MA 421 Lec 1 Intro to Opt

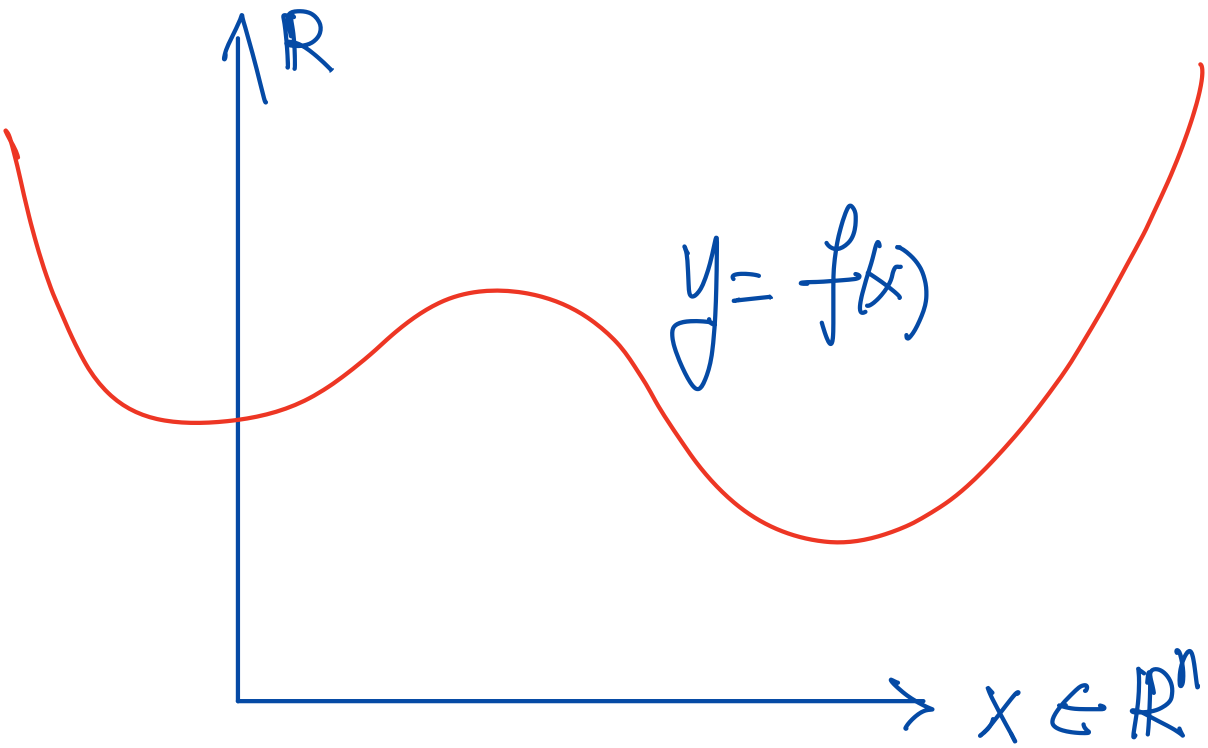
- max/min
- lin/nonlin functions
- with/without constraints
- equalities/inequalities

max $f(x)$ $\iff$ min $(-f(x))$

1 max (or min) $f(x)$

$x \in \mathbb{R}^n$, $x = (x_1, x_2, \dots, x_n) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

no-constraints



1^{st} derivative test $\Rightarrow$ $\nabla f(x) = 0$

critical pt(s)

2^{nd} derivative test $\Rightarrow$ $D^2 f(x) > 0$

(local min)

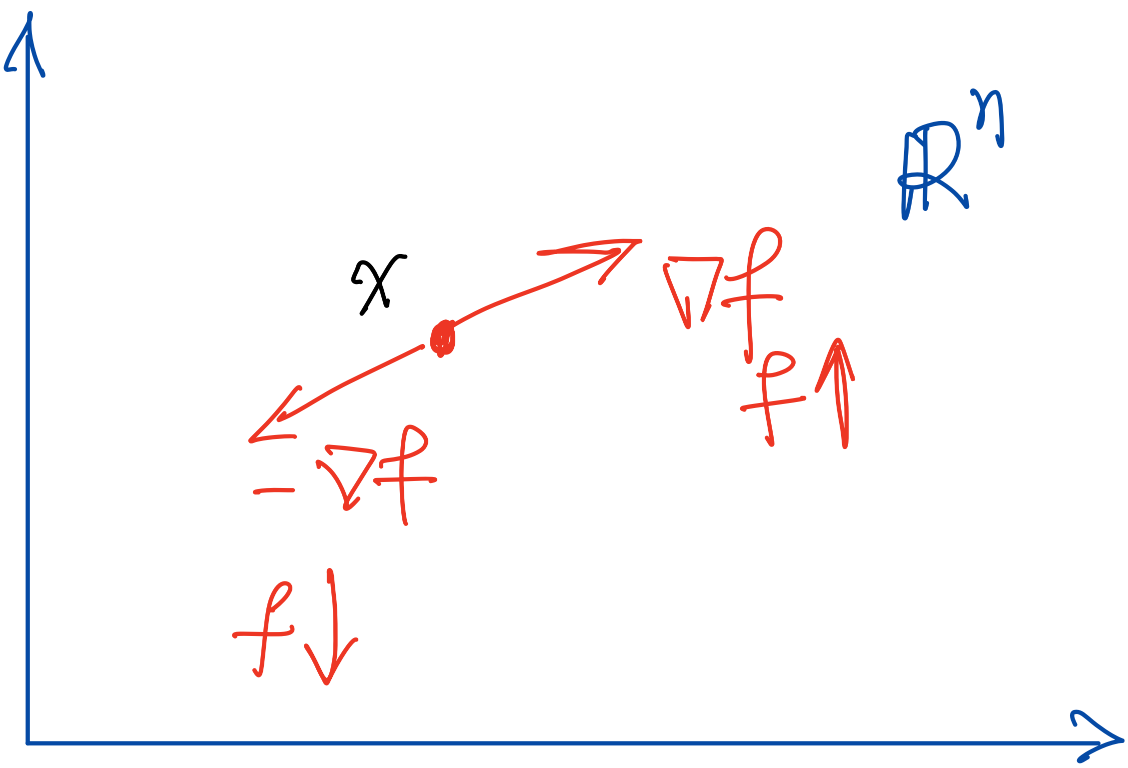
$[D^2 f] > 0$

$D^2 f(x) < 0$

(local max)

