[C] Chvatal

$x_2, \ldots, x_{n+m}$ and the objective function z in terms of the remaining n variables. (2.15)

The properties (2.14) and (2.15) are the defining properties of dictionaries.

In addition to these two properties, dictionaries (2.3), (2.8), and (2.10) have the following property:

> Setting the right-hand side variables at zero and evaluating the left-hand side variables, we arrive at a *feasible* solution.

Dictionaries with this additional property will be called *feasible dictionaries*. Hence, every feasible dictionary describes a feasible solution. However, not every feasible solution is described by a feasible dictionary; for instance, no dictionary describes the feasible solution $x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 2, x_5 = 5, x_6 = 3$ of (2.1). Feasible solutions that can be described by dictionaries are called *basic*. The characteristic feature of the simplex method is the fact that it works exclusively with basic feasible solutions and ignores all other feasible solutions.

SECOND EXAMPLE

We shall complete our preview of the simplex method by applying it to another LP problem:

$$\boxed{\begin{array}{llr}
\text{maximize} & 5x_1 + 5x_2 + 3x_3 & \text{objective fct} \\
\text{subject to} & x_1 + 3x_2 + x_3 \le 3 & x_4 \ \text{slack} \\
 & -x_1 + 3x_3 \le 2 & x_5 \\
 & 2x_1 - x_2 + 2x_3 \le 4 & x_6 \\
 & 2x_1 + 3x_2 - x_3 \le 2 & x_7 \\
 & x_1, x_2, x_3 \ge 0. &
\end{array}}$$

In this case, the initial feasible dictionary reads

$$\begin{array}{rcl l}
x_4 &=& 3 - x_1 - 3x_2 - x_3 & x_1 \le 3 \\
x_5 &=& 2 + x_1 - 3x_3 & \\
x_6 &=& 4 - 2x_1 + x_2 - 2x_3 & x_1 \le 2 \\
x_7 &=& 2 - 2x_1 - 3x_2 + x_3 & x_1 \le 1 \qquad x_1 \leftrightarrow x_7 \\
\hline
z &=& \underline{5x_1} + 5x_2 + 3x_3. &
\end{array} \tag{2.16}$$

(Even though the order of the equations in a dictionary is quite irrelevant, we shall make a habit of writing the formula for z last and separating it from the rest of the table by a solid line. Of course, that does *not* mean that the last equation is the sum of the previous ones.) This feasible dictionary describes the feasible solution

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 3, \quad x_5 = 2, \quad x_6 = 4, \quad x_7 = 2.$$

However, there is no need to write this solution down, as we just did: the solution is implicit in the dictionary.

In the first iteration, we shall attempt to increase the value of z by making one of the right-hand side variables positive. At this moment, any of the three variables x_1, x_2, x_3 would do. In small examples, it is common practice to choose the variable that, in the formula for z, has the largest coefficient: the increase in that variable will make z increase at the fastest rate (but not necessarily to the highest level). In our case, this rule leaves us a choice between x_1 and x_2; choosing arbitrarily, we decide to make x_1 positive. As the value of x_1 increases, so does the value of x_5. However, the values of x_4, x_6, and x_7 decrease, and none of them is allowed to become negative. Of the three constraints $x_4 \geq 0$, $x_6 \geq 0$, $x_7 \geq 0$ that impose upper bounds on the increment of x_1, the last constraint $x_7 \geq 0$ is the most stringent: it implies $x_1 \leq 1$. In the improved feasible solution, we shall have $x_1 = 1$ and $x_7 = 0$. Without writing the new solution down, we shall now construct the new dictionary. All we need to know is that x_1 just made its way from the right-hand side to the left, whereas x_7 went in the opposite direction. From the fourth equation in (2.16), we have

$$x_1 = 1 - \frac{3}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_7. \tag{2.17}$$

x_4 x_5 x_6

Substituting from (2.17) into the remaining equations of (2.16), we arrive at the desired dictionary

$$\begin{aligned}
x_1 &= 1 - \frac{3}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_7 \\
x_4 &= 2 - \frac{3}{2}x_2 - \frac{3}{2}x_3 + \frac{1}{2}x_7 \\
x_5 &= 3 - \frac{3}{2}x_2 - \frac{5}{2}x_3 - \frac{1}{2}x_7 \\
x_6 &= 2 + 4x_2 - 3x_3 + x_7 \\
\hline
z &= 5 - \frac{5}{2}x_2 + \frac{11}{2}x_3 - \frac{5}{2}x_7.
\end{aligned} \tag{2.18}$$

