Sensitivity Analysis of Simplex Method

(P) $\max \zeta(x) = c^T X$

s.t. $AX \le b$

$X \ge 0$

$W = b - AX$

$AX + W = b$

$$\begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} X \\ W \end{bmatrix} = b$$

$\xRightarrow{\text{simplex}}$

$\zeta = \bar{\zeta} + \bar{c}^T \bar{X}$

$\bar{W} = \bar{b} - \bar{A}\bar{X}$

$\bar{A}\bar{X} + \bar{W} = \bar{b}$

$$\begin{bmatrix} \bar{A} & I \end{bmatrix} \begin{bmatrix} \bar{X} \\ \bar{W} \end{bmatrix} = \bar{b}$$

$\bar{W}$ - basic, $\bar{X}$ - non-basic

$$\underbrace{\begin{bmatrix} A & I \end{bmatrix}}\begin{bmatrix} X \\ W \end{bmatrix} \qquad x_B, \; x_N \qquad = b$$

choose corresponding columns

$$\begin{bmatrix} B & N \end{bmatrix}\begin{bmatrix} x_B \\ x_N \end{bmatrix} = b$$

$$B x_B + N x_N = b$$

$$\bar{w} = \bar{b} - \bar{A}\bar{x}$$

$$\boxed{x_B = B^{-1}(b - N x_N) = B^{-1}b - B^{-1}N x_N}$$

$$
\begin{aligned}
\zeta &= c^T x = c^T x + 0^T w = \bar{c}^T \bar{x} \\
&= c_B^T x_B + c_N^T x_N \\
&= c_B^T \left(B^{-1} b - B^{-1} N x_N \right) + c_N^T x_N \\
&= c_B^T B^{-1} b - \left(c_B^T B^{-1} N - c_N^T \right) x_N \\
&= \underbrace{\left(B^{-T} c_B \right)^T b}_{\bar{\zeta}} - \underbrace{\left(\left(B^{-1} N \right)^T c_B - c_N \right)^T}_{\bar{z}_N} x_N
\end{aligned}
$$

$$
x_B = B^{-1} b - \left(B^{-1} N \right) x_N
$$

Simplex Iterations in Matrix Form

$$
\begin{aligned}
\max \quad & \zeta(x) = c^T x \\
\text{s.t.} \quad & w = b - Ax \\
& x, w \geq 0
\end{aligned}
$$

$x \in \mathbb{R}^n$,

$w \in \mathbb{R}^m$

Simplex iterations

$$
\begin{aligned}
\max \quad & \zeta(x) = \underline{c_B^T B^{-1} b} - \underline{\left((B^{-1}N)^T c_B - c_N \right)}^T x_N \\
\text{s.t.} \quad & x_B = \underline{B^{-1} b} - \underline{B^{-1} N x_N} \\
& x_B, x_N \geq 0
\end{aligned}
$$

B, N are basic and non-basic var. cols of $[A \ I]$

Simplex Iterations in Matrix Form

(P)

$c_B^T B^{-1} b$	$-\left((B^{-1}N)^T c_B - c_N\right)^T$
$B^{-1} b$	$-B^{-1} N$

negative transpose

(D)

$-c_B^T B^{-1} b$	$-(B^{-1} b)^T$
$(B^{-1}N)^T c_B - c_N$	$(B^{-1} N)^T$

Simplex Iterations in Matrix Form

(P)

ζ^*	$-(z_N^*)^T$
x_B^*	$-B^{-1}N$

negative transpose

(D)

$-\zeta^*$	$-(x_B^*)^T$
z_N^*	$(B^{-1}N)^T$

Simplex Iterations in Matrix Form

(P)

$$\begin{aligned} \max\ & \zeta(x) = \zeta^* - (z_N^*)^T x_N \\ \text{s.t.}\ & x_B = x_B^* - (B^{-1}N)\,x_N \\ & x_N, x_B \geq 0 \end{aligned}$$

(D)

$$\begin{aligned} \max\ & -\xi(z) = -\zeta^* - (x_B^*)^T z_B \\ \text{s.t.}\ & z_N = (z_N^*) + (B^{-1}N)^T z_B \\ & z_B, z_N \geq 0 \end{aligned}$$

$\zeta^* = c_B^T B^{-1} b$, $z_N^* = (B^{-1}N)^T c_B - c_N$, $x_B^* = B^{-1} b$

Simplex Iterations in Matrix Form

(P)

$$\begin{aligned} \max \quad & \zeta(x) = \zeta^* - (z_N^*)^T x_N \\ \text{s.t.} \quad & x_B = x_B^* - (B^{-1}N)\, x_N \\ & x_N, x_B \geq 0 \end{aligned}$$