$[D^2 f] < 0$

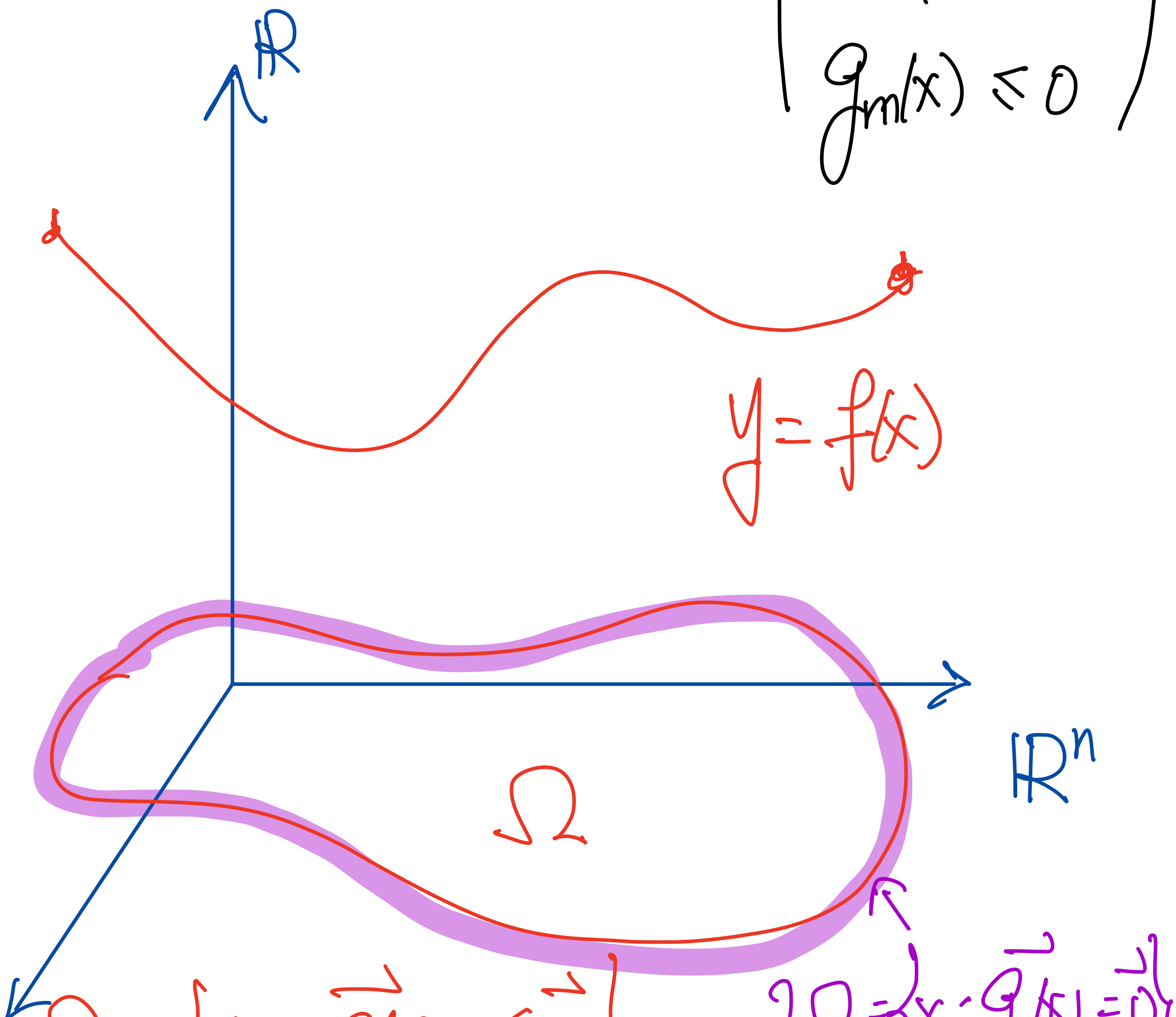
(in 1D, $x \in \mathbb{R}$, $f'(x) = 0$, $f''(x) \begin{matrix} >0 \\ <0 \end{matrix}$)



