{b}^T \tilde{y} < 0$$

Then

$$H_1 = \{ X : (y^T A) X \leq y^T b \} \quad (\leftarrow = b^T y)$$

$$H_2 = \{ X : (\tilde{y}^T \tilde{A}) X \leq \tilde{y}^T \tilde{b} \} \quad \leftarrow = \tilde{b}^T \tilde{y}$$

Note: H_1: $\quad y^T A X \leq y^T b \qquad y^T b < -\tilde{y}^T \tilde{b}$

$y^T A = -\tilde{y}^T \tilde{A}$

$$-\tilde{y}^T \tilde{A} X < -\tilde{y}^T \tilde{b}$$

i.e. $\quad \tilde{y}^T \tilde{b} < \tilde{y}^T \tilde{A} X \quad \leftarrow$ not in H_2

Hence $H_1 \cap H_2 = \phi$

$$\min \begin{pmatrix} b^T & \tilde{b}^T \end{pmatrix} \begin{pmatrix} y \\ \tilde{y} \end{pmatrix}$$

$$\text{s.t.} \quad \begin{pmatrix} A^T & \tilde{A}^T \end{pmatrix} \begin{pmatrix} y \\ \tilde{y} \end{pmatrix} = 0$$

$$y, \tilde{y} \geq 0$$

maximize $\zeta = 0\,x_1 + 5\,x_2 + 10\,x_3 + 0\,x_4 + 0\,x_5 + (-30)\,x_6$

subject to:

		x_1	x_2	x_3	x_4	x_5	x_6
w_1 =	0	- -1	- 1/2	- -1	- 0	- -1	- 3
w_2 =	0	- 0	- -1	- -1	- -1	- 1	- 1
w_3 =	0	- 1	- -1/2	- 1	- 0	- 1	- -3
w_4 =	0	- 0	- 1	- 1	- 1	- -1	- -1

$x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ \ w_1\ w_2\ w_3\ w_4 \geq 0$

maximize $\zeta = -10\,x_1 + 10\,x_2 + (-10)\,w_3 + 0\,x_4 + (-10)\,x_5 + 0\,x_6$

subject to:

		x_1	x_2	w_3	x_4	x_5	x_6
w_1 =	0	- 0	- 0	- 1	- 0	- 0	- 0
w_2 =	0	- 1	- -3/2	- 1	- -1	- 2	- -2
x_3 =	0	- 1	- -1/2	- 1	- 0	- 1	- -3
w_4 =	0	- -1	- 3/2	- -1	- 1	- -2	- 2

$x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ \ w_1\ w_2\ w_3\ w_4 \geq 0$

maximize $\zeta = -10/3\,x_1 + (-20/3)\,w_4 + (-10/3)\,w_3 + (-20/3)\,x_4 + 10/3\,x_5 + (-40/3)\,x_6$

subject to:

		x_1	w_4	w_3	x_4	x_5	x_6
w_1 =	0	- 0	- 0	- 1	- 0	- 0	- 0
w_2 =	0	- 0	- 1	- 0	- 0	- 0	- 0
x_3 =	0	- 2/3	- 1/3	- 2/3	- 1/3	- 1/3	- -7/3
x_2 =	0	- -2/3	- 2/3	- -2/3	- 2/3	- -4/3	- 4/3

$x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ \ w_1\ w_2\ w_3\ w_4 \geq 0$

maximize $\zeta = -10\,x_1 + (-10)\,w_4 + (-10)\,w_3 + (-10)\,x_4 + (-10)\,x_3 + 10\,x_6$

subject to:

		x_1	w_4	w_3	x_4	x_3	x_6
w_1 =	0	- 0	- 0	- 1	- 0	- 0	- 0
w_2 =	0	- 0	- 1	- 0	- 0	- 0	- 0
x_5 =	0	- 2	- 1	- 2	- 1	- 3	- -7
x_2 =	0	- 2	- 2	- 2	- 2	- 4	- -8

$x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ \ w_1\ w_2\ w_3\ w_4 \geq 0$

$(y_1 = 0,\ \underline{y_2 = 8},\ y_3 = 0)$, $\quad (\tilde{y}_1 = 0,\ \underline{\tilde{y}_2 = 7},\ \underline{\tilde{y}_3 = 1})$