$\leftarrow x_3 \leq 4/3$

$\leftarrow x_3 \leq 6/5$

$\leftarrow x_3 \leq 2/3 \leftarrow$

$x_3 \leftrightarrow x_6$

The construction of (2.18) completes the first iteration of the simplex method.

Digression on Terminology

The variables x_j that appear on the left-hand side of a dictionary are called *basic*; the variables x_j that appear on the right-hand side are *nonbasic*. The basic variables are said to constitute a *basis*. Of course, the basis changes with each iteration: for example, in the first iteration, x_1 entered the basis whereas x_7 left it. In each iteration, we first choose the nonbasic variable that is to enter the basis and then we find out which basic variable must leave the basis. The choice of the *entering* variable is motivated by our desire to increase the value of z; the determination of the *leaving* variable is based on the requirement that all variables must assume nonnegative values. The leaving variable is that basic variable whose nonnegativity imposes the most stringent upper bound on the increment of the entering variable. The formula for the leaving variable appears in the *pivot row* of the dictionary; the computational process of constructing the new dictionary is referred to as *pivoting*.

Back to the Second Example

In our example, the variable to enter the basis during the second iteration is quite unequivocally x_3. This is the only nonbasic variable in (2.18) whose coefficient in the last row is positive. Of the four basic variables, x_6 imposes the most stringent upper bound on the increase of x_3, and, therefore, has to leave the basis. Pivoting, we arrive at our third dictionary,

$$
\begin{aligned}
x_3 &= \frac{2}{3} + \frac{4}{3}x_2 + \frac{1}{3}x_7 - \frac{1}{3}x_6 \\
x_1 &= \frac{4}{3} - \frac{5}{6}x_2 - \frac{1}{3}x_7 - \frac{1}{6}x_6 && \leftarrow x_2 \le 8/5 \\
x_4 &= 1 - \frac{7}{2}x_2 \qquad\qquad + \frac{1}{2}x_6 && \leftarrow x_2 \le 2/7 \\
x_5 &= \frac{4}{3} - \frac{29}{6}x_2 - \frac{4}{3}x_7 + \frac{5}{6}x_6 && \leftarrow x_2 \le \frac{8}{29} \quad \left(\frac{8}{29} < \frac{2}{7}\right) \\
\hline
z &= \frac{26}{3} + \frac{29}{6}x_2 - \frac{2}{3}x_7 - \frac{11}{6}x_6. && x_2 \leftrightarrow x_5
\end{aligned}
\tag{2.19}
$$

In the third iteration, the entering variable is x_2 and the leaving variable is x_5. Pivoting yields the dictionary

$$
\begin{aligned}
x_2 &= \frac{8}{29} - \frac{8}{29}x_7 + \frac{5}{29}x_6 - \frac{6}{29}x_5 && x_2 = 8/29 \\
x_3 &= \frac{30}{29} - \frac{1}{29}x_7 - \frac{3}{29}x_6 - \frac{8}{29}x_5 && x_3 = 30/29 \\
x_1 &= \frac{32}{29} - \frac{3}{29}x_7 - \frac{9}{29}x_6 + \frac{5}{29}x_5 && x_1 = 32/29 \\
x_4 &= \frac{1}{29} + \frac{28}{29}x_7 - \frac{3}{29}x_6 + \frac{21}{29}x_5 && x_4 = 1/29 \\
\hline
z &= 10 - 2x_7 - x_6 - x_5. && \leftarrow \text{Optimal} \quad x_7, x_6, x_5 = 0
\end{aligned}
\tag{2.20}
$$

At this point, no nonbasic variable can enter the basis without making the value of z decrease. Hence, the last dictionary describes an optimal solution of our example. That solution is

$$x_1 = \frac{32}{29}, \quad x_2 = \frac{8}{29}, \quad x_3 = \frac{30}{29}$$

and it yields $z = 10$.