② With inequality constraints

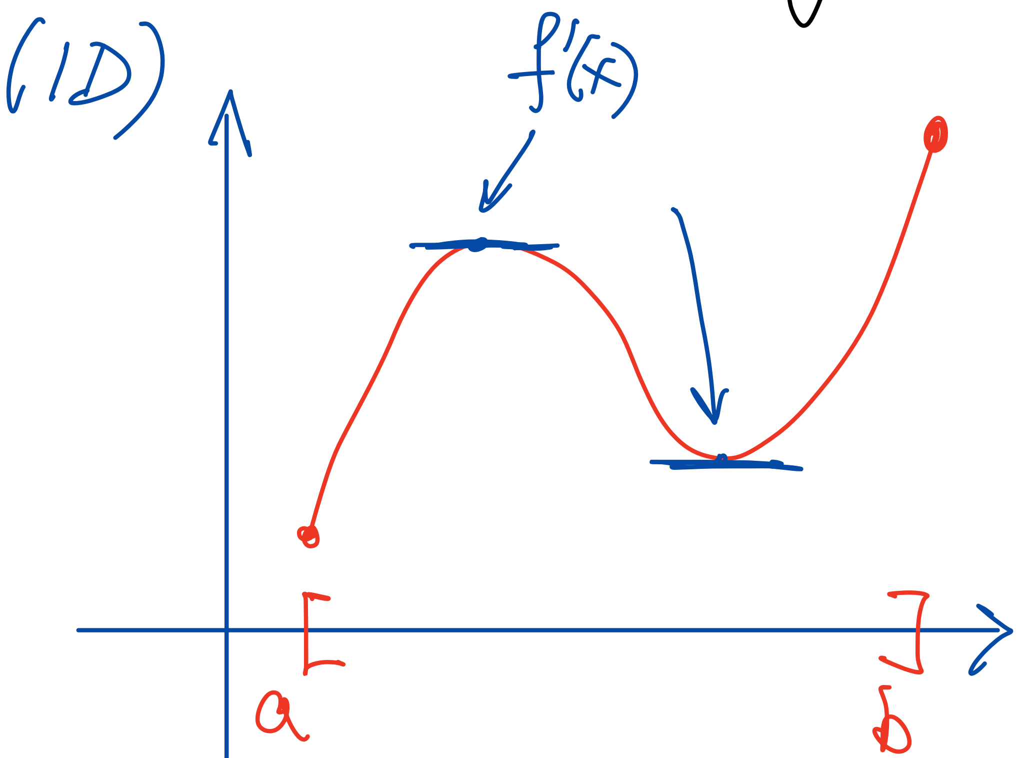
$$\min \ f(x)$$

$$\text{s.t.} \quad \vec{g}(x) \le 0 \nearrow \begin{pmatrix} g_1(x) \le 0 \\ g_2(x) \le 0 \\ \vdots \\ g_m(x) \le 0 \end{pmatrix}$$



$\Omega = \{x : \vec{g}(x) \le \vec{0}\}$ $\quad$ $\partial\Omega = \{x : \vec{g}(x) = \vec{0}\}$

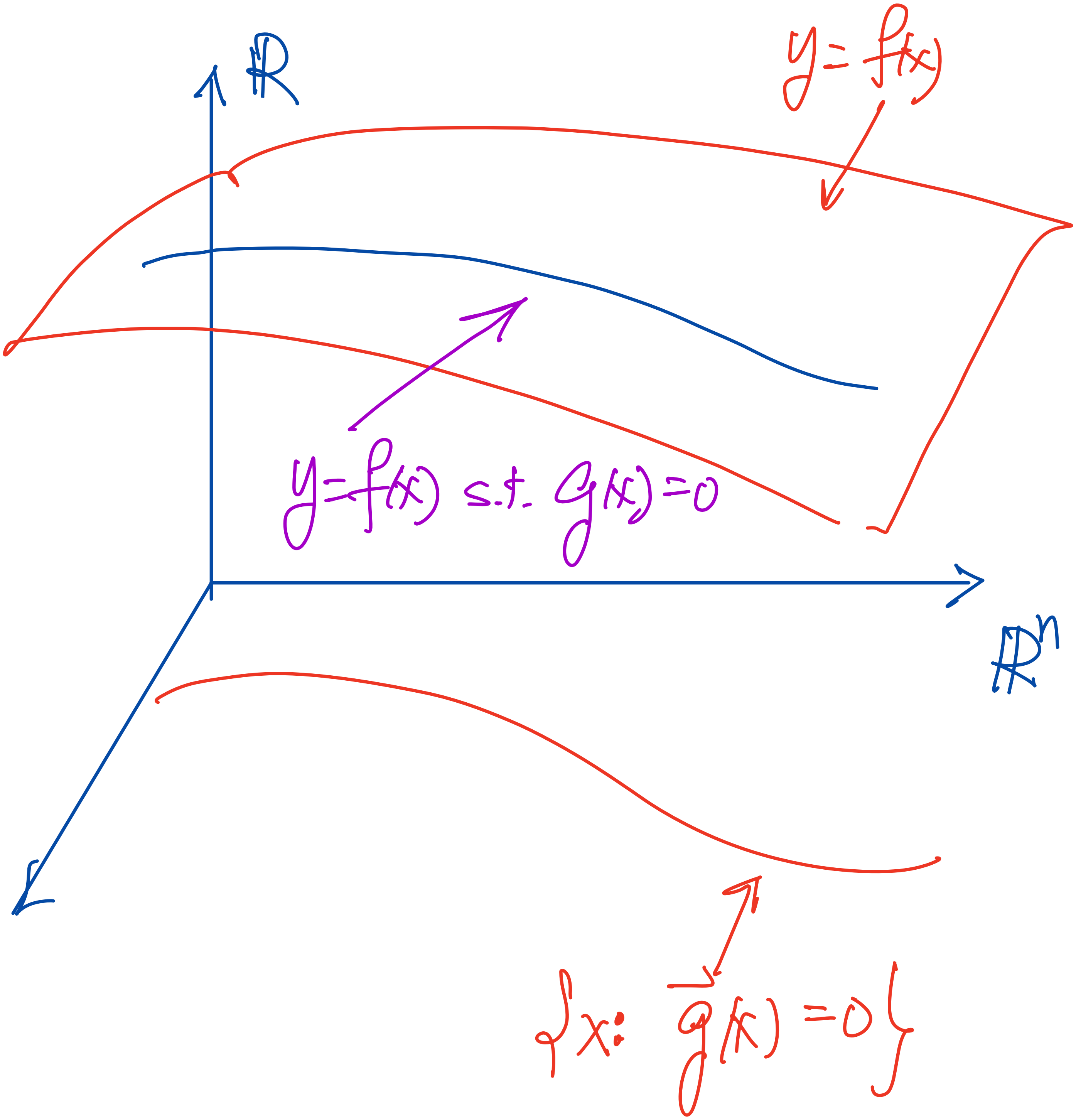
Interior vs Boundary pts



$f(a) = ?$ $\quad f(b) = ?$

③ Equality constraints

$$\min \; f(x)$$
$$\text{s.t.} \quad \vec{g}(x) = \vec{0} \rightarrow \begin{pmatrix} g_1(x) = 0 \\ g_2(x) = 0 \\ \vdots \\ g_m(x) = 0 \end{pmatrix}$$