$b = (0,\ -5,\ -10)$, $\quad \tilde{b} = (0,\ 0,\ 30)$

$$\boxed{b^T y + \tilde{b}^T \tilde{y} = -40 + 30 = -10 < 0}$$

$$H = \{x : y^T A x \leq y^T b\}$$

$$(0 \quad 8 \quad 0) \begin{pmatrix} -1 & 0 \\ \frac{1}{2} & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq (0 \quad 8 \quad 0) \begin{pmatrix} 0 \\ -5 \\ -10 \end{pmatrix}$$

$$(4 \quad -8) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq -40$$

$$\boxed{\boxed{x_1 - 2x_2 \leq -10} \quad H}$$

$$\tilde{H} = \{x : \tilde{y}^T \tilde{A} x \leq \tilde{y}^T \tilde{b}\}$$

$$(0 \quad 7 \quad 1) \begin{pmatrix} 0 & -1 \\ -1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq (0 \quad 7 \quad 1) \begin{pmatrix} 0 \\ 0 \\ 30 \end{pmatrix}$$

$$(-4 \quad 8) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq 30$$

$$\boxed{\boxed{x_1 - 2x_2 \geq -7.5} \quad \tilde{H}}$$

(b)

$$P_1 = \begin{cases} 2x_1 + 3x_2 + x_3 \leq 5 \\ 3x_1 + 4x_2 + 2x_3 \leq 8 \\ -x_1 \leq 0 \\ -x_2 \leq 0 \\ -x_3 \leq 0 \end{cases} = \underbrace{\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 2 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq \underbrace{\begin{bmatrix} 5 \\ 8 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{b}$$

$$P_2 = \begin{cases} -5x_1 - 4x_2 - 3x_3 \leq -14 \\ 4x_1 + x_2 + 2x_3 \leq 11 \\ -x_1 \leq 0 \\ -x_2 \leq 0 \\ -x_3 \leq 0 \end{cases} = \underbrace{\begin{bmatrix} -5 & -4 & -3 \\ 4 & 1 & 2 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq \underbrace{\begin{bmatrix} -14 \\ 11 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\tilde{b}}$$

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
maximize $\zeta =$		-5	-8	0	0	0	14	-11	0	0	0
subject to: $w_1 =$	0	2	3	-1	0	0	-5	4	-1	0	0
$w_2 =$	0	-2	-3	1	0	0	5	-4	1	0	0
$w_3 =$	0	3	4	0	-1	0	-4	1	0	-1	0
$w_4 =$	0	-3	-4	0	1	0	4	-1	0	1	0
$w_5 =$	0	1	2	0	0	-1	-3	2	0	0	-1
$w_6 =$	0	-1	-2	0	0	1	3	-2	0	0	1

$x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10}\quad w_1\ w_2\ w_3\ w_4\ w_5\ w_6 \geq 0$

		x_1	x_2	x_3	x_4	x_5	w_6	x_7	x_8	x_9	x_{10}
maximize $\zeta =$		-1/3	4/3	0	0	-14/3	-14/3	-5/3	0	0	-14/3
subject to: $w_1 =$	0	1/3	-1/3	-1	0	5/3	5/3	2/3	-1	0	5/3
$w_2 =$	0	-1/3	1/3	1	0	-5/3	-5/3	-2/3	1	0	-5/3
$w_3 =$	0	5/3	4/3	0	-1	4/3	4/3	-5/3	0	-1	4/3
$w_4 =$	0	-5/3	-4/3	0	1	-4/3	-4/3	5/3	0	1	-4/3
$w_5 =$	0	0	0	0	0	0	1	0	0	0	0
$x_6 =$	0	-1/3	-2/3	0	0	1/3	1/3	-2/3	0	0	1/3

$x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10}\quad w_1\ w_2\ w_3\ w_4\ w_5\ w_6 \geq 0$

		x_1	w_2	x_3	x_4	x_5	w_6	x_7	x_8	x_9	x_{10}
maximize $\zeta =$		1	-4	-4	0	2	2	1	-4	0	2
subject to: $w_1 =$	0	0	1	0	0	0	0	0	0	0	0
$x_2 =$	0	-1	3	3	0	-5	-5	-2	3	0	-5
$w_3 =$	0	3	-4	-4	-1	8	8	1	-4	-1	8
$w_4 =$	0	-3	4	4	1	-8	-8	-1	4	1	-8
$w_5 =$	0	0	0	0	0	0	1	0	0	0	0
$x_6 =$	0	-1	2	2	0	-3	-3	-2	2	0	-3

$x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10}\quad w_1\ w_2\ w_3\ w_4\ w_5\ w_6 \geq 0$

maximize $\zeta = \tfrac{1}{4}x_1 + (-3)w_2 + (-3)x_3 + \tfrac{1}{4}x_4 + 0x_5 + 0w_6 + \tfrac{3}{4}x_7 + (-3)x_8 + \tfrac{1}{4}x_9 + (-\tfrac{1}{4})w_3$

subject to:

		x_1	w_2	x_3	x_4	x_5	w_6	x_7	x_8	x_9	w_3
$w_1 =$	0	- 0	- 1	- 0	- 0	- 0	- 0	- 0	- 0	- 0	- 0
$x_2 =$	0	- 7/8	- 1/2	- 1/2	- -5/8	- 0	- 0	- -11/8	- 1/2	- -5/8	- 5/8
$x_{10} =$	0	- 3/8	- -1/2	- -1/2	- -1/8	- 1	- 1	- 1/8	- -1/2	- -1/8	- 1/8
$w_4 =$	0	- 0	- 0	- 0	- 0	- 0	- 0	- 0	- 0	- 0	- 1
$w_5 =$	0	- 0	- 0	- 0	- 0	- 0	- 1	- 0	- 0	- 0	- 0
$x_6 =$	0	- 1/8	- 1/2	- 1/2	- -3/8	- 0	- 0	- -13/8	- 1/2	- -3/8	- 3/8