FURTHER REMARKS

The reader may have noticed that, having first carefully laid down the definition of a dictionary, we then proceeded to refer to (2.18), (2.19), and (2.20) as dictionaries, without bothering to verify that they do indeed have property (2.14). Such carelessness can be easily justified. Take, for example, system (2.18). Since (2.18) arises from (2.16) by arithmetical operations (namely, pivoting with x_1 entering and x_7 leaving), every solution of (2.16) must be also a solution of (2.18). The converse is also true, since (2.16) can be obtained from (2.18) by pivoting with x_7 entering and x_1 leaving. Hence, *every solution of* (2.18) *is a solution of* (2.16), *and vice versa*. Similar arguments show that *every solution of* (2.19) *is a solution of* (2.18), *and vice versa*; and that *every solution of* (2.20) *is a solution of* (2.19), *and vice versa.*

Another point of concern is the question of the *uniqueness*, as opposed to the *existence*, of optimal solutions. This question will be of no great interest to us; nevertheless, it is easy to deal with and so we will get it out of the way now. Note that in each of our two examples, we not only found an optimal solution, but we also collected the evidence to prove that there is only one optimal solution. For instance, the final dictionary for our first problem reads

$$\begin{array}{rcl} x_3 &=& 1 + x_2 + 3x_4 - 2x_6 \\ x_1 &=& 2 - 2x_2 - 2x_4 + x_6 \\ x_5 &=& 1 + 5x_2 + 2x_4 \\ \hline z &=& 13 - 3x_2 - x_4 - x_6. \end{array}$$

The last row shows that every feasible solution with $z = 13$ satisfies $x_2 = x_4 = x_6 = 0$; the rest of the dictionary shows that every such solution satisfies $x_3 = 1, x_1 = 2, x_5 = 1$; therefore, there is just one optimal solution. A similar argument applies to the second problem.

Of course, there are LP problems with more than just one optimal solution; having solved

Tableau Method [V] p.20

$$
\begin{array}{rrrrrrl}
-\zeta & +3x_1 & +2x_2 & & & & = 0 \\
 & -x_1 & +3x_2 & +w_1 & & & = 12 \\
 & x_1 & +x_2 & & +w_2 & & = 8 \quad \leftarrow x_1 \le 8 \\
 & 2x_1 & -x_2 & & & +w_3 & = 10 \quad \leftarrow x_1 \le 5
\end{array}
$$

$$
\begin{array}{rrrrrrl}
-\zeta & +3x_1 & +2x_2 & & & & = 0 \\
 & -x_1 & +3x_2 & +w_1 & & & = 12 \\
 & x_1 & +x_2 & & +w_2 & & = 8 \\
 & x_1 & -\frac{x_2}{2} & & & +\frac{w_3}{2} & = 5
\end{array}
$$

$$
\begin{array}{rrrrrrl}
-\zeta & & \frac{7x_2}{2} & & & -\frac{3w_3}{2} & = -15 \\
 & & \frac{5x_2}{2} & +w_1 & & +\frac{w_3}{2} & = 17 \quad \leftarrow x_2 \le \frac{34}{5} \\
 & & \frac{3x_2}{2} & & w_2 & -\frac{w_3}{2} & = 3 \quad \leftarrow x_2 \le 2 \\
 & x_1 & -\frac{x_2}{2} & & & +\frac{w_3}{2} & = 5
\end{array}
$$

[C, p.25]

A tableau is nothing but a cryptic recording of a dictionary with all the variables collected on the left-hand side and the symbols for these variables omitted. We shall continue to use dictionaries instead, since they are more explicit. (Of course, nothing prevents the reader tired of writing the same symbols $x_1, x_2, \ldots$ over and over again from using the tableau shorthand.)