Optimal Dictionary

$\Updownarrow$

$z_N^* \geq 0$

(D)

$$\begin{aligned} \max \quad & -\xi(z) = -\zeta^* - (x_B^*)^T z_B \\ \text{s.t.} \quad & z_N = (z_N^*) + (B^{-1}N)^T z_B \\ & z_B, z_N \geq 0 \end{aligned}$$

feasibility

$\Updownarrow$

$x_B^* \geq 0$

$\zeta^* = c_B^T B^{-1} b$, $\quad z_N^* = (B^{-1}N)^T c_B - c_N$, $\quad x_B^* = B^{-1} b$

Complementary Variables

$$\left(x_1, x_2, \ldots, x_n, \overbrace{x_{n+1} \ldots x_{n+m}}^{(w_1 \ldots w_m)} \right)$$

$$\updownarrow \qquad\qquad \updownarrow$$

$$\left(z_1, z_2, \ldots, z_n, \overbrace{z_{n+1}, \ldots, z_{n+m}}^{(y_1, \ldots y_m)} \right)$$

① $x_j \sim z_j$ $(x_j z_j = 0)$ $\qquad w_i \sim y_i$ $(w_i y_i = 0)$

② x_j leaves/enters $\sim$ z_j enters/leaves

w_i leaves/enters $\sim$ y_j enters/leaves

③ $x_N \sim z_N$, $x_B \sim z_B$

[C] p.67

max. $\zeta = 40x_1 + 70x_2$

(P) s.t. $\begin{cases} x_1 + x_2 \leq 100 \\ 10x_1 + 30x_2 \leq 4000 \\ x_1, x_2 \geq 0 \end{cases}$

Opt. solution: $x_1^* = 25$, $x_2^* = 75$, $\zeta^* = 40(25) + 70(75)$
$= 6250$

Q. (1) Check optimality.

(2) What if 4000 is changed to $4000 + t$

How would x_1^*, x_2^*, ζ^* be changed?

How large can t be?

A (1) find y_1^*, y_2^* and check validity of Strong duality.

(D) $\min \ 100y_1 + 4000y_2$

$$\text{s.t.} \begin{cases} y_1 + 10y_2 \geq 40 \\ y_1 + 50y_2 \geq 70, \end{cases} \qquad y_1, y_2 \geq 0$$

Use complementary slackness:

$$x_1^* > 0 \Rightarrow z_1^* = 0 \Rightarrow y_1^* + 10y_2^* = 40$$

$$x_2^* > 0 \Rightarrow z_2^* = 0 \Rightarrow y_1^* + 50y_2^* = 70$$

$$y_1^* = 32.5 \geq 0, \quad y_2^* = 0.75 \geq 0$$

(Check:

$$\zeta^* = 40(25) + 70(75) = 100(32.5) + 4000(0.75) = \underline{6250} = \xi^*)$$

A(2) Use B, N method

$$\zeta = C_B^T B^{-1} b - \left((B^{-1}N)^T C_B - C_N \right)^T X_N$$

$$X_B = B^{-1} b - (B^{-1}N) X_N$$

$$[A \quad I] = \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 10 & 50 & 0 & 1 \end{bmatrix} \\ \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ & & (w_1) & (w_2) \end{array} \end{array}$$

Start

P

	0	x_1 40	x_2 70
$x_3\ (w_1)$	100	-1	-1
$x_4\ (w_2)$	4000	-10	-50

D

	0	(z_3) y_1 -100	(z_4) y_2 -4000
z_1	-40	1	10
z_2	-70	1	50

$\longrightarrow (-ve)^T$

End (Opt)

P

	6250	x_3 -32.5	x_4 -0.75
x_1	25	$-B^{-1}N$	
x_2	75		

D

	-6250	z_1 -25	z_2 -75
y_1	32.5	$(B^{-1}N)^T$	
y_2	0.75		

$\mathcal{B} = \{1, 2\}, \quad \mathcal{N} = \{3, 4\}$

$$B = \begin{bmatrix} 1 & 1 \\ 10 & 50 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$B^{-1} = \frac{\begin{pmatrix} 50 & -1 \\ -10 & 1 \end{pmatrix}}{40} = \begin{pmatrix} 5/4 & -1/40 \\ -1/4 & 1/40 \end{pmatrix}$$

End (Opt)