$x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10}\quad w_1\ w_2\ w_3\ w_4\ w_5\ w_6 \geq 0$

maximize $\zeta = -2x_1 + 0w_2 + 0x_3 + 1x_4 + (-6)x_5 + (-6)w_6 + (-6)x_{10} + 0x_8 + 1x_9 + (-1)w_3$

subject to:

		x_1	w_2	x_3	x_4	x_5	w_6	x_{10}	x_8	x_9	w_3
$w_1 =$	0	- 0	- 1	- 0	- 0	- 0	- 0	- 0	- 0	- 0	- 0
$x_2 =$	0	- 5	- -5	- -5	- -2	- 11	- 11	- 11	- -5	- -2	- 2
$x_7 =$	0	- 3	- -4	- -4	- -1	- 8	- 8	- 8	- -4	- -1	- 1
$w_4 =$	0	- 0	- 0	- 0	- 0	- 0	- 0	- 0	- 0	- 0	- 1
$w_5 =$	0	- 0	- 0	- 0	- 0	- 0	- 1	- 0	- 0	- 0	- 0
$x_6 =$	0	- 5	- -6	- -6	- -2	- 13	- 13	- 13	- -6	- -2	- 2

$x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10}\quad w_1\ w_2\ w_3\ w_4\ w_5\ w_6 \geq 0$

$$(y_1 = 0 \quad y_2 = 2 \quad y_3 = 0 \quad \underline{y_4 = 1} \quad y_5 = 0)$$

$$b = (5 \quad 8 \quad 0 \quad 0 \quad 0)$$

$$(\tilde{y}_1 = 2, \quad \tilde{y}_2 = 1, \quad \tilde{y}_3 = 0, \quad \tilde{y}_4 = 0, \quad \tilde{y}_5 = 0)$$

$$\tilde{b} = (-14, \quad 11, \quad 0, \quad 0, \quad 0)$$

$$\boxed{b^T y + \tilde{b}^T \tilde{y} = 16 - 28 + 11 = -1 < 0}$$

$$H = \{ x : y^T A x \leq y^T b \}$$

$$(0 \quad 2 \quad 0 \quad 1 \quad 0) \begin{pmatrix} 2 & 3 & 1 \\ 3 & 4 & 2 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq (0 \quad 2 \quad 0 \quad 1 \quad 0) \begin{pmatrix} 5 \\ 8 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 7 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq 16$$

