MA 421 Lec 2 Intro to Simplex

Linear algebra: solving $\overset{m\times n}{A}\overset{n\times 1}{X}=\overset{m\times 1}{b}$

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$$X=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad b=\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\begin{cases} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=b_1 \\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n=b_2 \\ \vdots \\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n=b_m \end{cases}$$

Linear Programming

max / min linear function

s.t. linear constraints

Objective function

$$Z = f(x_1, \dots, x_n) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

$$\left(= \langle c, x \rangle = c^T x \right)$$

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \begin{cases} \geq b_1 \\ = b_1 \\ \leq b_1 \end{cases}$$

$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \leq b_2$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \leq b_m$$

$$\boxed{\begin{aligned} \max \quad & Z = C^T X \\ \text{s.t.} \quad & A\vec{X} \le \vec{b} \\ & X \ge 0 \end{aligned}}$$

$c_1x_1 + \cdots + c_nx_n$

usual assumption

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \le \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \le b_i$$

$$i = 1, 2, \cdots, m$$

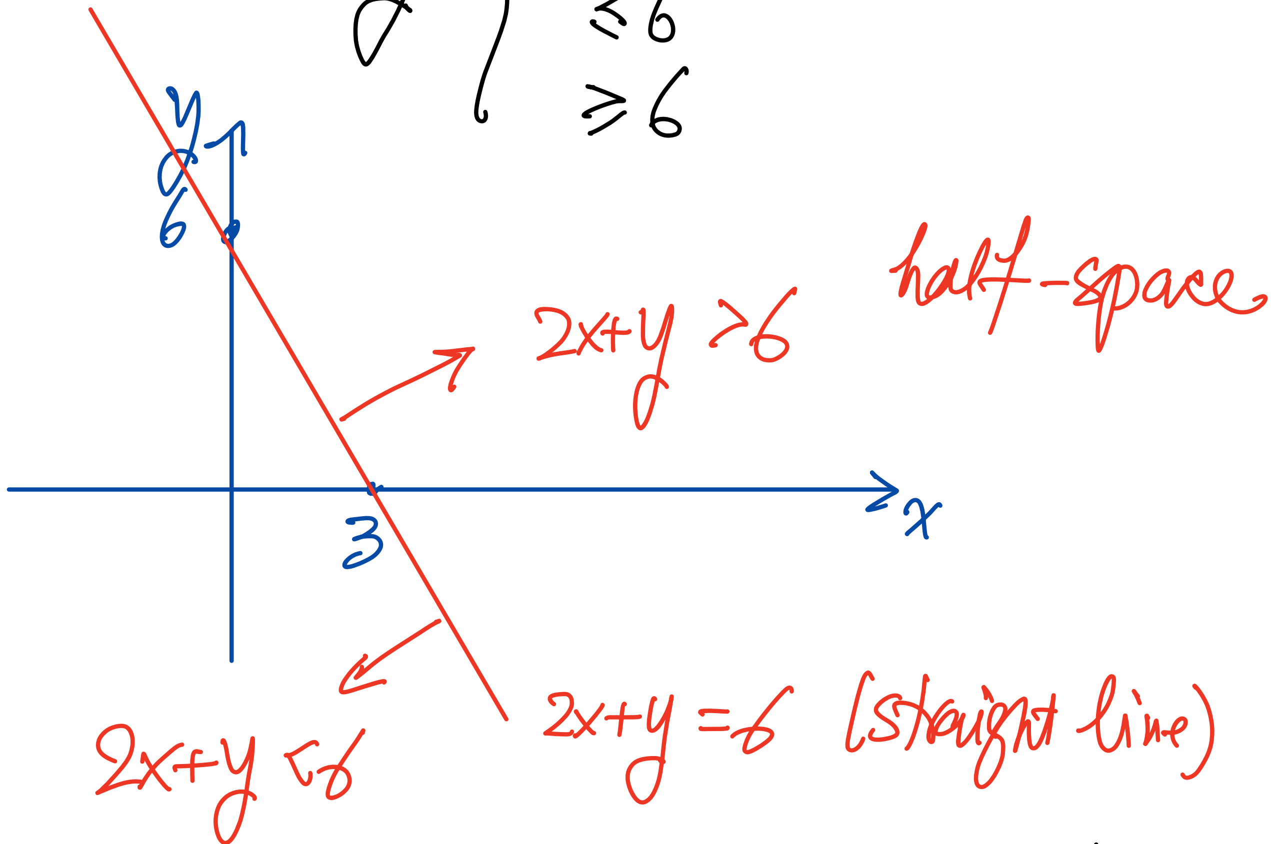
$$2x + 3y \ge 6 \iff -2x - 3y \le -6$$

$$2x + 3y = 6 \iff \begin{cases} 2x + 3y \le 6 \\ 2x + 3y \ge 6 \end{cases}$$

$$\iff \begin{cases} 2x + 3y \le 6 \\ -2x - 3y \le -6 \end{cases}$$

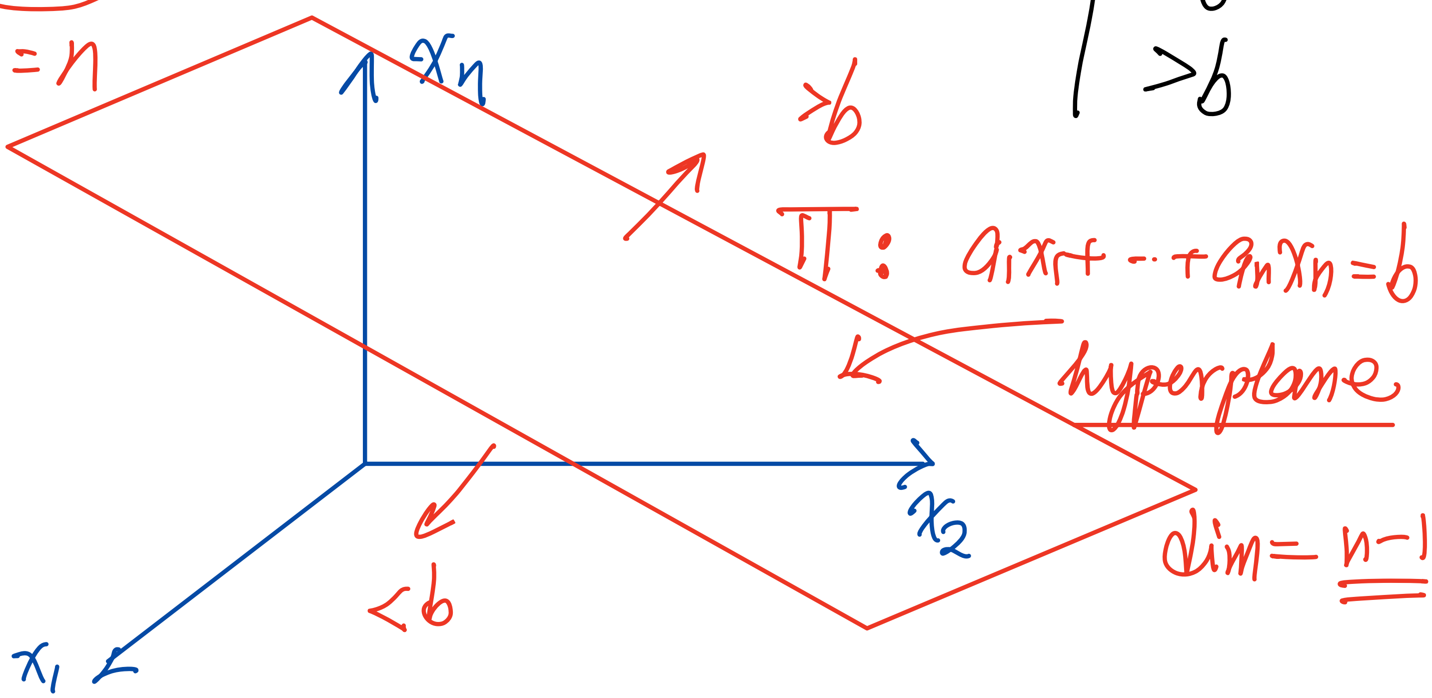
Geometry of (In)equality

$\mathbb{R}^2$: $\quad 2x+y \begin{cases} =6 \\ \leq 6 \\ \geq 6 \end{cases}$



$\mathbb{R}^n$ $\quad a_1x_1+a_2x_2+\cdots+a_nx_n \begin{cases} =b \\ <b \\ >b \end{cases}$