Lagrange Multiplier

$$\min_{x,\lambda} \; f(x) + \lambda g(x)$$

only 1 constraint

$g(x) = 0$

$$\nabla_x \Rightarrow \nabla f(x) + \lambda \nabla g(x) = 0$$

$$\partial_\lambda \Rightarrow g(x) = 0$$

$$\nabla f = -\lambda \nabla g$$

i.e. $\nabla f \parallel \nabla g$

$$\min f(x) + \lambda_1 g_1(x) + \cdots + \lambda_m g(x)$$

$x, \lambda_1, \cdots, \lambda_m$ — no constraints

$$\nabla_x \Rightarrow \nabla f(x) + \lambda_1 \nabla g(x) + \cdots + \lambda_m \nabla g(x) = 0$$

$$\partial_{\lambda_i} \Rightarrow g_i(x) = 0$$

Linear programming

Objective functions $f(x)$ and constraints $g(x)$ are all linear

Dietary Problem

	Vitamin A	Vitamin C	Price
Carrot (g)	2 (mg)	1 (mg)	2 ¢
Cabbage (g)	1 (mg)	3 (mg)	3 ¢
Min. Daily consumption	6 (mg)	8 (mg)	

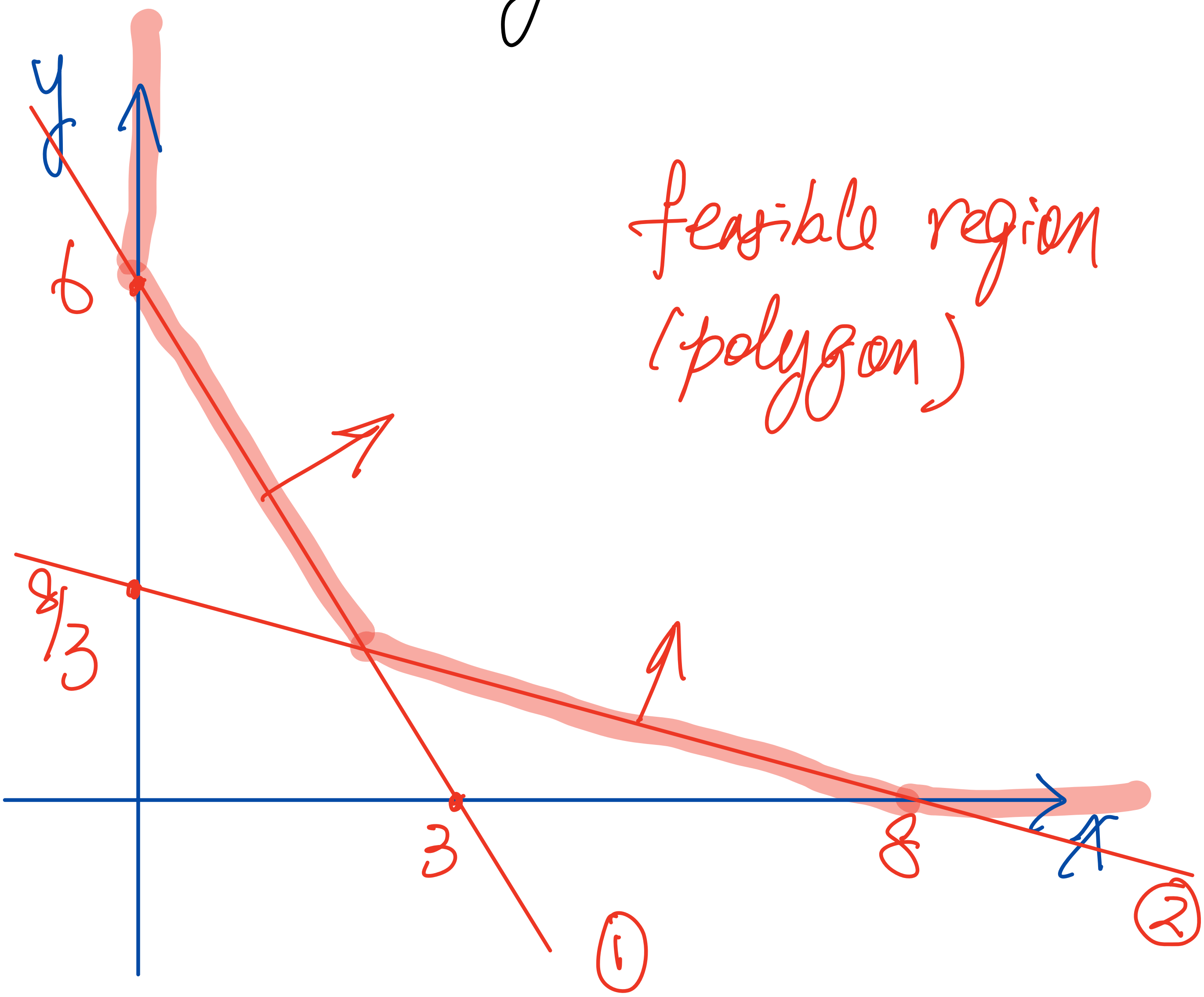
I 1 g of Carrot contains 2mg A & 1mg C

How much carrot & cabbage to consume so as to min cost?