$$\boxed{6x_1 + 7x_2 + 4x_3 \leq 16} \quad H$$

$$\tilde{H} = \{ x : \tilde{y}^T \tilde{A} x \leq \tilde{y}^T \tilde{b} \}$$

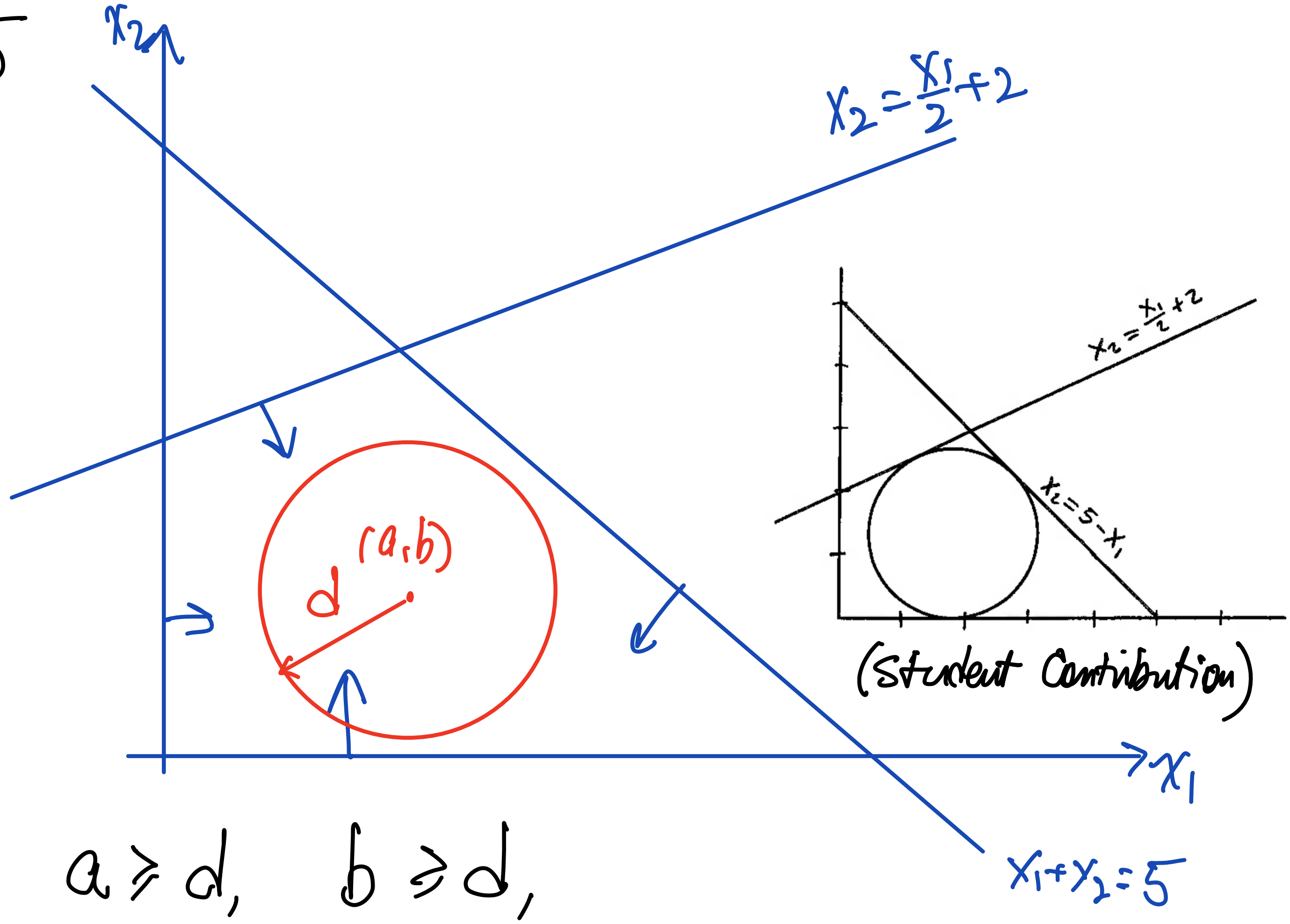
$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -5 & -4 & -3 \\ 4 & 1 & 2 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\leq \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -14 \\ 11 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -6 & -7 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq -17$$

$$\boxed{6x_1 + 7x_2 + 4x_3 \geq 17} \quad \tilde{H}$$

#5



$$a \geq d, \quad b \geq d,$$

$$\left| \frac{a+b-5}{\sqrt{2}} \right| \geq d \Longrightarrow \frac{5-a-b}{\sqrt{2}} \geq d$$

(need $a+b \leq 5$)

$$\left| \frac{\frac{a}{2} - b + 2}{\sqrt{\frac{1}{2^2} + 1^2}} \right| \geq d \Longrightarrow \frac{a-2b+4}{\sqrt{5}} \geq d$$

(need $\frac{a}{2} - b + 2 \geq 0$)

(LP)

$$\boxed{\begin{aligned} &\max \quad d \\ &\text{s.t.} \quad a \geq d, \ b \geq d, \\ &\qquad \frac{5-a-b}{\sqrt{2}} \geq d; \quad \frac{a-2b+4}{\sqrt{5}} \geq d \end{aligned}}$$

MA 421 Fall 2025 (Aaron N. K. Yip)
Homework 8, due on Thursday, Dec 4th, 11:59pm, in Gradescope
(Project paper due on Friday, Dec 5th, 11:59pm, in Brightspace)

1. [V]: (See Fall 2024 Final #3)
   p.168, Exercises: 10.2;
   p.414, Exercises: 23.4, 23.5

2. [C]:
   p.248, 16.10. (For each part, just do the "easy" direction of the "if and only if".)

3. In Homework 7, you were asked to find a separating hyperplane for two disjoint polyhedrons. Now you will do the same but with one of the polyhedrons being a "point". In particular, consider,

$$\begin{aligned} P_1 &= \{x_1 \geq 0;\ \ x_2 \geq \frac{x_1}{2} + 5;\ \ x_1 + x_2 \geq 10\}; \\ P_2 &= \{x_1 = -1,\ \ x_2 = 7\}. \end{aligned}$$

   By means of Farkas Lemma, find a hyperplane (a straight line in this case) that separates P_1 and P_2. Use your favorite graphing software to plot P_1, P_2 and the hyperplane in one single graph.