dim $= n$

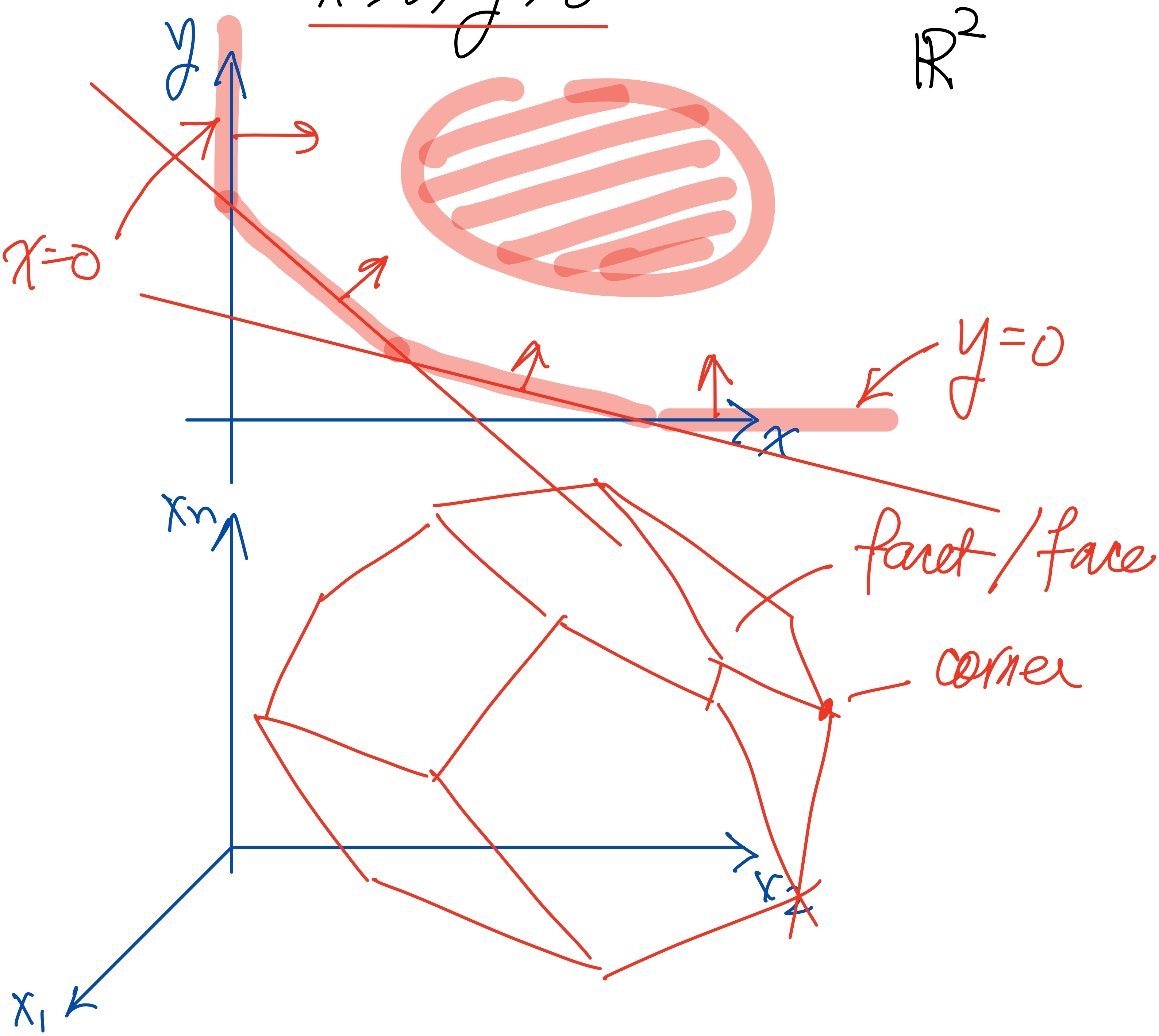


Polytope/Polyhedron

= intersection between half-spaces

eg

$$2x + y \geq 6$$
$$x + 3y \geq 8$$
$$x \geq 0, \ y \geq 0$$



Corner of a polytope ($\mathbb{R}^n$)

= intersection between n hyperplanes

($\mathbb{R}^2$, a corner = intersection between 2 straight lines.)

General (Standard) Formulation of LP:

$$\max \quad Z = C^T X = c_1 x_1 + \cdots + c_n x_n$$

$$\text{s.t.} \begin{cases} AX \le b & \text{— } m \text{ ineq's} \\ X \ge 0 & \text{— } n \text{ ineq's} \end{cases}$$

$A^{m\times n}, \; X^{n\times 1}, \; b^{m\times 1}$

total $m+n$ ineq.

