[V] p.95

0

$$
\begin{array}{llrl}
\text{maximize} & 4x_1 + 3x_2 & & \\
\text{subject to} & x_1 - \ \ x_2 & \leq & 1 \\
& 2x_1 - \ \ x_2 & \leq & 3 \\
& x_2 & \leq & 5 \\
& x_1,\ x_2 & \geq & 0.
\end{array}
$$

$$
\begin{aligned}
x_1 - x_2 + x_3 &= 1 \\
2x_1 - x_2 + x_4 &= 3 \\
x_2 + x_5 &= 5
\end{aligned}
$$

The matrix A is given by

$$
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{bmatrix} 1 & -1 & 1 & & \\ 2 & -1 & & 1 & \\ 0 & 1 & & & 1 \end{bmatrix}.
\end{array}
$$

(Note that some zeros have not been shown.) The initial sets of basic and nonbasic indices are

$$
\mathcal{B} = \{3, 4, 5\} \quad \text{and} \quad \mathcal{N} = \{1, 2\}.
$$

Corresponding to these sets, we have the submatrices of A:

$$
B = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & -1 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}.
$$

1

$x_1 \le 1$

$x_1 \le 3/2$

$x_1 \le \infty$

x_1 enters, largest coeff.

$$
\begin{array}{ll}
\text{maximize} & 4x_1 + 3x_2 \\
\text{subject to} & x_1 - x_2 \le 1 \\
& 2x_1 - x_2 \le 3 \\
& x_2 \le 5 \\
& x_1,\ x_2 \ge 0.
\end{array}
$$

$x_1 - x_2 + x_3 = 1$

$2x_1 - x_2 + x_4 = 3$

$x_2 + x_5 = 5$

The matrix A is given by

x_3 leaves

$$
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{bmatrix} 1 & -1 & 1 & & \\ 2 & -1 & & 1 & \\ 0 & 1 & & & 1 \end{bmatrix}.
\end{array}
$$

(Note that some zeros have not been shown.) The initial sets of basic and nonbasic indices are

$$
\mathcal{B} = \{3, 4, 5\} \quad \text{and} \quad \mathcal{N} = \{1, 2\}.
$$

Corresponding to these sets, we have the submatrices of A:

$$
B = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & -1 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}.
$$