4. [CZ]:

   21.2 Find local extremizers for:

   a. $x_1^2 + x_2^2 - 2x_1 - 10x_2 + 26$ subject to $\frac{1}{5}x_2 - x_1^2 \leq 0$, $5x_1 + \frac{1}{2}x_2 \leq 5$.

   b. $x_1^2 + x_2^2$ subject to $x_1 \geq 0$, $x_2 \geq 0$, $x_1 + x_2 \geq 5$.

   For each of the above parts, plot the feasible region, apply KKT and find the values of all the variables and the Lagrange multipliers corresponding to the extremizers.

5. Given the following points in $\mathbb{R}^2$:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

   Use the method of quadratic programming to find the smallest circle enclosing all of the above points. Use your favorite graphic package to plot the points and the circle you have found.

   (Hint: read pages 8-9 of the note Quadratic Programming from Week 12.)

[c]

Only do the "if" parts

16.10 Derive the following theorems (with the vector inequality $\mathbf{v} > \mathbf{w}$ meaning, as usual, $v_k > w_k$ for all k) from the result of problem 16.9.

(i) P. Gordan (1873): The system $\mathbf{Ax} < \mathbf{0}$ is unsolvable if and only if the system $\mathbf{yA} = \mathbf{0}$, $\mathbf{y} \geq \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$ is solvable.

(ii) J. Farkas (1902): The system $\mathbf{Ax} \leq \mathbf{0}$, $\mathbf{bx} > 0$ is unsolvable if and only if the system $\mathbf{yA} = \mathbf{b}$, $\mathbf{y} \geq \mathbf{0}$ is solvable.

(iii) E. Stiemke (1915): The system $\mathbf{Ax} = \mathbf{0}$, $\mathbf{x} > \mathbf{0}$ is unsolvable if and only if the system $\mathbf{yA} \geq \mathbf{0}$, $\mathbf{yA} \neq \mathbf{0}$ is solvable.

(iv) J. A. Ville (1938): The system $\mathbf{Ax} < \mathbf{0}$, $\mathbf{x} \geq \mathbf{0}$ is unsolvable if and only if the system $\mathbf{yA} \geq \mathbf{0}$, $\mathbf{y} \geq \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$ is solvable.

(v) A. W. Tucker (1956): The system $\mathbf{Ax} \geq \mathbf{0}$, $\mathbf{x} \geq \mathbf{0}$ has no solution with $x_k > 0$ if and only if the system $\mathbf{yA} \leq \mathbf{0}$, $\mathbf{y} \geq \mathbf{0}$ has a solution with

$$\sum_{i=1}^{m} a_{ik} y_i < 0.$$

(i) If there is a y s.t. $yA=0$, $y \geq 0$, $y \neq 0$.

then $AX < 0$ is not solvable.

pf Suppose $AX<0$ is solvable, i.e. such an X exists.

$AX < 0 \Rightarrow \underbrace{yA}_{0}X < \underbrace{y0}_{0}$ $(y \neq 0)$

$\Rightarrow 0 < 0$ Contradiction!

(ii) Suppose $AX \leq 0$, $bX > 0$ is solvable

$y \geq 0$ $\quad \underbrace{yA}_{b}X \leq 0 \Rightarrow bX \leq 0$ contradiction!

(iii) Suppose $AX=0$, $X>0$ is solvable

y $\quad \underbrace{yA}_{\geq 0,\ \text{but} \neq 0}X = 0 \Rightarrow (yA)X = 0$

$\Rightarrow$ Some component of X must be zero

contradiction

(iv) Suppose $AX < 0$, $X \geq 0$ is solvable

$y \geq 0$

$y \neq 0$

$$\underbrace{(yA)}_{\geq 0,} X < 0$$

≥ 0

$\Rightarrow$ $0 < 0$ contradiction!

(v) Suppose $AX \geq 0$, $X \geq 0$ has a solution with $x_k > 0$

y

$yA \leq 0$, $y \geq 0$, $\sum_i a_{ik} y_i < 0$

$$yAX \geq 0$$

$$0 \leq \sum_i y_i (AX)_i = \sum_i y_i \sum_j A_{ij} x_j$$

$$= \sum_j \sum_i y_i A_{ij} x_j$$

$$= \sum_j \underbrace{\left(\sum_i y_i A_{ij}\right)}_{\leq 0} \underbrace{x_j}_{\geq 0} \leq 0$$

but for index $j = k$, $\underbrace{\left(\sum_i y_i A_{ik}\right)}_{< 0} \underbrace{x_k}_{> 0} < 0$

Hence $0 < 0$ (Contradiction)

3. In Homework 7, you were asked to find a separating hyperplane for two disjoint polyhedrons. Now you will do the same but with one of the polyhedrons being a "point". In particular, consider,

$$\begin{aligned} P_1 &= \{x_1 \geq 0;\ x_2 \geq \frac{x_1}{2} + 5;\ x_1 + x_2 \geq 10\}; \\ P_2 &= \{x_1 = -1,\ x_2 = 7\}. \end{aligned}$$

By means of Farkas Lemma, find a hyperplane (a straight line in this case) that separates P_1 and P_2. Use your favorite graphing software to plot P_1, P_2 and the hyperplane in one single graph.