Phase I [C]

Initialization

The only remaining point that needs to be explained is getting hold of the initial feasible dictionary in a problem

$$\begin{array}{lll} \text{maximize} & \sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \le b_i & (i = 1, 2, \dots, m) \\ & x_j \ge 0 & (j = 1, 2, \dots, n) \end{array}$$

with an infeasible origin. The trouble with an infeasible origin is twofold. First, it may not be clear that our problem has any feasible solutions at all. Second, even if a feasible solution is apparent, a feasible dictionary may not be. One way of getting around both obstacles uses a so-called *auxiliary problem*,

$$\begin{array}{lll} \text{minimize} & x_0 & \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j - x_0 \le b_i & (i = 1, 2, \dots, m) \\ & x_j \ge 0 & (j = 0, 1, \dots, n). \end{array}$$

A feasible solution of the auxiliary problem is readily available: it suffices to set the value of each x_j with $1 \le j \le n$ at zero and make the value of x_0 sufficiently large. Furthermore, it is easy to see that the original problem has a feasible solution *if and only if* the auxiliary problem has a feasible solution with $x_0 = 0$. To put it differently, the original problem has a feasible solution if and only if the optimum value of the auxiliary problem is zero. Hence our plan is to solve the auxiliary problem first; the technical details are illustrated on the problem

$$\boxed{\begin{array}{lrcrcrcr} \text{maximize} & x_1 & - & x_2 & + & x_3 & & \\ \text{subject to} & 2x_1 & - & x_2 & + & 2x_3 & \le & 4 \\ & 2x_1 & - & 3x_2 & + & x_3 & \le & -5 \\ & -x_1 & + & x_2 & - & 2x_3 & \le & -1 \\ & & & & x_1, & x_2, x_3 & \ge & 0. \end{array}}$$

To avoid unnecessary confusion, we write the auxiliary problem in its maximization form:

$$\begin{array}{lrcrcrcrcr} \text{maximize} & -x_0 & & & & & & & & \\ \text{subject to} & 2x_1 & - & x_2 & + & 2x_3 & - & x_0 & \le & 4 \\ & 2x_1 & - & 3x_2 & + & x_3 & - & x_0 & \le & -5 \\ & -x_1 & + & x_2 & - & 2x_3 & - & x_0 & \le & -1 \\ & & & & x_0, & x_1, & x_2, & x_3 & \ge & 0. \end{array}$$

Writing down the formulas defining the slack variables x_4, x_5, x_6 and the objective function w, we obtain the dictionary

$$\begin{array}{rcl}
x_4 &=& 4 - 2x_1 + x_2 - 2x_3 + x_0 \\
x_5 &=& -5 - 2x_1 + 3x_2 - x_3 + x_0 \\
x_6 &=& -1 + x_1 - x_2 + 2x_3 + x_0 \\
\hline
w &=& \qquad\qquad\qquad\qquad - x_0
\end{array}$$

← choose the most negative

$x_0 \longleftrightarrow x_5$

which is infeasible. Nevertheless, this infeasible dictionary can be transformed into a feasible one by a single pivot, with x_0 entering and x_5 leaving the basis:

$$\begin{array}{rcl}
x_0 &=& 5 + 2x_1 - 3x_2 + x_3 + x_5 \\
x_4 &=& 9 \qquad\quad - 2x_2 - x_3 + x_5 \\
x_6 &=& 4 + 3x_1 - 4x_2 + 3x_3 + x_5 \\
\hline
w &=& -5 - 2x_1 + 3x_2 - x_3 - x_5.
\end{array}$$

← feasible

$x_2 \le 5/3$

$x_2 \le 9/2$

$x_2 \le 1$

$x_2 \longleftrightarrow x_6$

In general, the auxiliary problem may be written as

$$\begin{array}{lll}
\text{maximize} & -x_0 & \\
\text{subject to} & \sum_{j=1}^{n} a_{ij}x_j - x_0 \le b_i & (i = 1, 2, \ldots, m) \\
& x_j \ge 0 & (j = 0, 1, \ldots, n).
\end{array}$$

