Simplex in Matrix Form [V, ch. 6]

(P) $\max \ \zeta(x) = c^T x$ s.t. $Ax \le b$ $x \ge 0$	(D) $\max \ -\xi(y) = -b^T y$ s.t. $A^T y \ge c$ $y \ge 0$
$\max \ \zeta(x) = \zeta^* - (z_N^*)^T x_N$ $x_B = x_B^* - (B^{-1}N) x_N$	$\max \ -\xi(z) = -\zeta^* - (x_B^*)^T z_B$ s.t. $z_N = z_N^* + (B^{-1}N)^T z_B$

$$Ax \le b$$

$$\Longleftrightarrow \ Ax + w = b$$

$$\Longleftrightarrow \ \begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = b$$

$$\begin{bmatrix} B & N \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} = b$$

$$\Longleftrightarrow \ Bx_B + Nx_N = b$$

$$\Longleftrightarrow \ x_B = B^{-1}b - B^{-1}Nx_N$$

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s.t. $Ax \le b$

$x \ge 0$

(D) $\max\ -\xi(y) = -b^T y$

s.t. $A^T y \ge c$

$y \ge 0$

$$\max\ \zeta(x) = \zeta^* - (z_N^*)^T x_N$$

$$x_B = x_B^* - (B^{-1}N)x_N$$

$$\max\ -\xi(z) = -\zeta^* - (x_B^*)^T z_B$$

s.t. $z_N = z_N^* + (B^{-1}N)^T z_B$

$$\zeta(x) = c^T x = c_B^T x_B + c_N^T x_N$$

$$= c_B^T (B^{-1}b - B^{-1}N x_N) + c_N^T x_N$$

$$= c_B^T B^{-1} b - (c_B^T B^{-1} N - c_N^T) x_N$$

Simplex in Matrix Form [V, ch. 6]

$$\begin{pmatrix} \mathcal{B} - \text{Basic var indices} \\ \mathcal{N} - \text{Non-Basic var. indices} \end{pmatrix}$$

$$\boxed{\begin{aligned} \max\ \zeta(x) &= \zeta^* - \Big(\underbrace{(B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}}}_{z^*_{\mathcal{N}}}\Big)^T x_{\mathcal{N}} \\ x_{\mathcal{B}} &= \underbrace{B^{-1}b}_{x^*_{\mathcal{B}}} - (B^{-1}N)\,x_{\mathcal{N}} \end{aligned}}$$

Revised Simplex Method [V, Ch. 8], [C, Ch. 7]

Suppose $\mathcal{B}$, $\mathcal{N}$, B, B^{-1} are known

(1) From z_N^* (>0) $\Longrightarrow$ determine entering var.
(e.g. j^{th} col. of $(B^{-1}N)$)

$$\begin{array}{c|c} \zeta^* & -(z_N^*)^T \\ \hline x_B^* & -(B^{-1}N) \end{array}$$

j^{th} col of $B^{-1}N$

$= (B^{-1}N)e_j$

$= B^{-1}(\underbrace{Ne_j}_{a_j})$

$= \Delta x_B$

Revised Simplex Method [V, Ch.8], [C, Ch.7]

(2) Find $\underline{\max t}$ s.t. $\underline{x_B^* - t\,\Delta x_B \geq 0}$

$$
\begin{array}{c|c}
\zeta^* & -(z_N^*)^T \\
\hline
x_B^* & -(B^{-1}N)
\end{array}
$$

jth col of $B^{-1}N$

$$= (B^{-1}N)e_j$$

$$= B^{-1}(\underbrace{Ne_j}_{a_j})$$

$$= \Delta x_B$$

determine leaving variable.

(eg. i^{th} variable of x_B)

Revised Simplex Method [V, ch.8], [C, ch.7]