Hint Undo

Number format: Fraction Zero Visibility: Visible

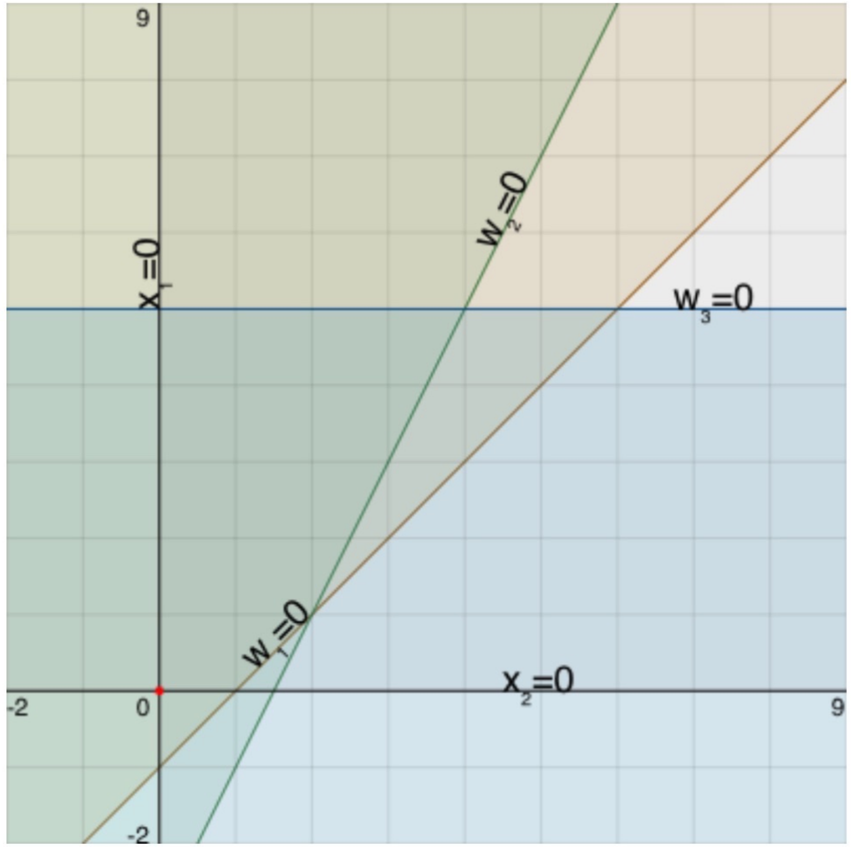
Current Dictionary

maximize	ζ	=			4	x_1	+	3	x_2
subject to:	w_1	=	1	-	1	x_1	-	-1	x_2
	w_2	=	3	-	2	x_1	-	-1	x_2
	w_3	=	5	-	0	x_1	-	1	x_2

$x_1 \; x_2 \; w_1 \; w_2 \; w_3 \geq 0$

Optimal Infeasible Unbounded

Pick a judge: Bart Simpson



$B_0 = I$,

$B_0^{-1} b = I^{-1} b = b$

1st col of $B^{-1}N$

Compare

2

Step 9. The new sets of basic and nonbasic indices are

$$\mathcal{B} = \{1, 4, 5\} \qquad \text{and} \qquad \mathcal{N} = \{3, 2\}.$$

Corresponding to these sets, we have the new basic and nonbasic submatrices of A,

$$B = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 0 & & 1 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & -1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix},$$

and the new basic primal variables and nonbasic dual variables:

$$x^*_{\mathcal{B}} = \begin{bmatrix} x^*_1 \\ x^*_4 \\ x^*_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} \qquad z^*_{\mathcal{N}} = \begin{bmatrix} z^*_3 \\ z^*_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}.$$

x_2 enters
x_4 leaves

$$\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & -1 & 1 & & \\ 2 & -1 & & 1 & \\ 0 & 1 & & & 1 \end{bmatrix} \end{array}$$

$$\begin{array}{lrcl} \text{maximize} & \zeta & = & 4 + -4\, w_1 + 7\, x_2 \\ \text{subject to:} & x_1 & = & 1 - 1\, w_1 - (-1)\, x_2 \\ & w_2 & = & 1 - (-2)\, w_1 - 1\, x_2 \\ & w_3 & = & 5 - 0\, w_1 - 1\, x_2 \end{array}$$

$$x_1 \quad x_2 \quad w_1 \quad w_2 \quad w_3 \geq 0$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \boxed{B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\begin{pmatrix} 4 \\ -7 \end{pmatrix} = \underline{(B^{-1}N)}^T C_B - C_N = \left(\underline{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & -1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \right)^T \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} = \underline{B^{-1}} b = \underline{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$$

$$\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \underline{\text{2nd col. of } B^{-1}N} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

enters

2

Step 9. The new sets of basic and nonbasic indices are

$$\mathcal{B} = \{1, 4, 5\} \qquad \text{and} \qquad \mathcal{N} = \{3, 2\}.$$

Corresponding to these sets, we have the new basic and nonbasic submatrices of A,

$$B = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 0 & & 1 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & -1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix},$$

and the new basic primal variables and nonbasic dual variables:

$$x^*_{\mathcal{B}} = \begin{bmatrix} x^*_1 \\ x^*_4 \\ x^*_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} \qquad z^*_{\mathcal{N}} = \begin{bmatrix} z^*_3 \\ z^*_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}.$$

x_2 enters
x_4 leaves

$$\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & -1 & 1 & & \\ 2 & -1 & & 1 & \\ 0 & 1 & & & 1 \end{bmatrix} \end{array}$$

maximize	ζ	=	4	+	-4	w_1	+	7	x_2
subject to:	x_1	=	1	-	1	w_1	-	-1	x_2
	w_2	=	1	-	-2	w_1	-	1	x_2
	w_3	=	5	-	0	w_1	-	1	x_2

$x_1 \quad x_2 \quad w_1 \quad w_2 \quad w_3 \geq 0$

$$B = \{1, 2, 5\}, \quad N = \{3, 4\}$$

2nd col. of $B^{-1}N$

$$B^{new} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = B^{old} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\left(B^{new}\right)^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} -1 & 1 & 0 \\ -2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}}$$

