Regression

Given $\{x_1, x_2, \cdots, x_m\}$ $x_i \in \mathbb{R}$

(1) $\min_x \frac{1}{m} \sum_{i=1}^{m} (x - x_i)^2 \Longrightarrow x = \frac{x_1 + \cdots + x_m}{m}$

L^2-error

(mean)

(2) $\min_x \frac{1}{m} \sum_{i=1}^{m} |x - x_i| \Longrightarrow x =$ median

L^1-error

(3) $\min_x \left(\max_i |x - x_i| \right) \Longrightarrow x = \frac{\min x_i + \max x_i}{2}$

L^∞-error

"mid-range"

Given attributes $\{a_1, a_2, \cdots, a_n\}$, b
find parameters $\{x_1, x_2, \cdots, x_n\}$ s.t.

$$b = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

$$\left(b = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n + \varepsilon \right)$$

error

Given data points (individuals)

$b_i, \ \{a_{ij}\}_{j=1}^{n}, \quad i = 1, \dots, m$

Find $x_1, x_2, \dots \dots x_n$ s.t.

$$b_i = \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, \dots, m$$

error

$$b_i = \sum_{j=1}^{n} a_{ij} x_j + \varepsilon_i \quad i = 1, \dots, m$$

$$\underline{b = AX + \varepsilon}$$

L^2-Regression (Least Square)

$$\min_{(x_1, \dots, x_n)} \sum_i \left[\left(b_i - \sum_j a_{ij} x_j \right)^2 \right]$$

$$\min_{x} \| b - Ax \|_2^2$$

$Ax = b$ (might not be solvable)

$A^T A \hat{x} = A^T b$ (normal equation, always solvable)

$\hat{x} = (A^T A)^{-1} A^T b$ (if $(A^T A)^{-1}$ exists)

L^1 - regression

$$\min_{(x_1,\dots,x_n)} \sum_i \left| b_i - \sum_j a_{ij} x_j \right|$$

$$\min_{x} \| b - Ax \|_1$$

$$\boxed{\begin{aligned} &\min_i \sum_{i=1}^{m} t_i \\ &\text{s.t.} \quad -t_i \le b_i - \sum_j a_{ij} x_j \le t_i \end{aligned}}$$