Feasible region $= \{X \in \mathbb{R}^n : AX \le b, X \ge 0\}$

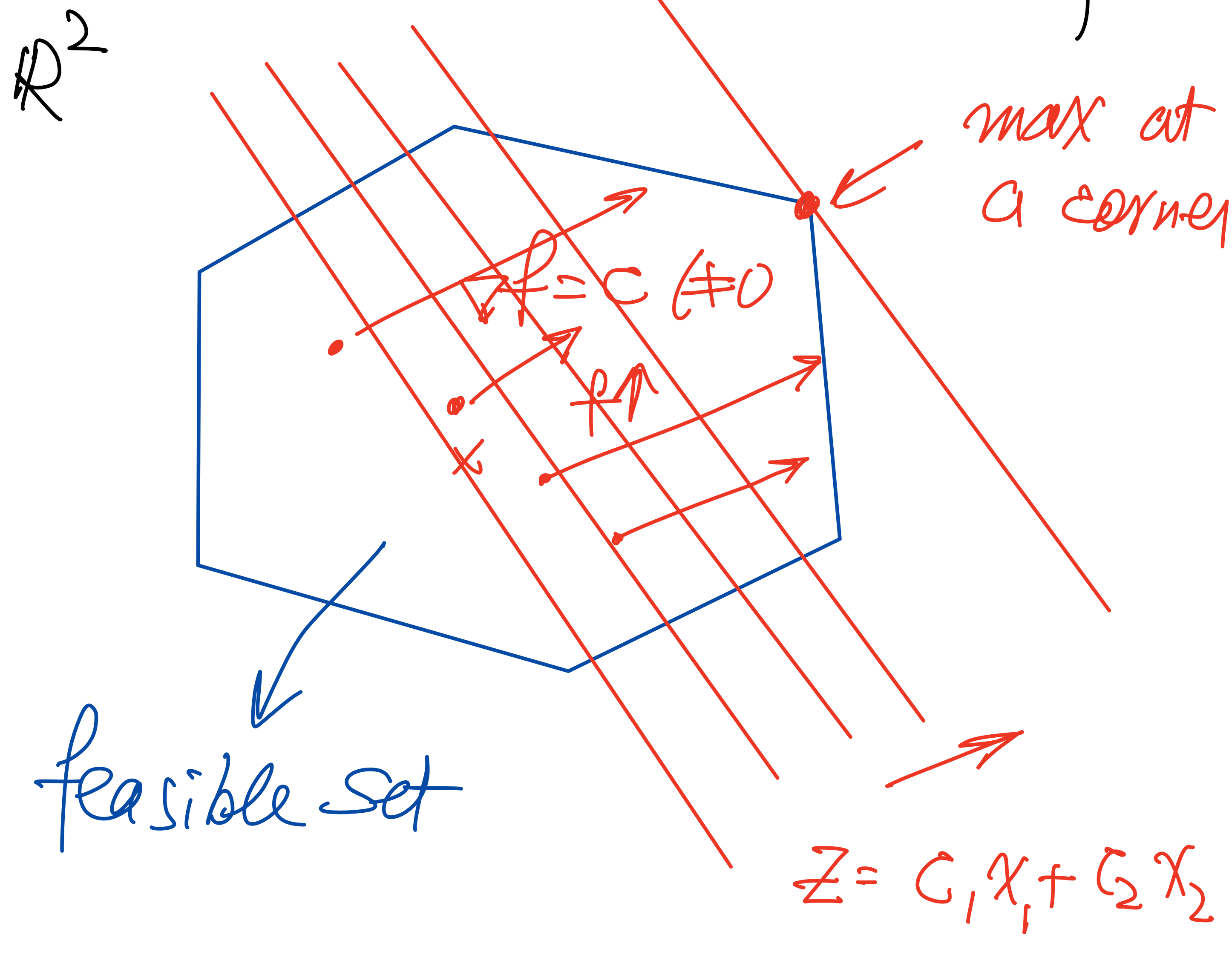
a polytope