Let x = carrot (g), y = cabbage (g)

obj: $\underline{\min}$ $z = 2x + 3y$ ← price/cost

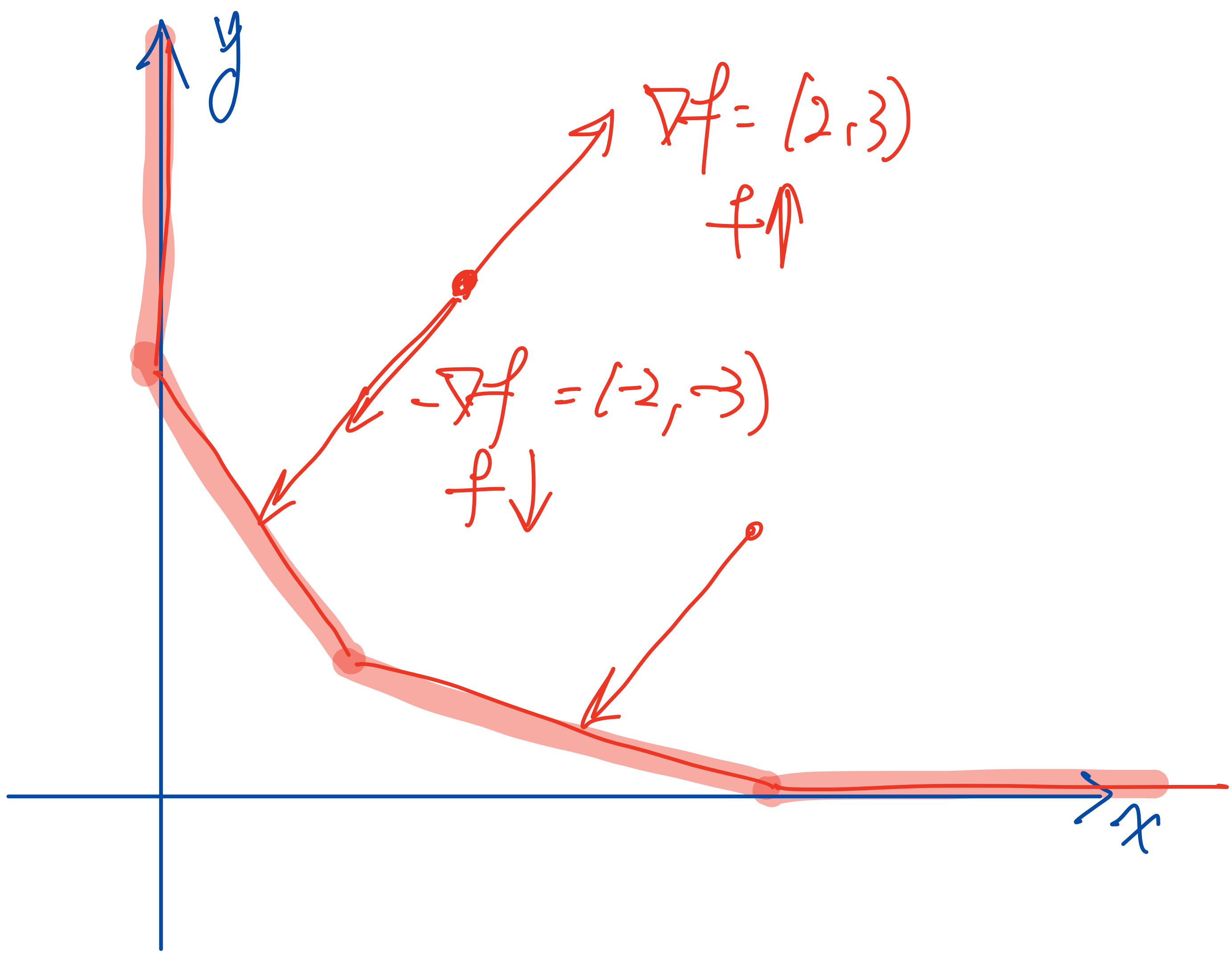
Constraints
$$\begin{cases} 2x + y \geq 6 & (1) \\ x + 3y \geq 8 & (2) \\ x \geq 0, y \geq 0 \end{cases}$$