$$\left| b_i - \sum_j a_{ij} x_j \right| \le t_i$$

L^∞ - Regression

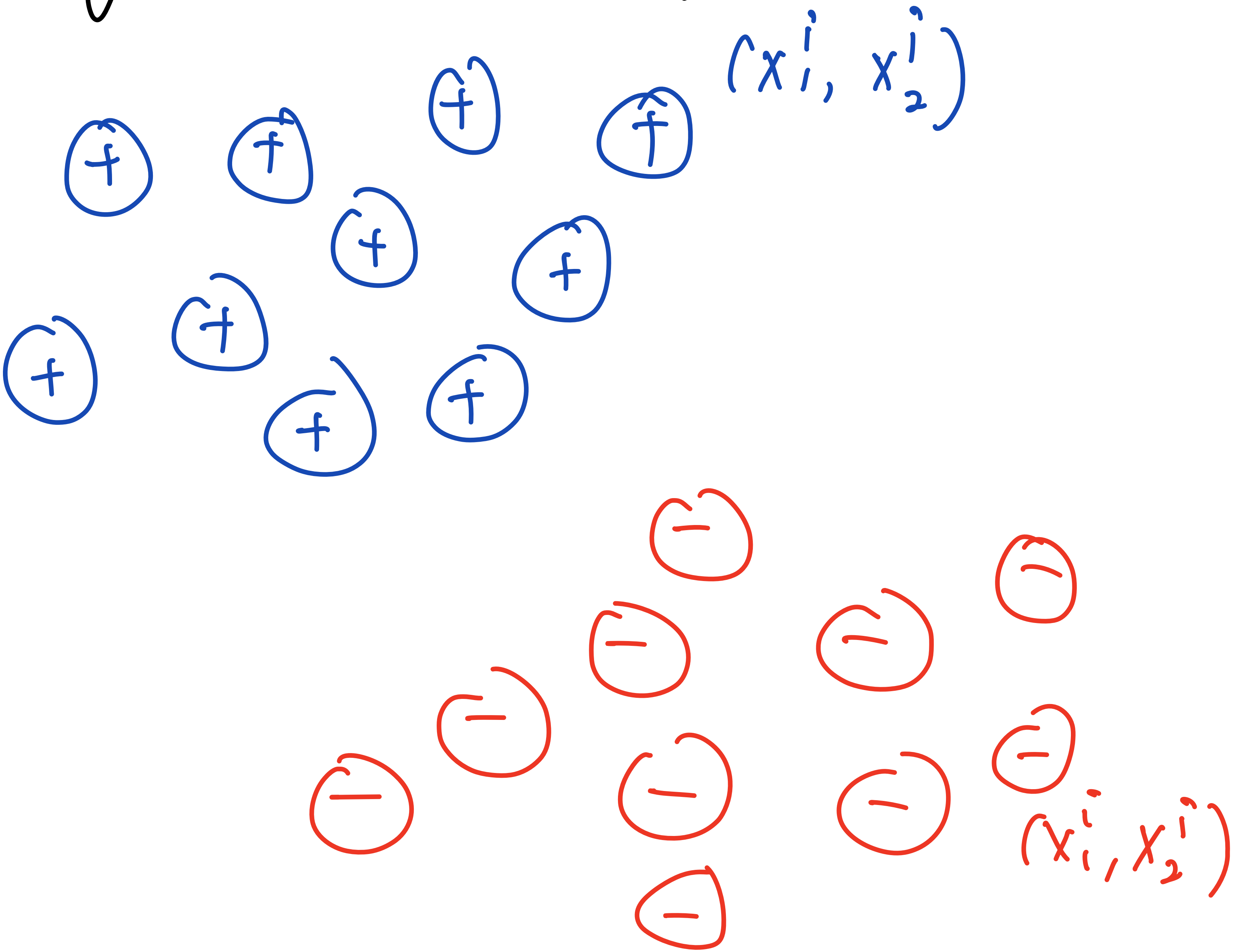
$$\min \left(\max_i \left| b_i - \sum_j a_{ij} x_j \right| \right)$$