Writing down the formulas defining the slack variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ and the objective function w gives us the dictionary

$$\begin{array}{rcll}
x_{n+i} &=& b_i - \sum_{j=1}^{n} a_{ij}x_j + x_0 & (i = 1, 2, \ldots, m) \\
\hline
w &=& \qquad\qquad - x_0 &
\end{array}$$

which is infeasible. Nevertheless, this infeasible dictionary can be transformed into a feasible one by a single pivot, with x_0 entering and the "most infeasible" x_{n+i} leaving the basis. More precisely, the leaving variable is that x_{n+k} whose negative value, b_k, has the largest magnitude among all the negative numbers b_i. After pivoting, the variable x_0 assumes the positive value of $-b_k$, whereas each basic x_{n+i} assumes the nonnegative value of $b_i - b_k$. Now we are set to solve the auxiliary problem by the simplex method. In our illustrative example, the computations go as follows.

After the first iteration, with x_2 entering and x_6 leaving:

$$\begin{array}{rcl}
x_2 &=& 1 + 0.75x_1 + 0.75x_3 + 0.25x_5 - 0.25x_6 \\
x_0 &=& 2 - 0.25x_1 - 1.25x_3 + 0.25x_5 + 0.75x_6 \\
x_4 &=& 7 - 1.5x_1 - 2.5x_3 + 0.5x_5 + 0.5x_6 \\
\hline
w &=& -2 + 0.25x_1 + 1.25x_3 - 0.25x_5 - 0.75x_6.
\end{array}$$

$x_3 \le \frac{2}{1.25}$ ←

$x_3 \le \frac{7}{2.5}$

$\frac{2}{1.25} < \frac{7}{2.5}$

After the second iteration, with x_3 entering and x_0 leaving:

$X_3 \leftrightarrow X_0$

$$\begin{aligned} x_3 &= 1.6 - 0.2x_1 + 0.2x_5 + 0.6x_6 - 0.8x_0 \\ x_2 &= 2.2 + 0.6x_1 + 0.4x_5 + 0.2x_6 - 0.6x_0 \\ x_4 &= 3 - x_1 - x_6 + 2x_0 \\ \hline w &= - x_0. \end{aligned} \tag{3.12}$$

$X_0 = 0$ is optimal

The last dictionary (3.12) is optimal. Since the optimal value of the auxiliary problem is zero, dictionary (3.12) points out a feasible solution of the original problem: $x_1 = 0, x_2 = 2.2, x_3 = 1.6$. Furthermore, (3.12) can be easily converted into the desired feasible dictionary of the original problem. To obtain the first three rows of the desired dictionary, we simply copy down the first three rows of (3.12), omitting all the terms involving x_0:

$$\begin{aligned} x_3 &= 1.6 - 0.2x_1 + 0.2x_5 + 0.6x_6 \\ x_2 &= 2.2 + 0.6x_1 + 0.4x_5 + 0.2x_6 \\ x_4 &= 3 - x_1 - x_6. \end{aligned} \tag{3.13}$$

$(X_0 = 0)$

To obtain the last row, we have to express the original objective function

$$z = x_1 - x_2 + x_3 \tag{3.14}$$

in terms of the nonbasic variables x_1, x_5, x_6. For this purpose, we simply substitute from (3.13) into (3.14), obtaining

$$\begin{aligned} z &= x_1 - (2.2 + 0.6x_1 + 0.4x_5 + 0.2x_6) + (1.6 - 0.2x_1 + 0.2x_5 + 0.6x_6) \\ &= -0.6 + 0.2x_1 - 0.2x_5 + 0.4x_6. \end{aligned}$$

In short, the desired dictionary reads

$$\begin{aligned} x_3 &= 1.6 - 0.2x_1 + 0.2x_5 + 0.6x_6 \\ x_2 &= 2.2 + 0.6x_1 + 0.4x_5 + 0.2x_6 \\ x_4 &= 3 - x_1 - x_6 \\ \hline z &= -0.6 + 0.2x_1 - 0.2x_5 + 0.4x_6. \end{aligned}$$

$\leftarrow$ feasible "origin"

Clearly, the same procedure will transform an optimal dictionary of the auxiliary problem into a feasible dictionary of the original problem whenever x_0 is nonbasic in the former.