Objective function

$$Z = f(x,y) = 2x + 3y$$

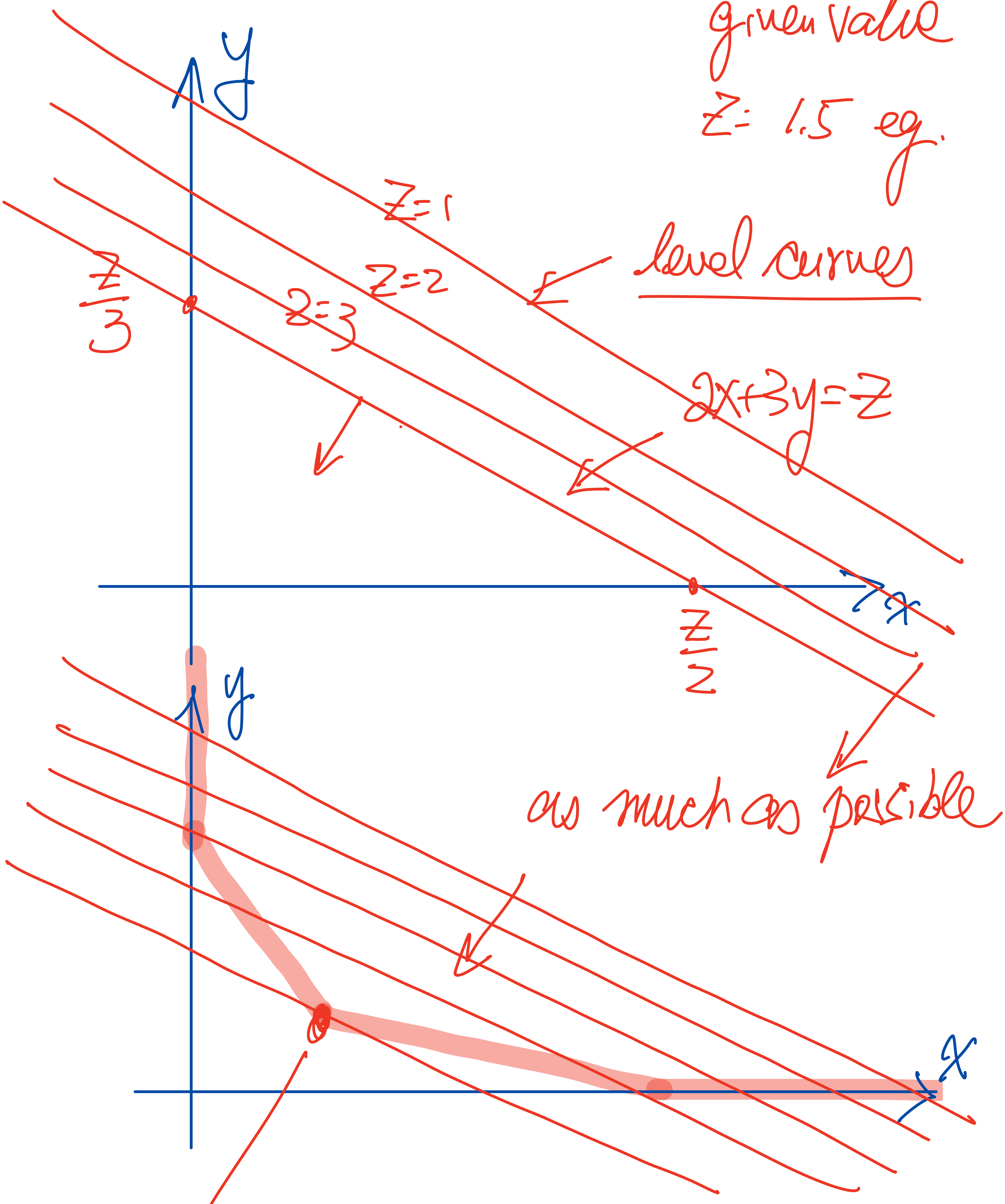
(i) $\nabla f = (\partial_x f, \partial_y f) = (2,3)$



(ii) $f(x,y) = 2x+3y = z$

given value

$z = 1.5$ eg.



intersection between ① & ②:

$$2x+y=6$$
$$x+3y=8$$
$$5x=10 \Longrightarrow x=2$$
$$\Longrightarrow y=2$$

But at $(2,2)$, $\nabla f(2,2)=(2,3)\neq 0$

Hence $(2,2)$ is not a critical pt.

Note: $(2,2)$ is at the boundary of the feasible set, not an interior pt.

(2,2) is at corner pt.

$$\boxed{f\underbrace{(2,2)}_{\text{min pt}} = 2(2)+3(2) = \underbrace{10}_{\text{min value}} \not{c}}$$

min pt

min value.

II Dual Problem

Pharmaceutical company

- develop Vitamin A, C pills (to replace carrot & cabbage)
- determine the prices of A, C pills.

Let p = price of A
q = price of C

objective function: max $Z = 6p + 8q$

constraint:
$$\begin{cases} 2p + q \leq 2 & \leftarrow \text{carrot price} \ (1) \\ p + 3q \leq 3 & \leftarrow \text{cabbage price} \ (2) \\ p, q \geq 0 \end{cases}$$

The price of carrot from the company's pt. of view

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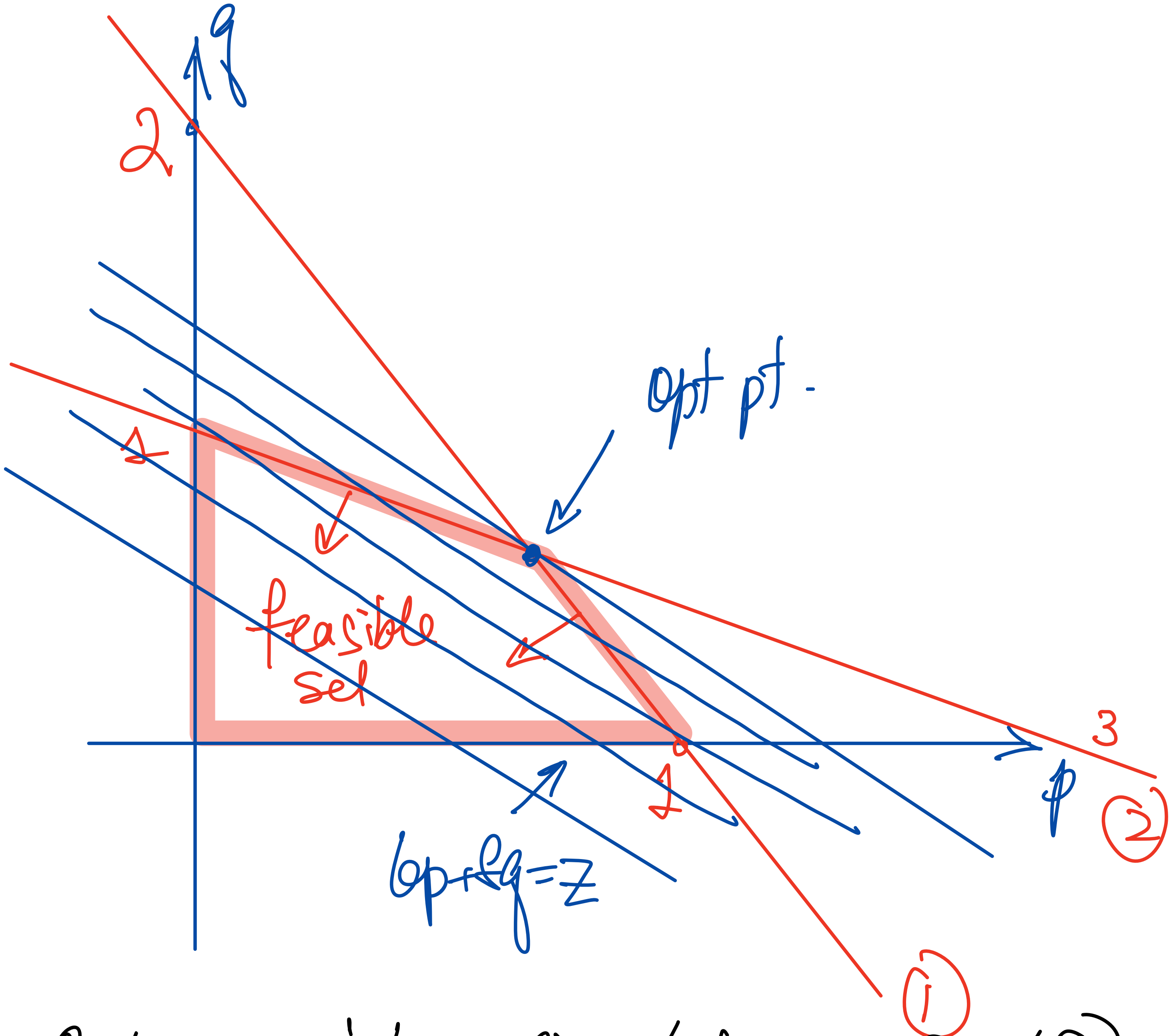