$$\min_X \| b - AX \|_\infty$$

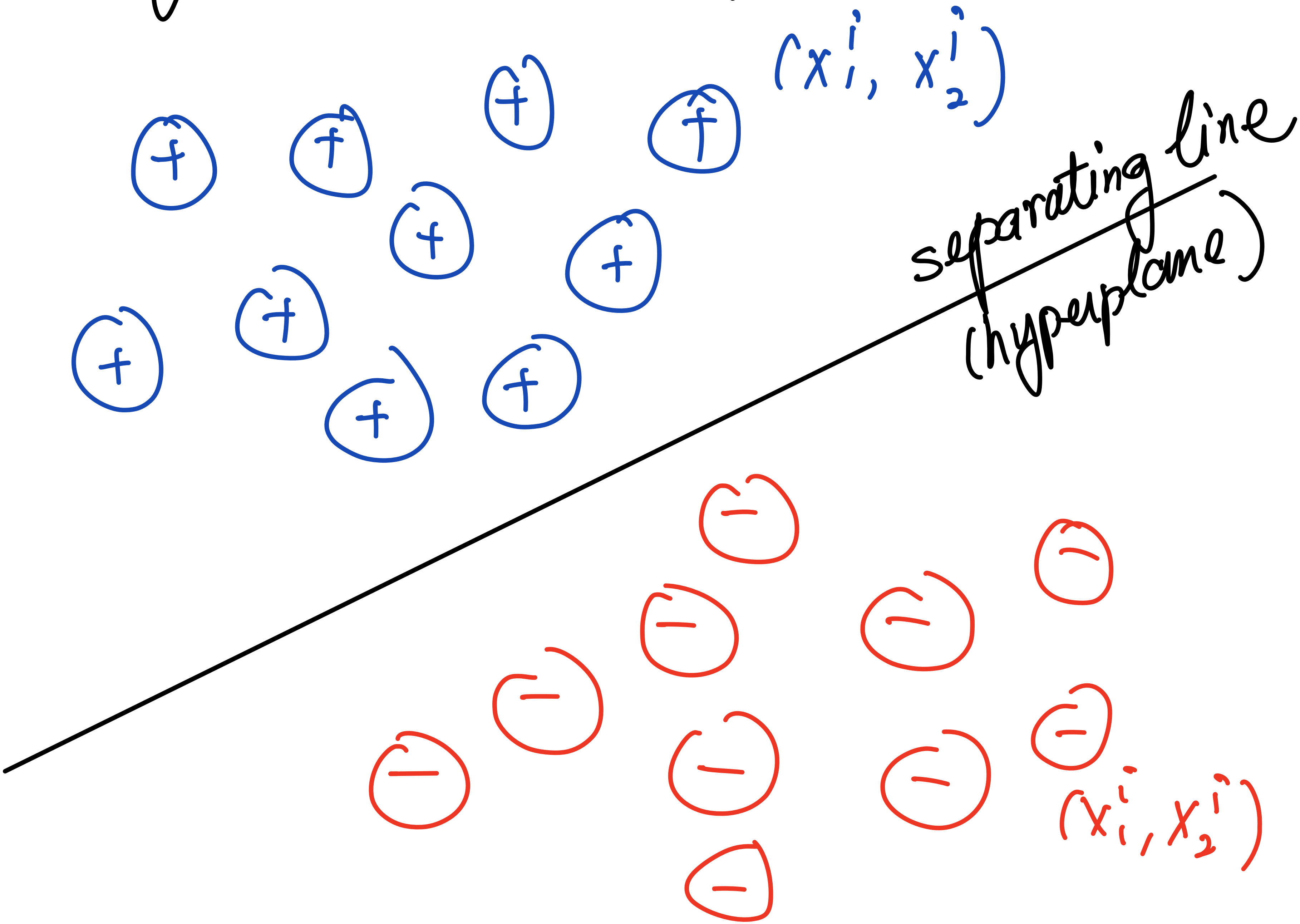
$$\boxed{\begin{aligned} &\min \quad \delta \\ &\text{s.t.} \quad -\delta \leq b_i - \sum_j a_{ij} x_j \leq \delta \end{aligned}}$$