Now, let us review the general situation. We have learned how to construct the auxiliary problem and its first feasible dictionary. In the process of solving the auxiliary problem, we may encounter a dictionary where x_0 competes with other variables for leaving the basis. If and when that happens, it is only natural to choose x_0 as the actual leaving variable; immediately after pivoting, we obtain a dictionary where

x_0 is nonbasic, and so the value of w is zero. (3.15)

Clearly, a feasible dictionary with this property is optimal. However, we may also reach the optimum of the auxiliary problem while x_0 is still basic. Thus, we may obtain an optimal dictionary where

$$x_0 \text{ is basic and the value of } w \text{ is nonzero} \tag{3.16}$$

or, conceivably, an optimal dictionary where

$$x_0 \text{ is basic and the value of } w \text{ is zero.} \tag{3.17}$$

Let us examine case (3.17). Since the next-to-last dictionary was not yet optimal, the value of $w = -x_0$ must have changed from some negative level to zero in the last iteration. To put it differently, the value of the basic variable x_0 must have dropped from some positive level to zero in the last iteration. But then x_0 was a candidate for leaving the basis; yet, contrary to our policy, we did not pick it. This contradiction shows that (3.17) cannot occur. Hence the optimal dictionary of the auxiliary problem has either property (3.15) or property (3.16). In the former case, we construct a feasible dictionary of the original problem as illustrated previously and proceed to solve the original problem by the simplex method; in the latter case, we simply conclude that the original problem is infeasible.

This strategy is known as the *two-phase simplex method*. In the *first phase*, we set up and solve the auxiliary problem; if the optimal dictionary turns out to have property (3.15) then we proceed to the *second phase*, solving the original problem itself. We shall return to the two-phase simplex method in Chapter 8.

THE FUNDAMENTAL THEOREM OF LINEAR PROGRAMMING

This name is given to the following result.

THEOREM 3.4. Every LP problem in the standard form has the following three properties:

(i) If it has no optimal solution, then it is either infeasible or unbounded.

(ii) If it has a feasible solution, then it has a basic feasible solution.

(iii) If it has an optimal solution, then it has a basic optimal solution.

PROOF. The first phase of the two-phase simplex method either discovers that the problem is infeasible or else it delivers a basic feasible solution. The second phase of the two-phase simplex method either discovers that the problem is unbounded or else it delivers a basic optimal solution. ■

Unboundedness

[V] p. 20

What if all of the ratios, $\bar{a}_{ik}/\bar{b}_i$, are nonpositive? In that case, none of the basic variables will become zero as the entering variable increases. Hence, the entering variable can be increased indefinitely to produce an arbitrarily large objective value. In such situations, we say that the problem is *unbounded*. For example, consider the following dictionary:

$$
\begin{array}{rcrcrrcrr}
\zeta & = & 5 & + & 1 & x_3 & - & 1 & x_1 \\
\hline
x_2 & = & 5 & + & 2 & x_3 & - & 3 & x_1 \\
x_4 & = & 7 & & & & - & 4 & x_1 \\
x_5 & = & & & & & & & x_1
\end{array}
$$

The entering variable is x_3 and the ratios are

$$-2/5, \quad -0/7, \quad 0/0.$$

Since none of these ratios is positive, the problem is unbounded.