$$z = f(X) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

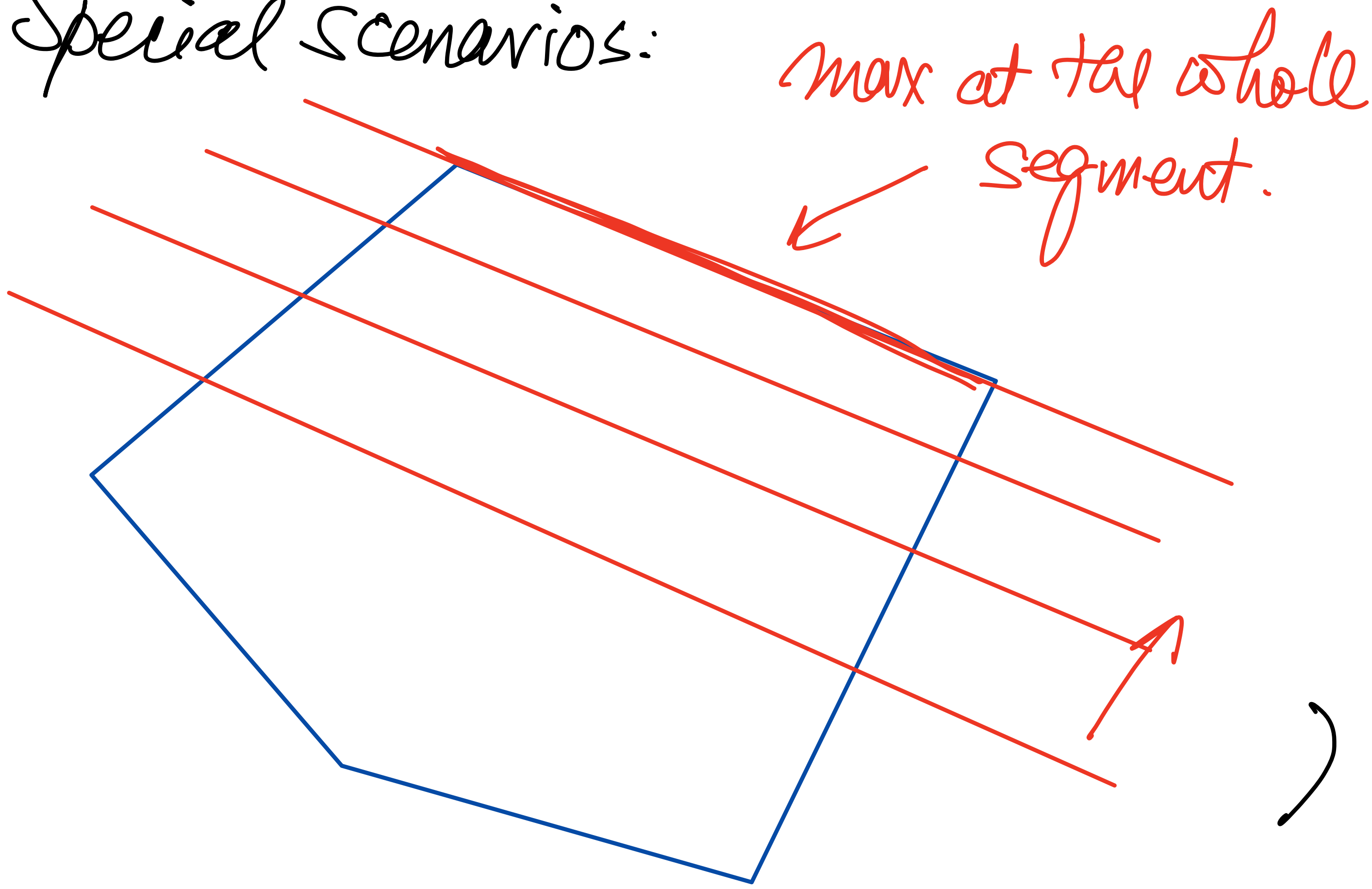
Searching for critical pt:

$$\nabla f = (c_1, c_2, \ldots, c_n) = c \ (\neq 0)$$

(never zero)



(Special scenarios:



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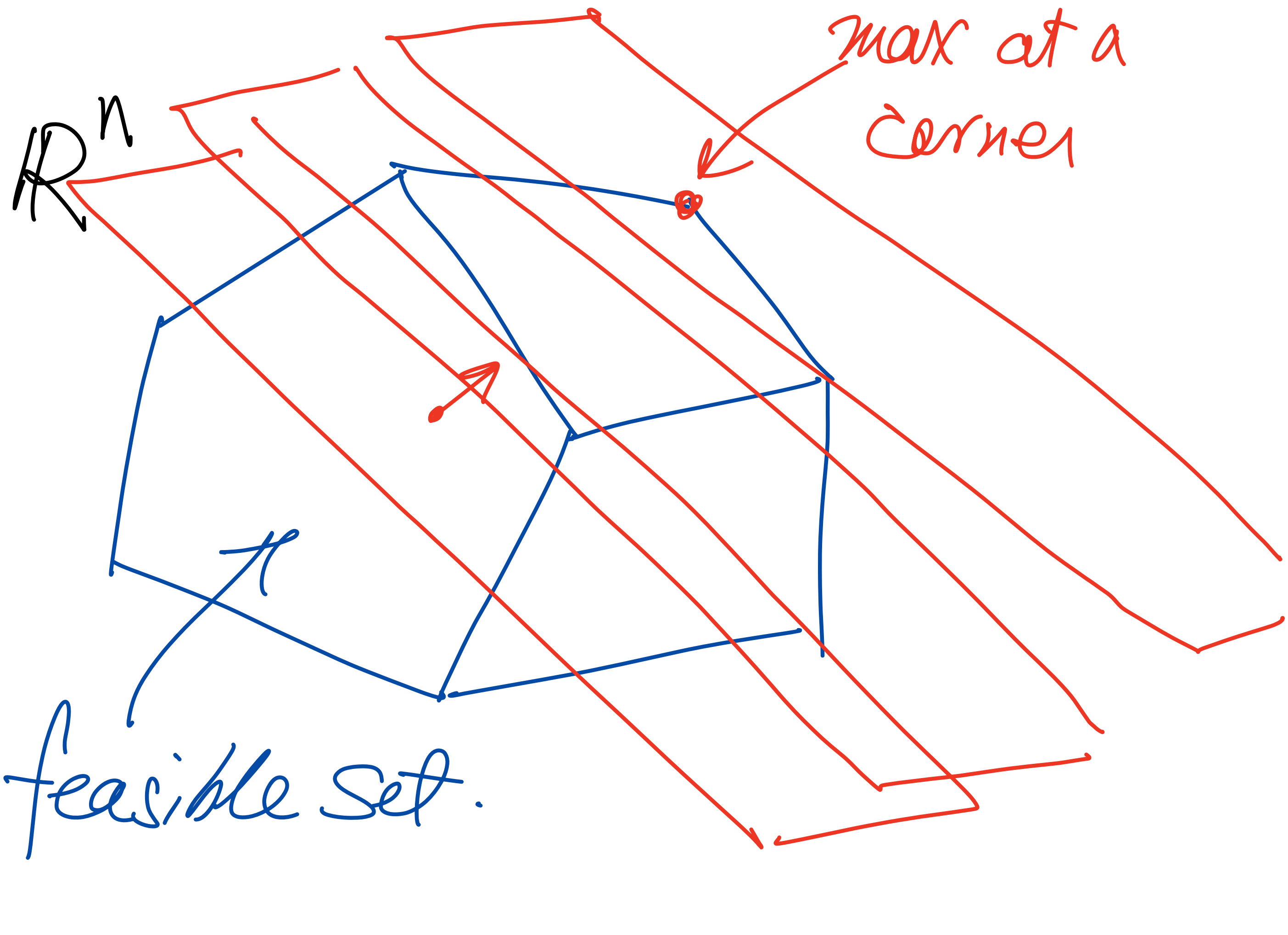


In general,

$$\max \quad Z = c^T X$$

$$\text{s.t.} \quad \begin{cases} AX \leq b & \text{— } m \text{ ineq} \\ X \geq 0 & \text{— } n \text{ ineq} \end{cases}$$