$\left| b_i - \sum_j a_{ij} x_j \right| \leq \delta$ for all i

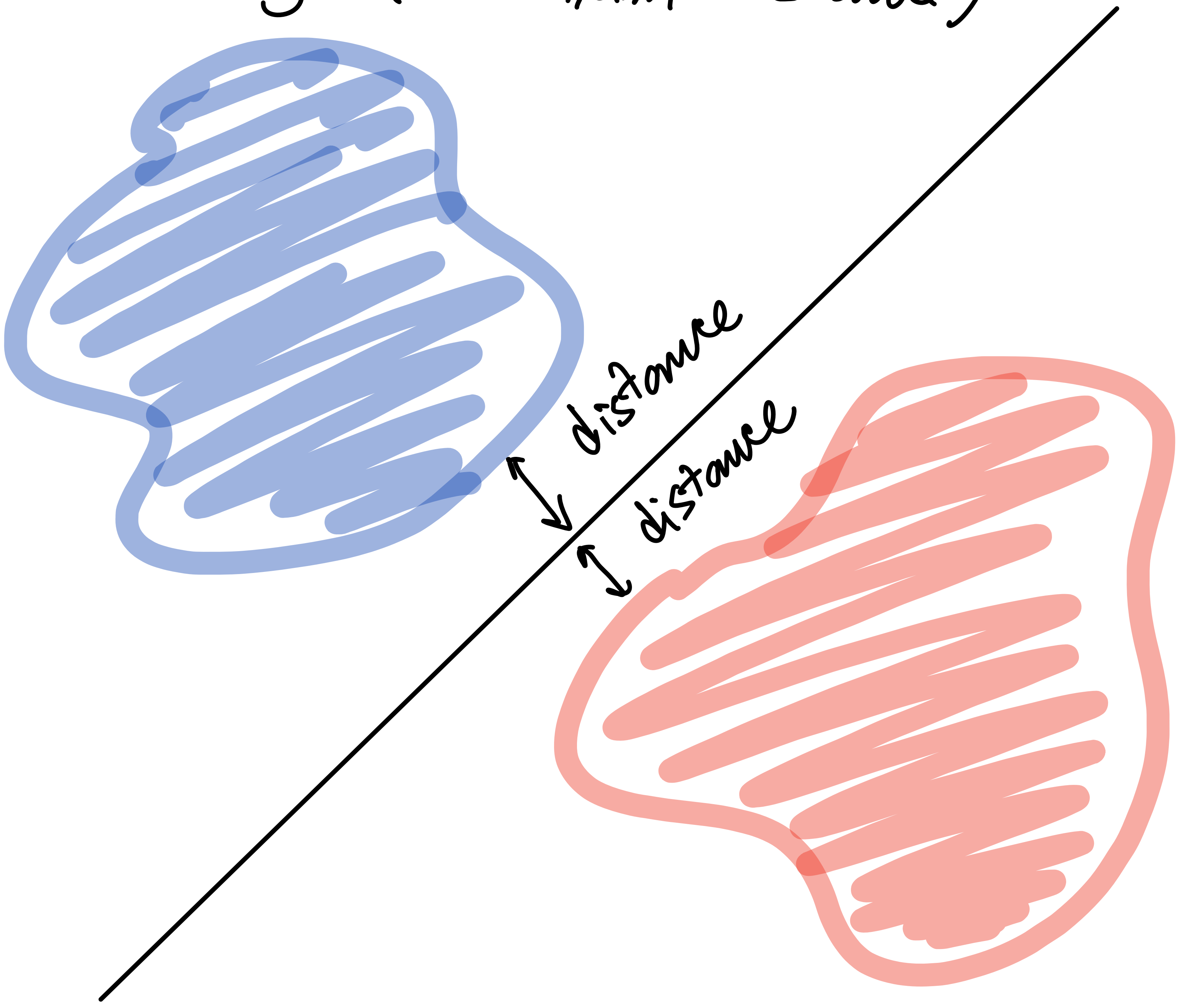
Binary Classification given labelled points