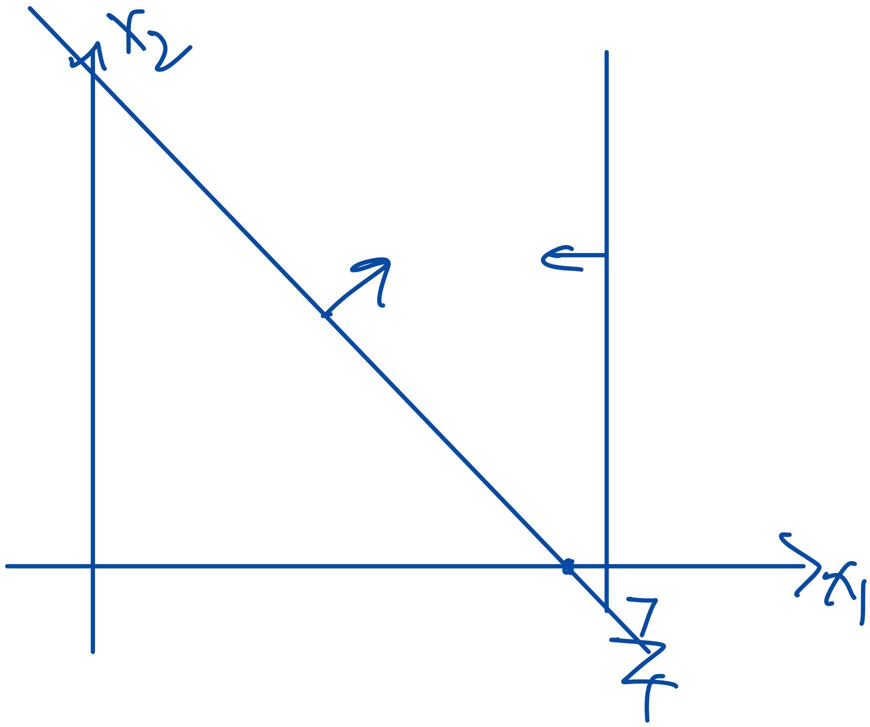
x_3 can be as large as possible

$$\left\{\begin{array}{l} x_3 = -\frac{5}{2} + \frac{3}{2}x_1 + \frac{x_2}{2} \\ x_4 = 7 - 4x_1 \\ x_5 = x_1 \end{array}\right. \Longleftrightarrow \left\{\begin{array}{l} 3x_1 + x_2 \geq 5 \\ x_1 \leq 7/4 \\ x_1 \geq 0 \end{array}\right.$$