(M1) Use Farkas Lemma just like HW 7 #4

$$P_1 = \left\{ \begin{array}{l} -x_1 \leq 0 \\ +\frac{x_1}{2} - x_2 \leq -5 \\ -x_1 - x_2 \leq -10 \end{array} \right\}, \quad P_2 = \left\{ \begin{array}{l} x_1 \leq -1 \\ -x_1 \leq 1 \\ x_2 \leq 7 \\ -x_2 \leq -7 \end{array} \right\}$$

(M2) In fact, the question is trivial.

As P_2 is a point, just test P_2 against the inequalities of P_1 and see which is violated.

eg: $-(-1) \cancel{\leq} 0 \Rightarrow$ $x_1 \geq 0$ works

$\frac{1}{2}(-1) - (7) \cancel{\leq} -5 \Rightarrow$ $\frac{x_1}{2} - x_2 \leq -5$ works also

$-(-1) - (7) \cancel{\leq} -10 \Rightarrow$ $-x_1 - x_2 \leq -10$ works also

#4 **a.** $x_1^2 + x_2^2 - 2x_1 - 10x_2 + 26$ subject to $\frac{1}{5}x_2 - x_1^2 \le 0$, $5x_1 + \frac{1}{2}x_2 \le 5$.

M1 Use vector calculus (MA 261)

$$f(x_1, x_2) = x_1^2 + x_2^2 - 2x_1 - 10x_2 + 26$$

(1) $$\nabla f = \begin{pmatrix} 2x_1 - 2 \\ 2x_2 - 10 \end{pmatrix} = 0 \Rightarrow \begin{array}{l} x_1 = 1 \\ x_2 = 5 \end{array}$$

$$\frac{1}{5}x_2 - x_1^2 \le 0$$

but $5x_1 + \frac{1}{2}x_2 \cancel{\le} 5$

Look at boundary:

(2) $\frac{1}{2}x_2 - x_1^2 = 0$ and $5x_1 + \frac{1}{2}x_2 < 5$

$$\Downarrow$$

$$x_2 = 2x_1^2$$

$$f(x_1, x_2) = \underbrace{x_1^2 + 4x_1^4 - 2x_1 - 20x_1^2 + 26}_{\text{1D min problem}}$$

(3) $5x_1 + \frac{1}{2}x_2 = 5$, $\frac{1}{2}x_2 - x_1^2 < 0$

$$x_2 = 10(1 - x_1)$$

$$f(x_1, x_2) = \overbrace{x_1^2 + 10^2(1-x_1)^2 - 2x_1 - 100(1-x_1) + 26}^{\text{1D min problem}}$$

(M2) Use KKT (Essentially the same as (M1))

$L(x_1, x_2, \mu_1, \mu_2)$

$$= x_1^2 + x_2^2 - 2x_1 - 10x_2 + 26 + \mu_1\left(\frac{1}{5}x_2 - x_1^2\right) + \mu_2\left(5x_1 + \frac{1}{2}x_2 - 5\right)$$

$$\partial_{x_1} L = 0 \Rightarrow \quad 2x_1 - 2 - 2x_1\mu_1 + 5\mu_2 = 0 \quad (1)$$

$$\partial_{x_2} L = 0 \Rightarrow \quad 2x_2 - 10 + \frac{\mu_1}{5} + \frac{\mu_2}{2} = 0 \quad (2)$$

$$\mu_1\left(\frac{1}{5}x_2 - x_1^2\right) + \mu_2\left(5x_1 + \frac{1}{2}x_2 - 5\right) = 0 \quad (3)$$

$$\frac{1}{5}x_2 - x_1^2 \le 0, \quad 5x_1 + \frac{1}{2}x_2 \le 5$$

$$\mu_1, \mu_2 \ge 0$$

Case 1: $\mu_1 > 0, \mu_2 > 0 \Rightarrow$ (3)

$\frac{1}{5}x_2 = x_1^2$, $5x_1 + \frac{1}{2}x_2 = 5 \Rightarrow$ solve for x_1, x_2

using (1), (2) to solve for μ_1, μ_2 to see if $\mu_1, \mu_2 > 0$

Case 2: $\mu_1 = 0, \mu_2 > 0 \overset{3}{\Longrightarrow}$ $5x_1 + \frac{1}{2}x_2 = 5$

with 1 and 2

to see if

$$\boxed{\begin{array}{c} \frac{1}{5}x_2 - x_1^2 \leq 0 \\ \text{and } \mu_2 > 0 \end{array}}$$