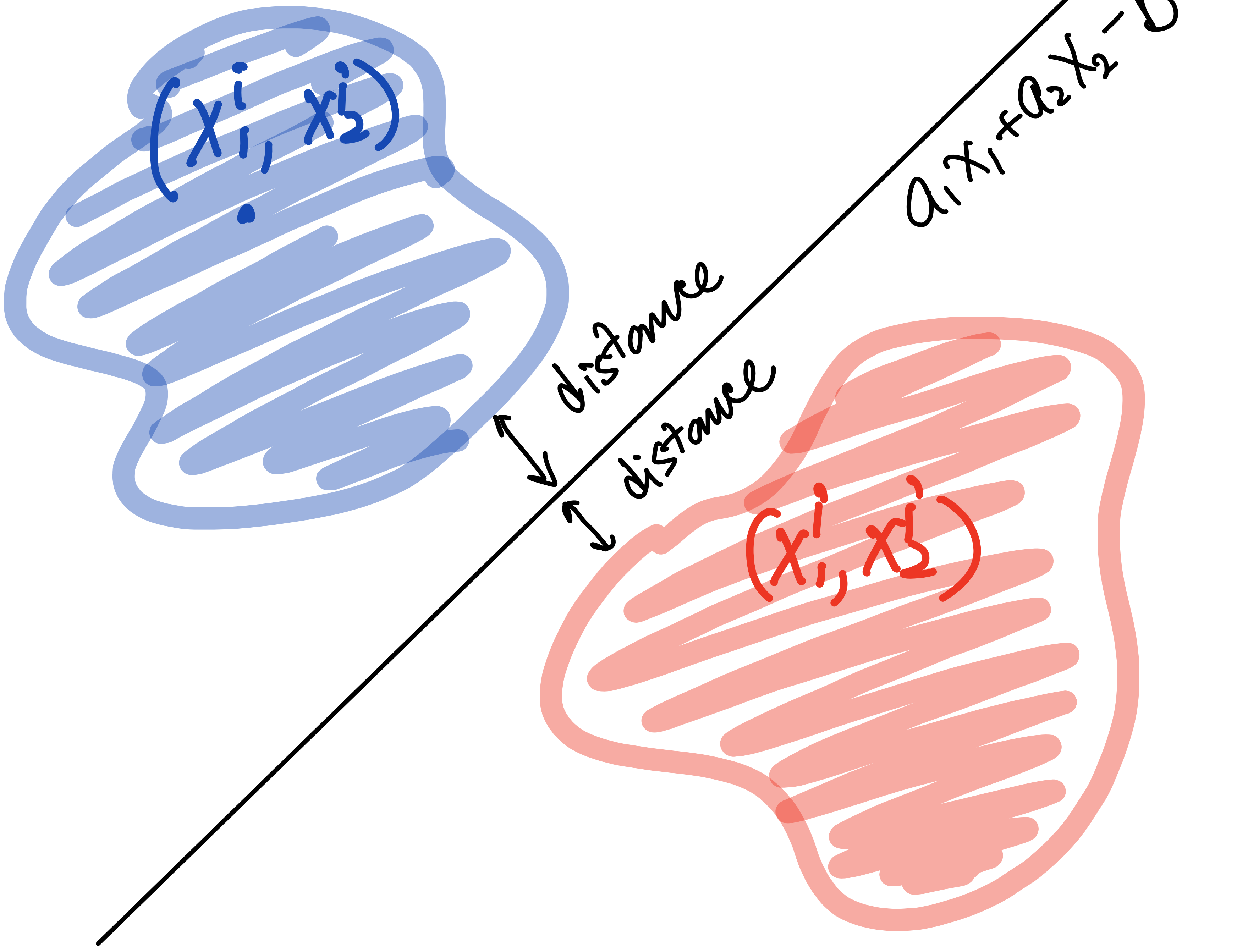
Binary Classification given labelled points