Opt pt: intersection between (1) & (2)

$$2p+q=2$$
$$p+3q=3$$

$$5p=3 \Rightarrow p=\frac{3}{5}$$
$$q=\frac{4}{5}$$

Opt value $\quad Z = 6p + 8q$

$$= 6\left(\frac{3}{5}\right) + 8\left(\frac{4}{5}\right)$$

$$= \underline{10}$$

the Same as the $\underbrace{\text{origin problem}}_{\text{primal}}$

$$\boxed{\text{opt primal value} = \text{opt. dual value}}$$

III Change of Parameters

	Vitamin A	Vitamin C	Price
Carrot (g)	2 (mg)	1 (mg)	2 ¢
Cabbage (g)	1 (mg)	3 (mg)	3 ¢
Min. Daily consumption	6 (mg)	8 (mg)	

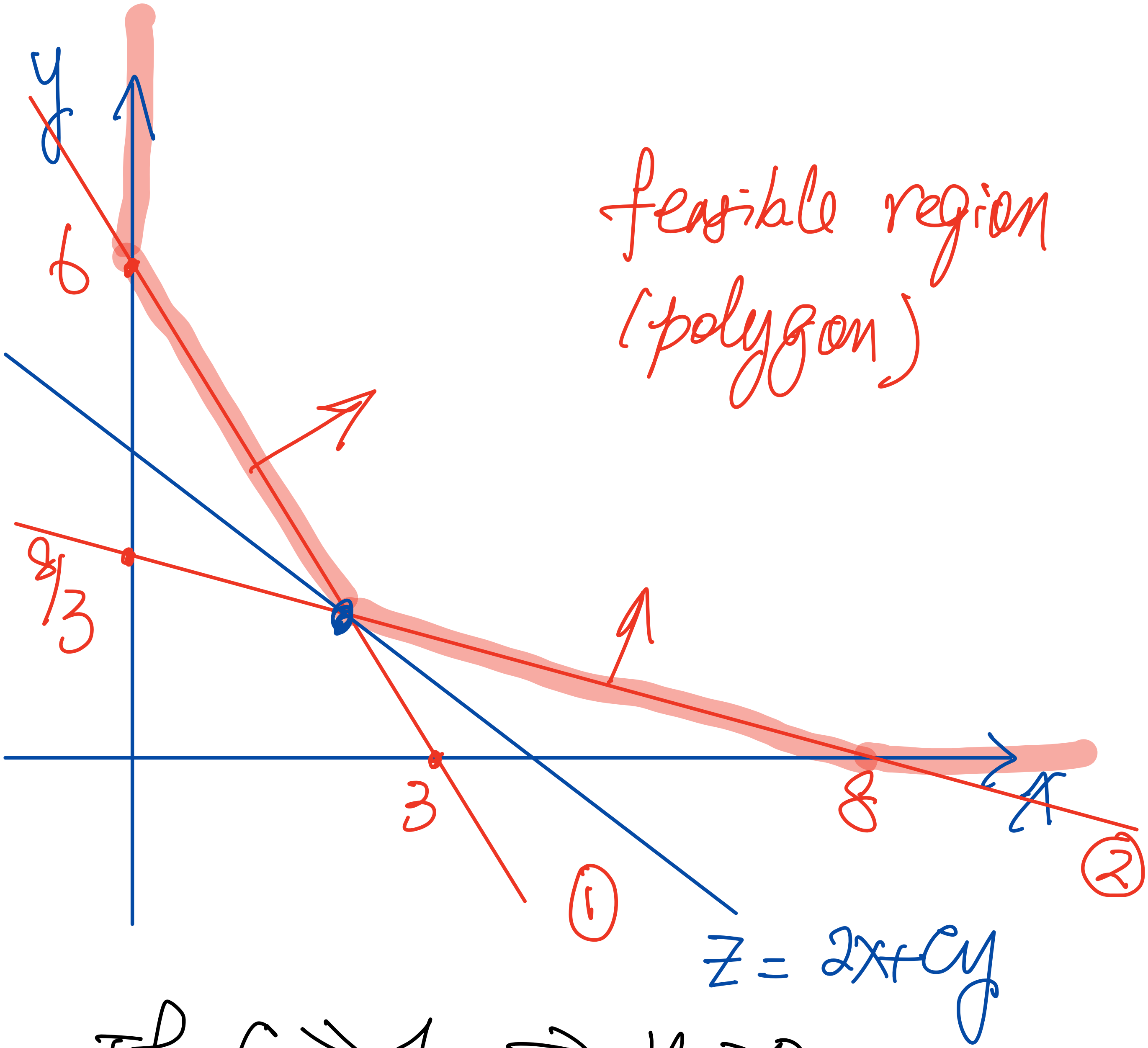
What if the price of cabbage has changed?

$$\min Z = 2x + cy$$

$$2x + y \geq 6$$

$$x + 3y \geq 8$$

For what range of c is $(2,2)$ still the opt. pt?