Case 3: $\mu_1 > 0, \mu_2 = 0 \overset{3}{\Longrightarrow}$ $\frac{1}{5}x_2 - x_1^2 = 0$

with 1 and 2

to see if

$$\boxed{\begin{array}{c} 5x_1 + \frac{1}{2}x_2 \leq 5 \\ \text{and } \mu_1 > 0 \end{array}}$$

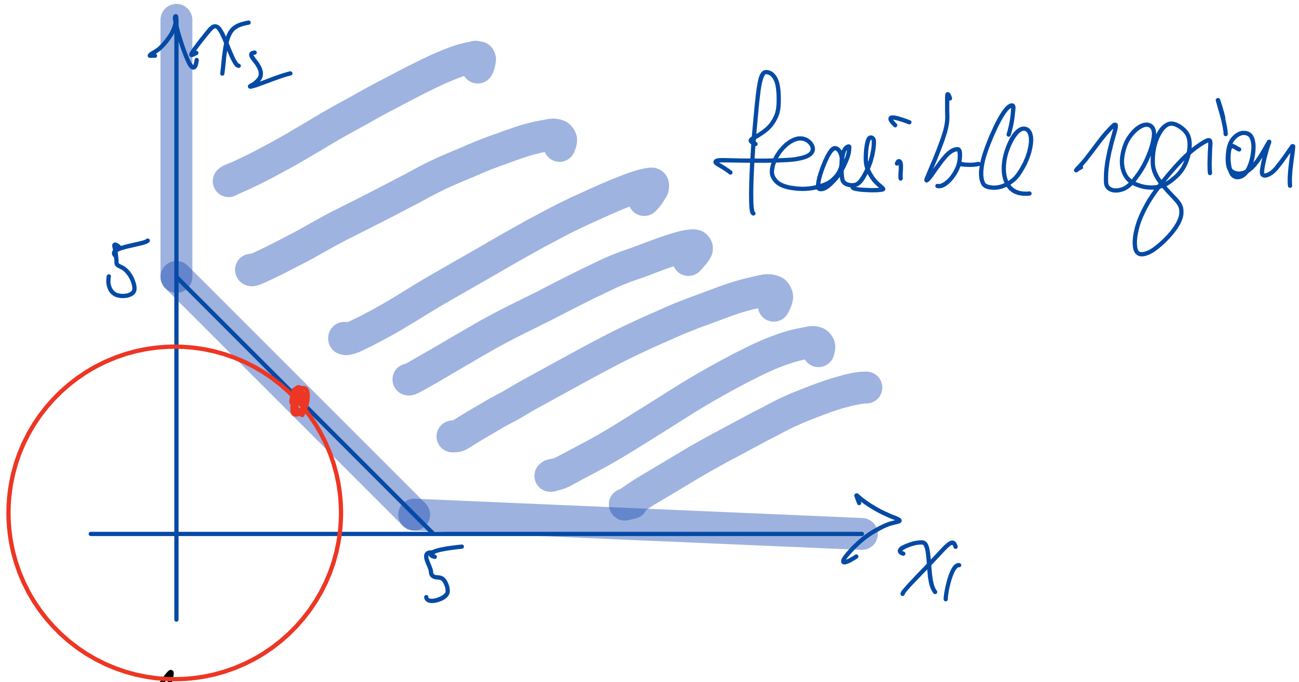
Case 4: $\mu_1, \mu_2 = 0 \Longrightarrow$ 1 and 2

solve for x_1, x_2

see if

$$\boxed{\begin{array}{c} \frac{1}{5}x_2 - x_1^2 \leq 0, \\ 5x_1 + \frac{1}{2}x_2 \leq 5 \end{array}}$$

b. $x_1^2 + x_2^2$ subject to $x_1 \geq 0$, $x_2 \geq 0$, $x_1 + x_2 \geq 5$.



(It is clear that the solution is at $(\frac{5}{2}, \frac{5}{2})$.)

M1 Use 261:

Case 1: $x_1 = 0$, $x_2 \geq 5$

Case 2: $x_2 = 0$, $x_1 \geq 5$

Case 3: $x_1 + x_2 = 5$, $x_1, x_2 \geq 0$

Case 4: $x_1 > 0$, $x_2 > 0$, $x_1 + x_2 > 5$
(interior point.)