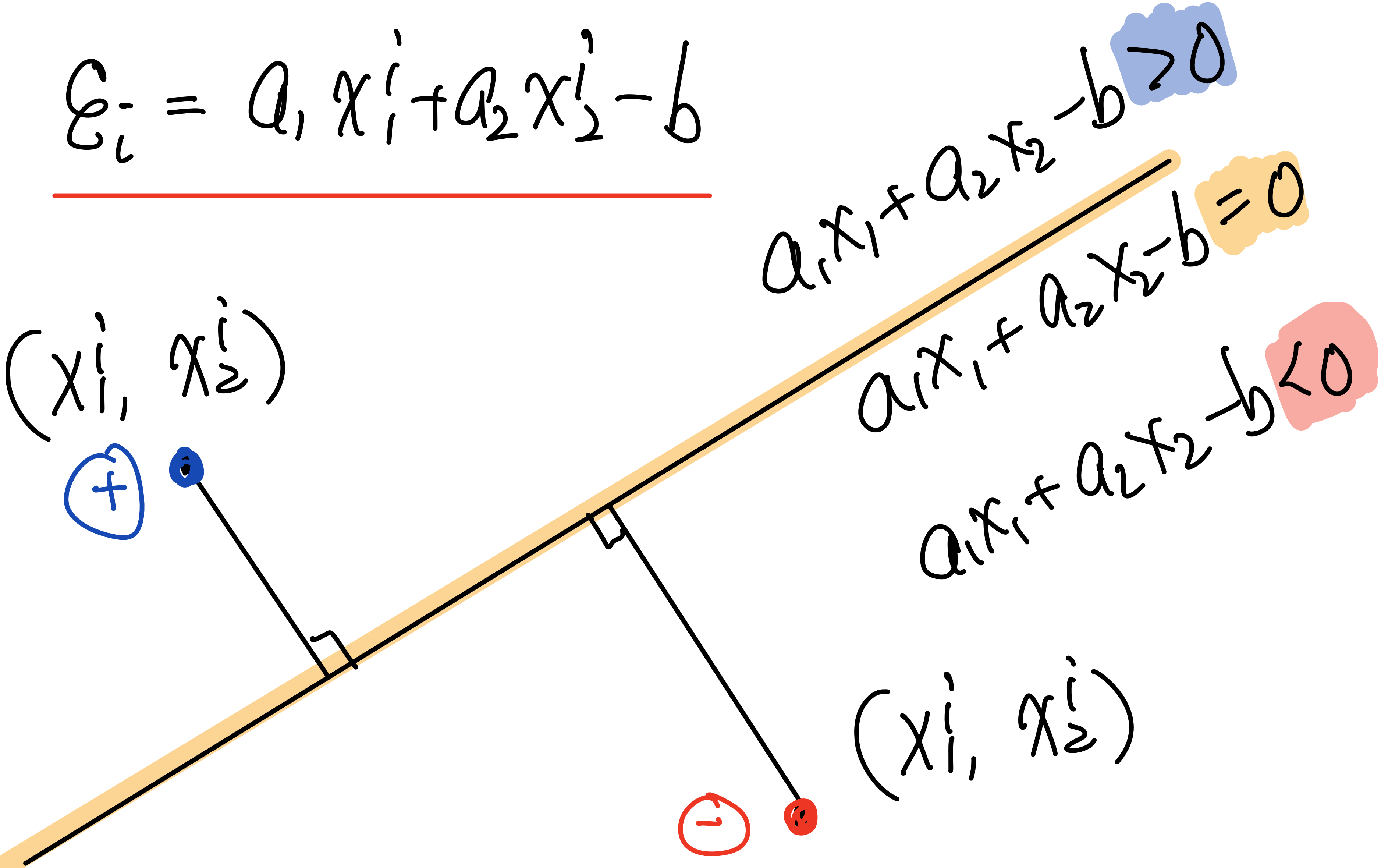
maximize (minimum distance)



maximize (minimum distance)



$$\varepsilon_i = a_1 x_1^i + a_2 x_2^i - b$$



$$\varepsilon_i = b - (a_1 x_1^i + a_2 x_2^i)$$

Distance of $\{+\}$ from $a_1x_1 + a_2x_2 - b$

$$= \min_{i,\,+} \left(a_1 x_1^i + a_2 x_2^i - b \right)$$

Distance of $\{-\}$ from $a_1x_1 + a_2x_2 - b$

$$= \min_{i,\,-} \left(b - \left(a_1 x_1^i + a_2 x_2^i \right) \right)$$

$$\max_{\delta,\ \varepsilon_i,\ a_1,\ a_2,\ b} \delta$$

s.t.

$$\varepsilon_i = a_1 x_1^i + a_2 x_2^i - b \quad \text{if } Label(i) = +$$

$$\varepsilon_i = b - a_1 x_1^i - a_2 x_2^i \quad \text{if } Label(i) = -$$

$$\delta \le \varepsilon_i$$

$$\max_{\delta,\ \varepsilon_i,\ a_1,\ a_2,\ b} \delta$$

s.t.