$$\zeta = 5 - \frac{5}{2} + \frac{3x_1}{2} + \frac{x_2}{2} - x_1$$

$$= \frac{5}{2} + \frac{x_1}{2} + \frac{x_2}{2}$$

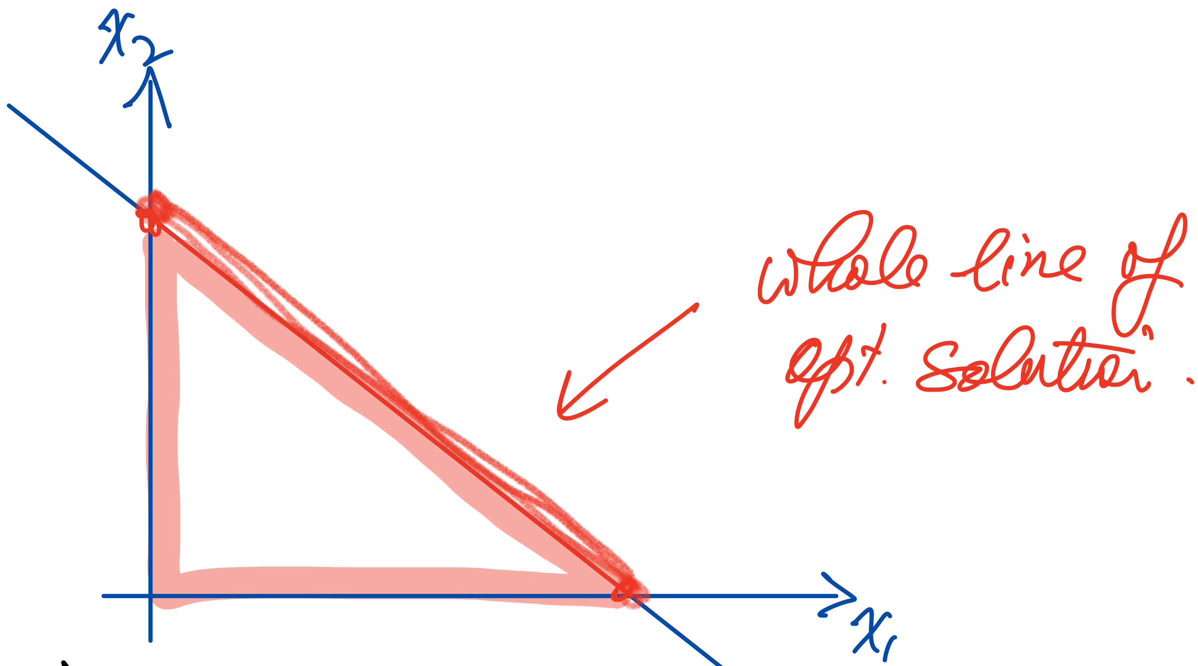
$$\frac{5}{3} < \frac{7}{4}$$



x2
x1
7
Z*

Uniqueness of Solution

$$\max \zeta = x_1 + x_2$$
$$\text{s.t.} \quad x_1 + x_2 \le 1$$
$$x_1, x_2 \ge 0$$



$$\zeta = x_1 + x_2$$
$$w = 1 - x_1 - x_2 \qquad x_1 \longleftrightarrow w$$

$$\Rightarrow \quad x_1 = 1 - w - x_2$$
$$\zeta = 1 - w - x_2 + x_2 = 1 - w$$

Optimal Solution: $w = 0, \ \zeta = 1$

$$\Longrightarrow \quad x_1 + x_2 = 1, \quad \underline{\underline{x_1, x_2 \ge 0}}$$

[C, p.23]

such problems by the simplex method, we can effectively describe all the optimal solutions. For example, consider the following dictionary:

$$\begin{aligned}
x_4 &= 3 + x_2 - 2x_5 + 7x_3 \\
x_1 &= 1 - 5x_2 + 6x_5 - 8x_3 \\
x_6 &= 4 + 9x_2 + 2x_5 - x_3 \\
\hline
z &= 8 \qquad\qquad\qquad - x_3.
\end{aligned}$$

The last row shows that every optimal solution satisfies $x_3 = 0$ (but not necessarily $x_2 = 0$ or $x_5 = 0$). For such solutions, the rest of the dictionary implies

$$\begin{aligned}
x_4 &= 3 + x_2 - 2x_5 \\
x_1 &= 1 - 5x_2 + 6x_5 \\
x_6 &= 4 + 9x_2 + 2x_5.
\end{aligned} \tag{2.21}$$

We conclude that every optimal solution arises by the substitution formulas (2.21) from some x_2 and x_5 such that

$$\begin{aligned}
-x_2 + 2x_5 &\le 3 \\
5x_2 - 6x_5 &\le 1 \\
-9x_2 - 2x_5 &\le 4 \\
x_2, x_5 &\ge 0.
\end{aligned}$$

(In fact, the inequality $-9x_2 - 2x_5 \le 4$ is clearly redundant; its validity is forced by $x_2 \ge 0$ and $x_5 \ge 0$.)

[C, p.42]

THE FUNDAMENTAL THEOREM OF LINEAR PROGRAMMING

This name is given to the following result.

THEOREM 3.4. Every LP problem in the standard form has the following three properties:

(i) If it has no optimal solution, then it is either infeasible or unbounded.

(ii) If it has a feasible solution, then it has a basic feasible solution.

(iii) If it has an optimal solution, then it has a basic optimal solution.

PROOF. The first phase of the two-phase simplex method either discovers that the problem is infeasible or else it delivers a basic feasible solution. The second phase of the two-phase simplex method either discovers that the problem is unbounded or else it delivers a basic optimal solution. ■

[V] p.36

THEOREM 3.4. *For an arbitrary linear program in standard form, the following statements are true:*

(1) *If there is no optimal solution, then the problem is either infeasible or unbounded.*
(2) *If a feasible solution exists, then a basic feasible solution exists.*
(3) *If an optimal solution exists, then a basic optimal solution exists.*

PROOF. The Phase I algorithm either proves that the problem is infeasible or produces a basic feasible solution. The Phase II algorithm either discovers that the problem is unbounded or finds a basic optimal solution. These statements depend, of course, on applying a variant of the simplex method that does not cycle, which we now know to exist. □