M2 KKT (Essentially the same as M1)

$$L(x_1, x_2, \mu_1, \mu_2, \mu_3)$$
$$= x_1^2 + x_2^2 + \mu_1(-x_1) + \mu_2(-x_2) + \mu_3(5 - x_1 - x_2)$$

$\partial_{x_1} L = 0 \Rightarrow$

$$\boxed{\begin{gathered} 2x_1 - \mu_1 - \mu_3 = 0 \\ 2x_2 - \mu_2 - \mu_3 = 0 \\ -\mu_1 x_1 - \mu_2 x_2 + \mu_3(5 - x_1 - x_2) = 0 \\ x_1 \geq 0, \; x_2 \geq 0, \; x_1 + x_2 \geq 5 \\ \mu_1, \mu_2, \mu_3 \geq 0 \end{gathered}}$$

M3 (Quadratic Programming)

Formulate as an LP:

$$\min \quad w_1 + w_2 + w_3$$

$$\text{s.t.} \quad w_1 = 2x_1 - \mu_1 - \mu_3$$

$$w_2 = 2x_2 - \mu_2 - \mu_3$$

$$w_3 = x_1$$

$$w_4 = x_2$$

$$w_5 = x_1 + x_2 - 5$$

$$w_1, w_2, w_3, w_4, w_5, \mu_1, \mu_2, \mu_3 \geq 0$$

$$\mu_1 w_3 + \mu_2 w_4 + \mu_3 w_5 = 0$$

During Simplex, make sure

μ_1, w_3 are not basic variables together;

μ_2, w_4 are not basic variables together;

μ_3, w_5 are not basic variables together;

5. Given the following points in $\mathbb{R}^2$:

$$a_1 \begin{pmatrix} 0 \\ 0 \end{pmatrix}, a_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, a_3 \begin{pmatrix} 1 \\ 2 \end{pmatrix}, a_4 \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Use the method of quadratic programming to find the smallest circle enclosing all of the above points. Use your favorite graphic package to plot the points and the circle you have found.

(Hint: read pages 8-9 of the note Quadratic Programming from Week 12.)

$$\min \quad \lambda + x_1^2 + x_2^2$$

$$\text{s.t.} \quad 2\left\langle \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle + \lambda \geq 0^2 + 0^2$$

$$2\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle + \lambda \geq 0^2 + 1^2$$

$$2\left\langle \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle + \lambda \geq 1^2 + 2^2$$

$$2\left\langle \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle + \lambda \geq 2^2 + (-1)^2$$

quadratic

i.e. $\min \quad x_1^2 + x_2^2 + \lambda$

$$\text{s.t.} \quad \lambda \geq 0$$

$$2x_2 + \lambda \geq 1$$

$$2x_1 + 4x_2 + \lambda \geq 5$$

$$4x_1 - 2x_2 + \lambda \geq 5$$

quadratic prog.

lin

$$\min \quad x_1^2 + x_2^2 + \lambda$$

s.t.

$$-\lambda \leq 0$$
$$-2x_2 - \lambda + 1 \leq 0$$
$$-2x_1 - 4x_2 - \lambda + 5 \leq 0$$
$$-4x_1 + 2x_2 - \lambda + 5 \leq 0$$

$$L(x_1, x_2, \lambda, \mu_1, \mu_2)$$
$$= x_1^2 + x_2^2 + \lambda + \mu_1(-\lambda) + \mu_2(-2x_2 - \lambda + 1)$$
$$+ \mu_3(-2x_1 - 4x_2 - \lambda + 5)$$
$$+ \mu_4(-4x_1 + 2x_2 - \lambda + 5)$$

$$\partial_{x_1} L = 0 \quad 2x_1 - 2\mu_3 - 4\mu_4 = 0$$
$$\partial_{x_2} L = 0 \quad 2x_2 - 2\mu_2 - 4\mu_3 + 2\mu_4 = 0$$
$$\partial_{\lambda} L = 0 \quad 1 - \mu_1 - \mu_2 - \mu_3 - \mu_4 = 0$$

$$\mu_1(-\lambda) + \mu_2(-2x_2 - \lambda + 1) + \mu_3(-2x_1 - 4x_2 - \lambda + 5) + \mu_4(-4x_1 + 2x_2 - \lambda + 5) = 0$$

$$\mu_1, \mu_2, \mu_3, \mu_4 \geq 0$$

LP formulation:

$$\min \quad w_1 + w_2 + w_3$$

$$\text{s.t.} \quad w_1 = 2x_1 - 2\mu_3 - 4\mu_4$$

$$w_2 = 2x_2 - 2\mu_2 - 4\mu_3 + 2\mu_4$$

$$w_3 = 1 - \mu_1 - \mu_2 - \mu_3 - \mu_4$$

$$w_4 = \lambda$$

$$w_5 = 2x_2 + \lambda - 1$$

$$w_6 = 2x_1 + 4x_2 + \lambda - 5$$

$$w_7 = 4x_1 - 2x_2 + \lambda - 5$$

$$w_1, \dots, w_7, \mu_1, \mu_2, \mu_3, \mu_4 \geq 0$$

$$\mu_1 w_4 + \mu_2 w_5 + \mu_3 w_6 + \mu_4 w_7 = 0$$