a polytope in $\mathbb{R}^n$

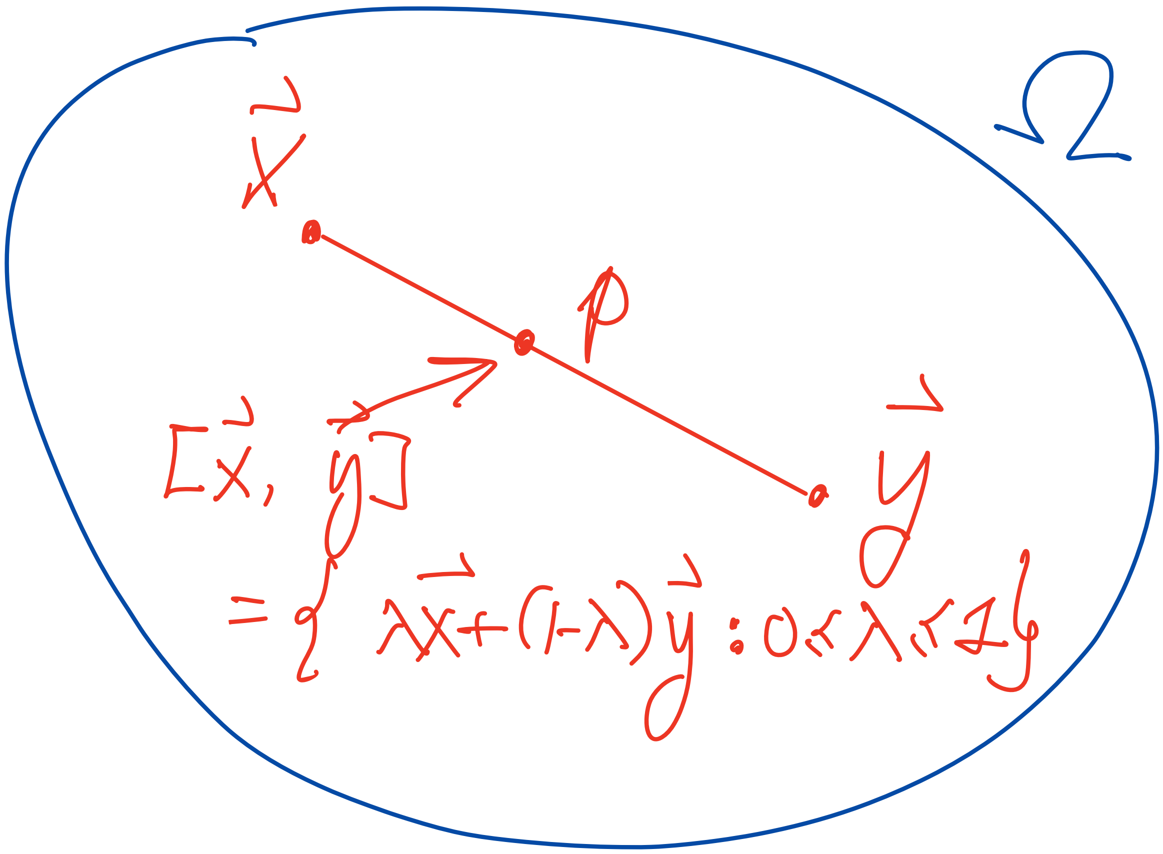
How many corners?

$\binom{m+n}{n}$ huge (how large?)

Convexity of feasible set

Def: A set $\Omega \subseteq \mathbb{R}^n$ is convex if $\forall \vec{x}, \vec{y} \in \Omega$, then

$$\lambda\vec{x} + (1-\lambda)\vec{y} \in \Omega \text{ for } 0 \le \lambda \le 1$$



$\vec{XP} \parallel \vec{PY}$ $\qquad \vec{XP} = \alpha \vec{PY} \quad \alpha > 0$

$$\vec{p} - \vec{x} = \alpha(\vec{y} - \vec{p})$$

$$(1+\alpha)\vec{p} = \vec{x} + \alpha\vec{y}$$

$$\vec{P} = \frac{1}{1+\alpha}\vec{X} + \frac{\alpha}{1+\alpha}\vec{Y}$$

λ $\quad$ $1-\lambda$

$$\lambda\vec{X} + (1-\lambda)\vec{Y} = a\vec{X} + b\vec{Y}$$

$0 \leq a, b \leq 1$, $a+b=1$.

Lemma: Feasible set Ω for LP is convex

Pf:
$$\begin{cases} A\vec{X} \leq \vec{b} \\ \vec{X} \geq 0 \end{cases} \quad (*)$$

Suppose $\vec{X}, \vec{Y}$ satisfy $(*)$,
so does $\alpha X + \beta Y$, $0 \leq \alpha, \beta \leq 1$,
$\alpha + \beta = 1$

$$\left.\begin{aligned}&\alpha(AX \le b)\\ &\beta(AY \le b)\end{aligned}\right\} \Rightarrow \begin{aligned} A(\alpha X) &\le \alpha b\\ A(\beta Y) &\le \beta b\end{aligned}$$

$$A(\alpha X) + A(\beta Y) \le \alpha b + \beta b$$

$$A(\alpha X + \beta Y) \le \underbrace{\cancel{(\alpha + \beta)}}_{=b}\, b$$

i.e. $A(\alpha X + \beta Y) \le b$

$$X \ge 0, \quad Y \ge 0$$

$$\Rightarrow \alpha X + \beta Y \ge 0$$