(3) Update $\mathcal{B}, \mathcal{N}$ and B, N

(interchange j^{th} col. of N $= Ne_j = a_j$

and i^{th} var of $x_B \Rightarrow i^{th}$ col. of B

$Be_i = a_i$, $e_i = B^{-1}a_i$

i^{th} col. of B, a_i

old $\qquad\qquad\qquad\qquad$ new

$$\begin{bmatrix} B & N \end{bmatrix} \longrightarrow \begin{bmatrix} \tilde{B} & \tilde{N} \end{bmatrix}$$

j^{th} col. of N, a_j

Revised Simplex Method [V, ch.8], [C, ch.7]

$$\text{old } [\, B \quad N \,] \longrightarrow \text{new } [\, \tilde{B} \quad \tilde{N} \,]$$

$$a_i \leftrightarrow a_j$$

$$\left(\begin{array}{l} N e_j = a_j \\ B e_i = a_i \end{array} \right)$$

$$\begin{aligned}
\tilde{B} &= B + (\vec{a}_j - \vec{a}_i)\, \vec{e}_i^{\,T} \\
&= B\left(I + B^{-1}(a_j - a_i)\, e_i^T \right) \\
&= B\left(I + (B^{-1}a_j - B^{-1}a_i)\, e_i^T \right) \\
&= B\left(I + (B^{-1}N e_j - e_i)\, e_i^T \right)
\end{aligned}$$

Revised Simplex Method [V, ch.8], [C, ch.7]

Suppose B, B^{-1} is known

$$\tilde{B} = B \underbrace{\left(I + (B^{-1}Ne_j - e_i)e_i^T\right)}_{E}$$

$$\tilde{B}^{-1} = \underbrace{\left(I + (B^{-1}Ne_j - e_i)e_i^T\right)^{-1}}_{E^{-1}} \underbrace{B^{-1}}_{\text{already known}}$$

E^{-1}: easy to compute

Revised Simplex Method [V, Ch.8], [C, Ch.7]

PROPOSITION 8.1. *Given two column vectors u and v for which $1 + v^T u \neq 0$,*

$$(I + uv^T)^{-1} = I - \frac{uv^T}{1 + v^T u}.$$

(Proof by direct verification)

PROOF. The proof is trivial. We simply multiply the matrix by its supposed inverse and check that we get the identity:

$$\begin{aligned}(I + uv^T)\left(I - \frac{uv^T}{1 + v^T u}\right) &= I + uv^T - \frac{uv^T}{1 + v^T u} - \frac{uv^T uv^T}{1 + v^T u} \\ &= I + uv^T\left(1 - \frac{1}{1 + v^T u} - \frac{v^T u}{1 + v^T u}\right) \\ &= I,\end{aligned}$$

Revised Simplex Method [V, ch.8], [C, ch.7]

Proof by computation

Recall : $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots \quad (|x| < 1)$

or $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

$$(I + uv^T)^{-1} = I - uv^T + (uv^T)^2 - (uv^T)^3 + \cdots$$

Let $X = uv^T$, $\quad X^2 = uv^T uv^T = \underbrace{(v^T u)}_{\text{a number}} uv^T = \underbrace{c}_{\text{a number}} X$

Hence $X^3 = c^2 X$, $X^4 = c^3 X \cdots\cdots$

$$\begin{aligned}
(I + uv^T)^{-1} &= I - X + cX - c^2 X + c^3 X - \cdots \\
&= I - (1 - c + c^2 - c^3 \cdots) X \\
&= I - \frac{1}{1+c} X = I - \frac{uv^T}{1 + v^T u}
\end{aligned}$$

Revised Simplex Method [V, ch.8], [C, ch.7]

$$E^{m\times m} = \begin{bmatrix} 1 & & \alpha_1 & & \\ & \ddots & \alpha_2 & & \\ & & \vdots & & \\ & & \alpha_i & & \\ & & \vdots & & \\ & & \alpha_m & \ddots & 1 \end{bmatrix}, \quad E^{-1} = \begin{bmatrix} 1 & & -\alpha_1/\alpha_i & & \\ & \ddots & \vdots & & \\ & & 1/\alpha_i & & \\ & & \vdots & \ddots & \\ & & -\alpha_m/\alpha_i & & 1 \end{bmatrix}$$

eg

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 5 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -3/2 & 0 \\ 0 & 1 & -4/2 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & -5/2 & 1 \end{bmatrix}$$

Revised Simplex Method [V, ch.8], [C, ch.7]

$$E = I + (B^{-1}Ne_j - e_i)e_i^T = \underbrace{I - e_ie_i^T + B^{-1}Ne_je_i^T}_{\text{replace } i^{th} \text{ col. of } I \text{ by}}$$

$B^{-1}Ne_j = j^{th}$ col of $B^{-1}N$

$$\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & \boxed{1} & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix}$$