If $c \gg 1, \Rightarrow y = 0$

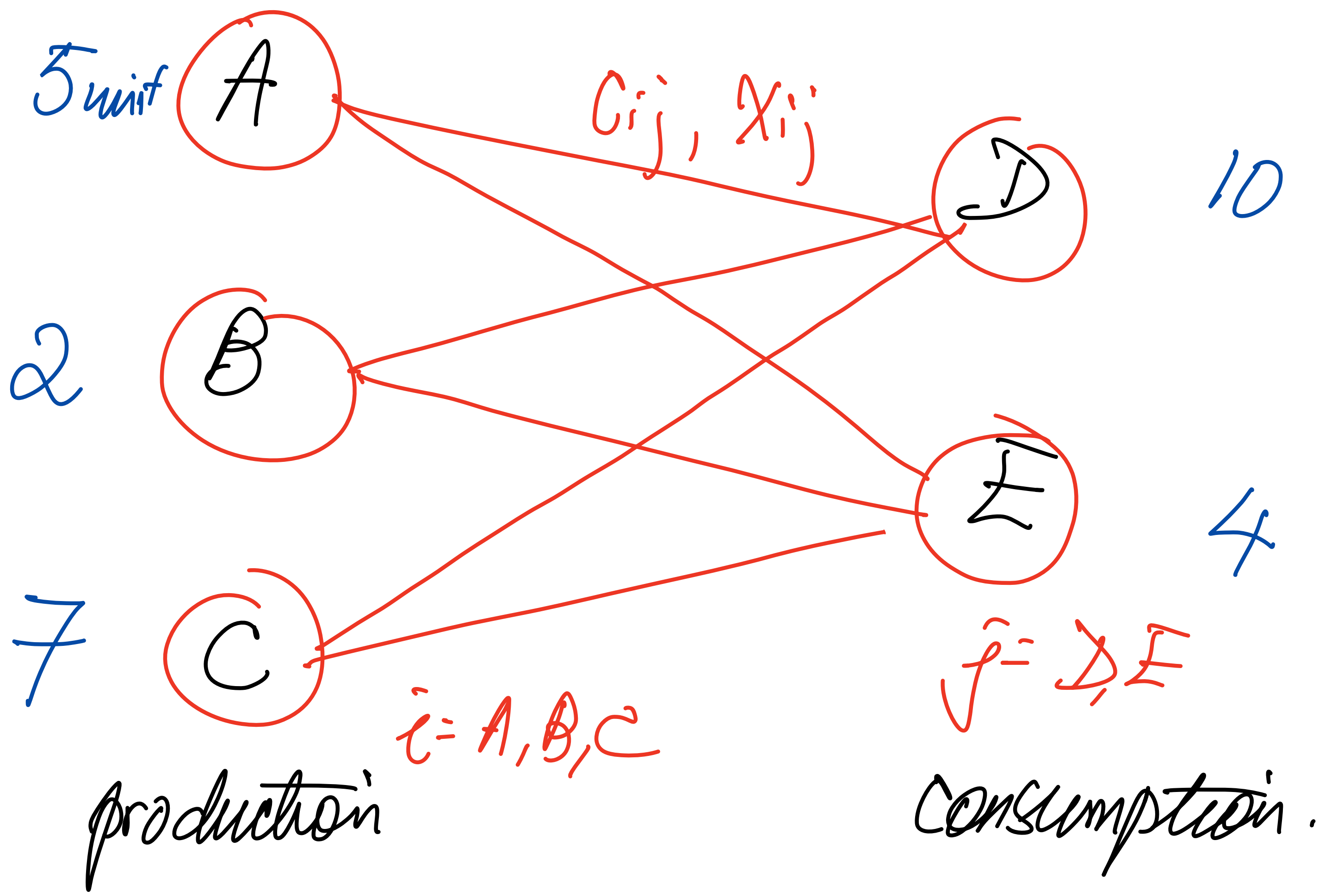
$\frac{2}{c} = \frac{1}{3} \Rightarrow \underline{\underline{c = 6}}$

If $c \ll 1 \Rightarrow x = 0$

$\frac{2}{c} = 2 \Rightarrow c = 1$

$$\boxed{1 < c < 6}$$

Transportation Problem



($5+2+7 = 10+4$, production balances consumption.)

x_{ij} = amount of transportation from i to j

c_{ij} = unit cost of transportation from i to j

Total cost $Z = \sum_{i,j} c_{ij} x_{ij}$

Constraints:

$$\begin{pmatrix} \forall i, & \sum_j x_{ij} = p_i \\ \forall j & \sum_i x_{ij} = q_j \end{pmatrix}$$

$$x_{AD} + x_{AE} = 5$$

$$x_{BD} + x_{BE} = 2$$

$$x_{CD} + x_{CE} = 7$$

$$x_{AD} + x_{BD} + x_{CD} = 10$$

$$x_{AE} + x_{BE} + x_{CE} = 4$$

$$x_{ij} \geq 0$$