3

Step 9. The new sets of basic and nonbasic indices are

$$\mathcal{B} = \{1, 2, 5\} \quad \text{and} \quad \mathcal{N} = \{3, 4\}.$$

Corresponding to these sets, we have the new basic and nonbasic submatrices of A,

$$B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and the new basic primal variables and nonbasic dual variables:

$$x_{\mathcal{B}}^* = \begin{bmatrix} x_1^* \\ x_2^* \\ x_5^* \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \qquad z_{\mathcal{N}}^* = \begin{bmatrix} z_3^* \\ z_4^* \end{bmatrix} = \begin{bmatrix} -10 \\ 7 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & -1 & 1 & & \\ 2 & -1 & & 1 & \\ 0 & 1 & & & 1 \end{bmatrix}$$

Current Dictionary

maximize	ζ	=	11	+	10	w_1	+	-7	w_2
subject to:	x_1	=	2	-	-1	w_1	-	1	w_2
	x_2	=	1	-	-2	w_1	-	1	w_2
	w_3	=	4	-	2	w_1	-	-1	w_2

$x_1 \quad x_2 \quad w_1 \quad w_2 \quad w_3 \geq 0$

$$\begin{pmatrix} -10 \\ 7 \end{pmatrix} = \left(\underline{B^{-1}N}\right)^T c_B - c_N = \left(\begin{bmatrix} -1 & 1 & 0 \\ -2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \right)^T \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \underline{B^{-1}b} = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

$$\begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix} = \underline{1^{st} \text{ col. of } B^{-1}N} = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

③

Step 9. The new sets of basic and nonbasic indices are

$$\mathcal{B} = \{1, 2, 5\} \qquad \text{and} \qquad \mathcal{N} = \{3, 4\}.$$

Corresponding to these sets, we have the new basic and nonbasic submatrices of A,

$$B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and the new basic primal variables and nonbasic dual variables:

$$x^*_{\mathcal{B}} = \begin{bmatrix} x^*_1 \\ x^*_2 \\ x^*_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \qquad z^*_{\mathcal{N}} = \begin{bmatrix} z^*_3 \\ z^*_4 \end{bmatrix} = \begin{bmatrix} -10 \\ 7 \end{bmatrix}.$$

x_3 enters

x_5 leaves

$$\begin{bmatrix} 1 & -1 & 1 & & \\ 2 & -1 & & 1 & \\ 0 & 1 & & & 1 \end{bmatrix}$$

Current Dictionary

maximize	ζ	=	11	+	10	w	+	-7	w		
subject to:	x_1	=	2	-	-1	w_1	-	1	w_2		
	x_2	=	1	-	-2	w_1	-	1	w_2		
	w_3	=	4	-	2	w_1	-	-1	w_2		

$x_1 \quad x_2 \quad w_1 \quad w_2 \quad w_3 \geq 0$

Opt.

Step 9. The new sets of basic and nonbasic indices are

$$\mathcal{B} = \{1, 2, 3\} \qquad \text{and} \qquad \mathcal{N} = \{5, 4\}.$$

Corresponding to these sets, we have the new basic and nonbasic submatrices of A,

$$B = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad N = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},$$

Optimal as $z_{\mathcal{N}}^* \geq 0$ ($x_{\mathcal{B}}^* \geq 0$)

and the new basic primal variables and nonbasic dual variables:

$$x_{\mathcal{B}}^* = \begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} \qquad z_{\mathcal{N}}^* = \begin{bmatrix} z_5^* \\ z_4^* \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

$\zeta^* = 31$

$$\begin{bmatrix} 1 & -1 & 1 & & \\ 2 & -1 & & 1 & \\ 0 & 1 & & & 1 \end{bmatrix}$$

Current Dictionary

maximize $\zeta = 31 + -5\, w + -2\, w$

subject to:

$x_1 = 4 - 1/2\, w_3 - 1/2\, w_2$

$x_2 = 5 - 1\, w_3 - 0\, w_2$

$w_1 = 2 - 1/2\, w_3 - -1/2\, w_2$

$x_1 \;\; x_2 \;\; w_1 \;\; w_2 \;\; w_3 \geq 0$

1st col. of $B^{-1}N$

$$B^{new} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = B^{old} \begin{bmatrix} 1 & 0 & \boxed{-1} \\ 0 & 1 & \boxed{-2} \\ 0 & 0 & \boxed{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ -2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \\ 1 & -1/2 & 1/2 \end{bmatrix}$$

$$\begin{pmatrix}5\\2\end{pmatrix} = (B^{-1}N)^T C_B - C_N$$

$$= \left(\begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \\ 1 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \right)^T \begin{pmatrix}4\\3\\0\end{pmatrix} - \begin{pmatrix}0\\0\end{pmatrix} \overset{\checkmark}{=} \begin{pmatrix}5\\2\end{pmatrix}$$

$$\begin{pmatrix}4\\5\\2\end{pmatrix} = B^{-1}\begin{pmatrix}1\\3\\5\end{pmatrix} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \\ 1 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix}1\\3\\5\end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix}4\\5\\2\end{bmatrix}$$

$$31 = C_B^T B^{-1} b = (4 \quad 3 \quad 0)\begin{pmatrix}4\\5\\2\end{pmatrix} \overset{\checkmark}{=} 31$$

Hint Undo

Number format: Fraction   Zero Visibility: Visible

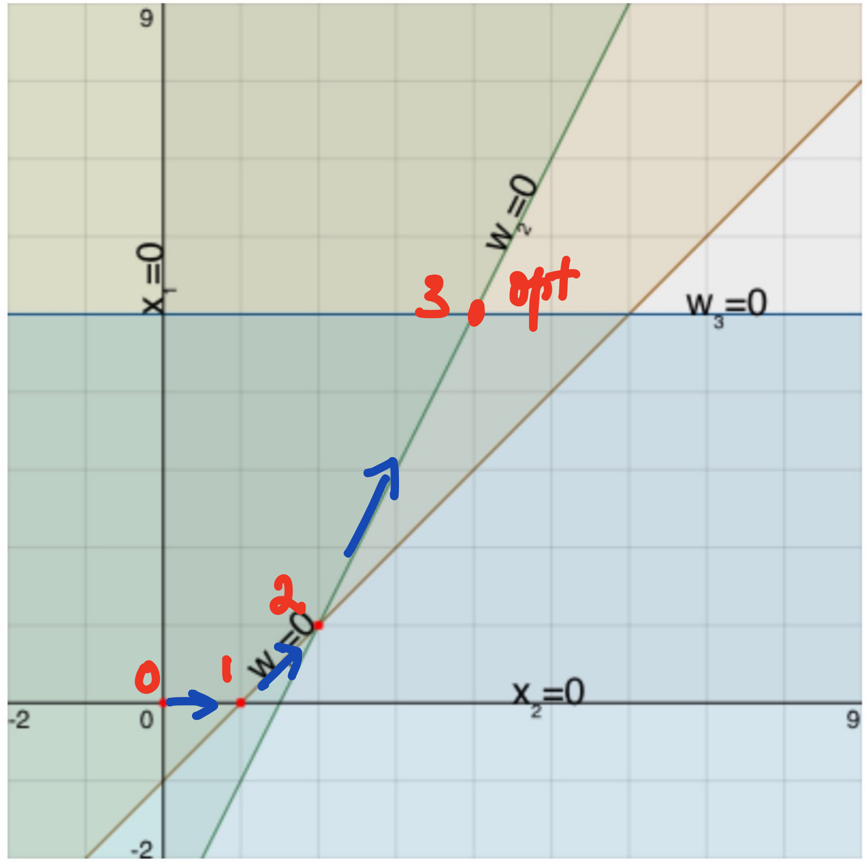
Current Dictionary

maximize	ζ	=	31	+	-5	w	+	-2	w	
subject to:	x_1	=	4	-	1/2	w_3	-	1/2	w_2	
	x_2	=	5	-	1	w_3	-	0	w_2	
	w_1	=	2	-	1/2	w_3	-	-1/2	w_2	

$x_1 \quad x_2 \quad w_1 \quad w_2 \quad w_3 \geq 0$

Optimal   Infeasible   Unbounded

Pick a judge: Bart Simpson



$$\zeta^* = 31 \left(= 4x_1^* + 3x_2^* = 4(4) + 3(5) \right)$$