(P)

		x_3	x_4
	6250	-32.5	-0.75
x_1	25	$-5/4$	$1/40$
x_2	75	$1/4$	$-1/40$

(D)

		z_1	z_2
	-6250	-25	-75
y_1	32.5	$5/4$	$-1/4$
y_2	0.75	$-1/40$	$1/40$

$$\zeta = c_B^T B^{-1} b - \left((B^{-1} N)^T c_B - c_N \right)^T x_N$$

$$x_B = B^{-1} b - (B^{-1} N) x_N$$

$$\boxed{-3000 \leq t \leq 1000}$$

Changed to $B^{-1}\left(b + \begin{pmatrix} 0 \\ t \end{pmatrix} \right)$

$$= \begin{pmatrix} 5/4 & -\frac{1}{40} \\ -1/4 & \frac{1}{40} \end{pmatrix} \begin{pmatrix} 100 \\ 4000+t \end{pmatrix} = \begin{pmatrix} 25 - \frac{t}{40} \\ 75 + \frac{t}{40} \end{pmatrix} \begin{matrix} \geq 0 \\ \geq 0 \end{matrix}$$

$x_1^* \longrightarrow 25 - t/40, \quad x_2^* \longrightarrow 75 + t/40$

(M1) $$z^* = 40x_1^* + 70x_2^*$$

$$= 40\left(25 - \frac{t}{40}\right) + 70\left(75 + \frac{t}{40}\right)$$

$$= 6250 + \underline{0.75t}$$

(M2) $$z^* = c_B^T B^{-1}\left(b + \begin{pmatrix} 0 \\ t \end{pmatrix}\right)$$

$$= \begin{pmatrix} 40 & 70 \end{pmatrix} \begin{pmatrix} 5/4 & -1/40 \\ -1/4 & 1/40 \end{pmatrix} \begin{pmatrix} 100 \\ 4000+t \end{pmatrix}$$

$$= \underline{\begin{pmatrix} 32.5 & 0.75 \end{pmatrix}} \begin{pmatrix} 100 \\ 4000+t \end{pmatrix}$$

(y_1^*, y_2^*)

no accident.

$$= 6250 + \underline{0.75t}$$

M3

$$\zeta^*_{\text{new}} = 40x_1^* + 70x_2^*$$

$$\le (y_1^* + 10y_2^*)x_1^* + (y_1^* + 50y_2^*)x_2^*$$

$$= y_1^*(x_1^* + x_2^*) + y_2^*(10x_1^* + 50x_2^*)$$

$$\le y_1^*(100) + y_2^*(4000 + t)$$

$$= \underbrace{100y_1^* + 4000y_2^*}_{\text{old}} + \underbrace{y_2^* t}_{\text{change}}$$

$= 0.75t$

Note: $\zeta^* = c_B^T B^{-1} b = b^T B^{-T} c_B$

$$\Delta\zeta^* = (\Delta b)^T \underline{(B^{-T} c_B)} \qquad y^* = B^{-T} c_B$$

In fact $\boxed{\zeta^*_{new} = \zeta^*_{old} + (\Delta b)^T y^*_{old}}$

(if non-degenerate, + $|\Delta b| \ll 1$)

[C] p. 66 Thm 5.5

$$(\Delta b)^T y^*_{old} = (\Delta b)_1 y_1^* + (\Delta b)_2 y_2^* + \cdots + (\Delta b)_m y_m^*$$

shadow prices

pf (P) $\max \zeta = c^T x$

s.t. $Ax \le b$

$x \ge 0$

at Opt:

(P)

$$\zeta = \zeta^* - z_N^{*} x_N$$

$$x_B = x_B^* - (B^{-1}N) x_N$$

$$z_N^* = (B^{-1}N)^T c_B - c_N \ge 0$$

$$x_B^* = B^{-1} b \ge 0$$

$$\zeta^* = c_B^T B^{-1} b = c_B^T x_B^* = b^T B^{-T} c_B$$

(D) $\min \ \xi = b^T y$

s.t. $A^T y \geq c$

$y \geq 0$

at Opt. $-\xi = -\zeta^* + (x_B^*)^T z_B$

$z_N = z_N^* + (B^{-1}N)^T z_B$

Compare: $\zeta^* = c^T x^* = \xi^* = b^T y^*$

$= b^T B^{-T} c_B$

Guess/Prove $\boxed{y^* = B^{-T} c_B}$

(1) $B^T(B^{-T} c_B)$ does gives opt. objective value ✓

(2) check for feasibility of $B^{-T} c_B$

$$A^T(B^{-T} c_B) \geq c \quad ?$$

$$B^{-T} c_B \geq 0 \quad ?$$

We need $A^T y \geq \vec{c} \iff y^T A \geq c^T$

$y \geq \vec{0} \iff y^T \geq \vec{0}$

$$y^T [A \quad I] \geq [c^T \quad 0^T]$$

$$y^T [B \quad N] \geq [c_B^T \quad c_N^T]$$

$y^T B \overset{?}{\geq} c_B^T$ $\qquad$ $y^T N \overset{?}{\geq} c_N^T$

$$y = B^{-T} C_B \,, \qquad y^T B = C_B^T B^{-1} B = C_B^T \geq C_B^T$$

$$y^T N = \underbrace{C_B^T B^{-1} N \geq C_N}_{\text{Optimality condition}}$$

$$z_N^* = (B^{-1} N)^T C_B - C_N \geq 0$$

Sensitivity and Parametric Analysis

[V] p. 111

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 5x_1 & + & 4x_2 & + & 3x_3 & & \\
\text{subject to} & 2x_1 & + & 3x_2 & + & x_3 & \leq & 5 \\
& 4x_1 & + & x_2 & + & 2x_3 & \leq & 11 \\
& 3x_1 & + & 4x_2 & + & 2x_3 & \leq & 8 \\
& x_1, & x_2, & x_3 & \geq & 0 & . &
\end{array}
$$

Sensitivity and Parametric Analysis

Opt.
Dict.

$$
\begin{array}{rcrcrcrcr}
\xi & = & 13 & - & 3\,x_2 & - & 1\,x_4 & - & 1\,x_6 \\
\hline
x_3 & = & 1 & + & x_2 & + & 3\,x_4 & - & 2\,x_6 \\
x_1 & = & 2 & - & 2\,x_2 & - & 2\,x_4 & + & x_6 \\
x_5 & = & 1 & + & 5\,x_2 & + & 2\,x_4 & &
\end{array}
$$

Sensitivity and Parametric Analysis

Opt. Dict. $z_N^* = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = (B^{-1}N)^T c_B - c_N$, $x_N = \begin{pmatrix} x_2 \\ x_4 \\ x_6 \end{pmatrix}$

$$\xi = 13 \boxed{- 3x_2 - 1x_4 - 1x_6}$$

$$x_B = \begin{pmatrix} x_3 \\ x_1 \\ x_5 \end{pmatrix} \begin{array}{l} = \\ = \\ = \end{array} \boxed{\begin{array}{c} 1 \\ 2 \\ 1 \end{array}} \begin{array}{l} + x_2 + 3x_4 - 2x_6 \\ - 2x_2 - 2x_4 + x_6 \\ + 5x_2 + 2x_4 \end{array}$$

$x_B^* = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = B^{-1}b$

$-B^{-1}N = \begin{pmatrix} 1 & 3 & -2 \\ -2 & -2 & 1 \\ 5 & 2 & 0 \end{pmatrix}$

Sensitivity and Parametric Analysis

what if

$(5+t)$

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 5x_1 & + & 4x_2 & + & 3x_3 & & \\
\text{subject to} & 2x_1 & + & 3x_2 & + & x_3 & \leq & 5 \\
& 4x_1 & + & x_2 & + & 2x_3 & \leq & 11 \\
& 3x_1 & + & 4x_2 & + & 2x_3 & \leq & 8 \\
& & & x_1, & x_2, & x_3 & \geq & 0 .
\end{array}
$$

Sensitivity and Parametric Analysis

Still optimal?

$$
\begin{array}{rcrrrrrrr}
\xi & = & 13 & - & 3x_2 & - & 1x_4 & - & 1x_6 \\
\hline
x_3 & = & 1 & + & x_2 & + & 3x_4 & - & 2x_6 \\
x_1 & = & 2 & - & 2x_2 & - & 2x_4 & + & x_6 \\
x_5 & = & 1 & + & 5x_2 & + & 2x_4 & &
\end{array}
$$

Sensitivity and Parametric Analysis

Still optimal ?

$$
\begin{array}{rcrcrcrcr}
\xi & = & 13 & - & \boxed{3 x_2 - 1 x_4 - 1 x_6} \\
\hline
x_3 & = & 1 & + & x_2 & + & 3 x_4 & - & 2 x_6 \\
x_1 & = & 2 & - & 2 x_2 & - & 2 x_4 & + & x_6 \\
x_5 & = & 1 & + & 5 x_2 & + & 2 x_4 & &
\end{array}
$$

Sensitivity and Parametric Analysis

$$x_B^* = B^{-1} b \geq 0 \quad \text{not changed}$$

$$z_N^* = (B^{-1}N)^T c_B - c_N$$

$$\Delta z_N^* = (B^{-1}N)^T \Delta c_B - \Delta c_N$$

$$\boxed{\text{We need} \quad z_N^* + \Delta z_N^* \geq 0}$$

Sensitivity and Parametric Analysis

what if

$$
\begin{array}{lrcrcrcl}
\text{maximize} & 5x_1 & + & 4x_2 & + & 3x_3 & & \\
\text{subject to} & 2x_1 & + & 3x_2 & + & x_3 & \leq & 5 + \Delta b_1 \\
 & 4x_1 & + & x_2 & + & 2x_3 & \leq & 11 + \Delta b_2 \\
 & 3x_1 & + & 4x_2 & + & 2x_3 & \leq & 8 + \Delta b_3 \\
 & & & x_1, \; x_2, \; x_3 & \geq & 0 & . &
\end{array}
$$

Sensitivity and Parametric Analysis

Still optimal ?

$$
\begin{array}{rcrrrrrrr}
\xi & = & 13 & - & 3\,x_2 & - & 1\,x_4 & - & 1\,x_6 \\
\hline
x_3 & = & 1 & + & x_2 & + & 3\,x_4 & - & 2\,x_6 \\
x_1 & = & 2 & - & 2\,x_2 & - & 2\,x_4 & + & x_6 \\
x_5 & = & 1 & + & 5\,x_2 & + & 2\,x_4 & &
\end{array}
$$

Sensitivity and Parametric Analysis

Still optimal?

$$
\begin{array}{rcrrrrrrr}
\xi & = & 13 & - & 3 & x_2 & - \; 1 & x_4 & - \; 1 \; x_6 \\
\hline
x_3 & = & 1 & + & & x_2 & + \; 3 & x_4 & - \; 2 \; x_6 \\
x_1 & = & 2 & - & 2 & x_2 & - \; 2 & x_4 & + \quad x_6 \\
x_5 & = & 1 & + & 5 & x_2 & + \; 2 & x_4 &
\end{array}
$$

Sensitivity and Parametric Analysis

$$Z_N^* = (B^{-1}N)^T C_B - C_N \geq 0 \text{ not changed}$$

$$X_B^* = B^{-1} b$$

$$\Delta X_B^* = B^{-1} \Delta b$$

$$\Delta \vec{b} = \begin{pmatrix} \Delta b_1 \\ \Delta b_2 \\ \Delta b_3 \end{pmatrix}$$

$$\boxed{\text{We need } X_B^* + \Delta X_B^* \geq 0}$$

What if $A \to A + \Delta A$?

$$B \longrightarrow B + \Delta B \longleftarrow \text{new } B$$

$$N \longrightarrow N + \Delta N \longleftarrow \text{new } N$$

Need ① $x_B^* = (B + \Delta B)^{-1} b \geq 0$

② $z_N^* = \left((B+\Delta B)^{-1}(N+\Delta N)\right)^T c_B - c_N \geq 0$

New ③ $\zeta^* = c_B^T (B + \Delta B)^{-1} b$

In general $(M+\Delta M)^{-1} = M^{-1} + \boxed{?}$

$$\simeq M^{-1} + \underbrace{\Delta(M^{-1})}_{=\,?}$$

(if ΔM is small)

$$M M^{-1} = I$$

$$\Delta(M M^{-1}) = \Delta I = 0$$

$$(\Delta M) M^{-1} + M(\Delta(M^{-1})) = 0$$

$$\underline{\Delta(M^{-1}) = -M^{-1}(\Delta M) M^{-1}}$$

Hence $\underline{(M+\Delta M)^{-1} \simeq M^{-1} - M^{-1}(\Delta M) M^{-1}}$

$$x_B^* = (B+\Delta B)^{-1} b$$

$$\cong \left(B^{-1} - B^{-1}(\Delta B)B^{-1}\right) b$$

$$= B^{-1}b - B^{-1}(\Delta B)B^{-1}b \quad (\geq 0)$$

$$z_N^* = \left((B+\Delta B)^{-1}(N+\Delta N)\right)^T c_B - c_N$$

$$\cong \left[\left(B^{-1} - B^{-1}(\Delta B)B^{-1}\right)(N+\Delta N)\right]^T c_B - c_N$$

$$\zeta^* = c_B^T (B+\Delta B)^{-1} b \cong c_B^T \left(B - B^{-1}(\Delta B)B^{-1}\right) b$$