$$\varepsilon_i = a_1 x_1^i + a_2 x_2^i - b \quad \text{if } Label(i) = +$$

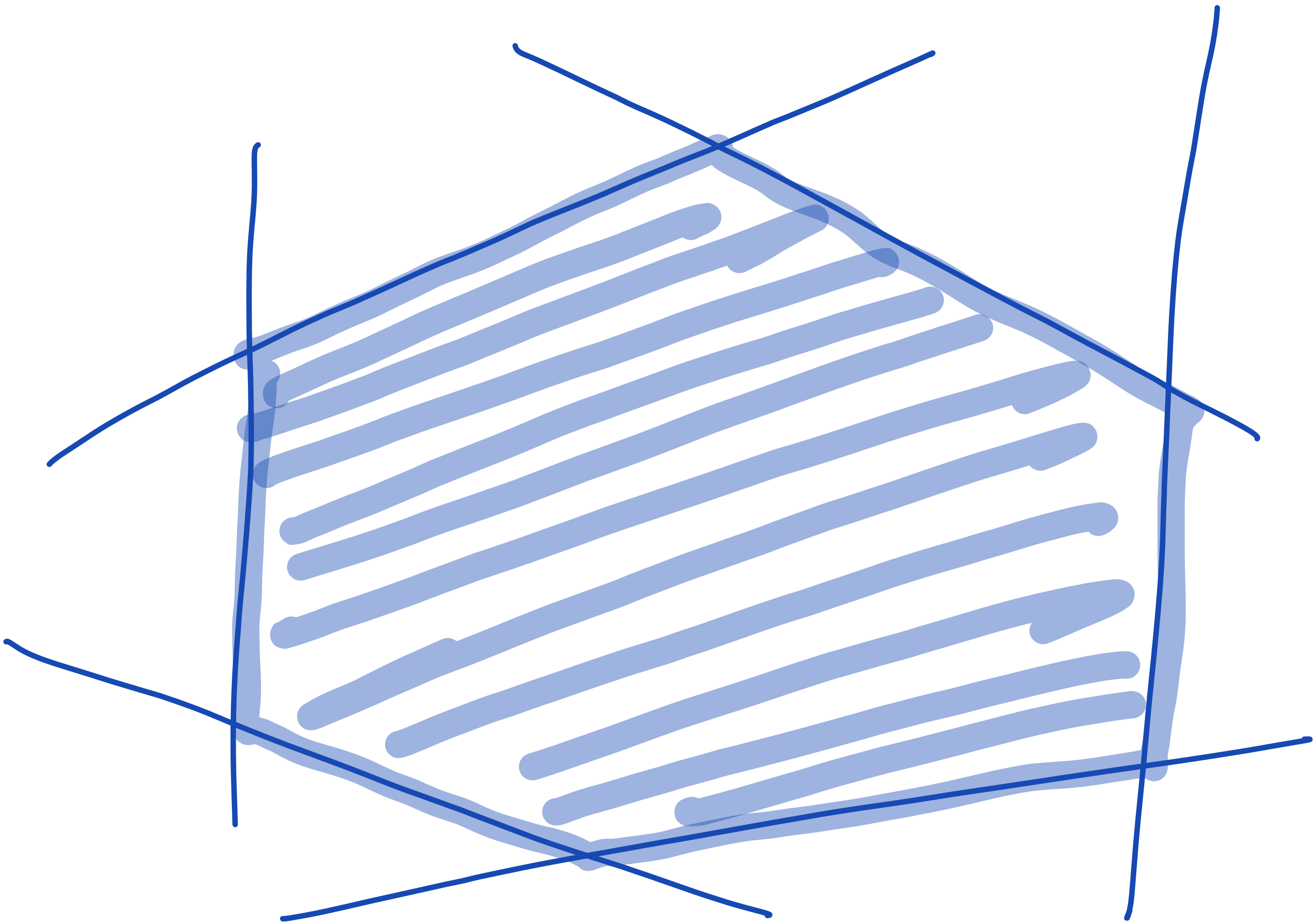
$$\varepsilon_i = b - a_1 x_1^i - a_2 x_2^i \quad \text{if } Label(i) = -$$

$$\delta \leq \varepsilon_i$$

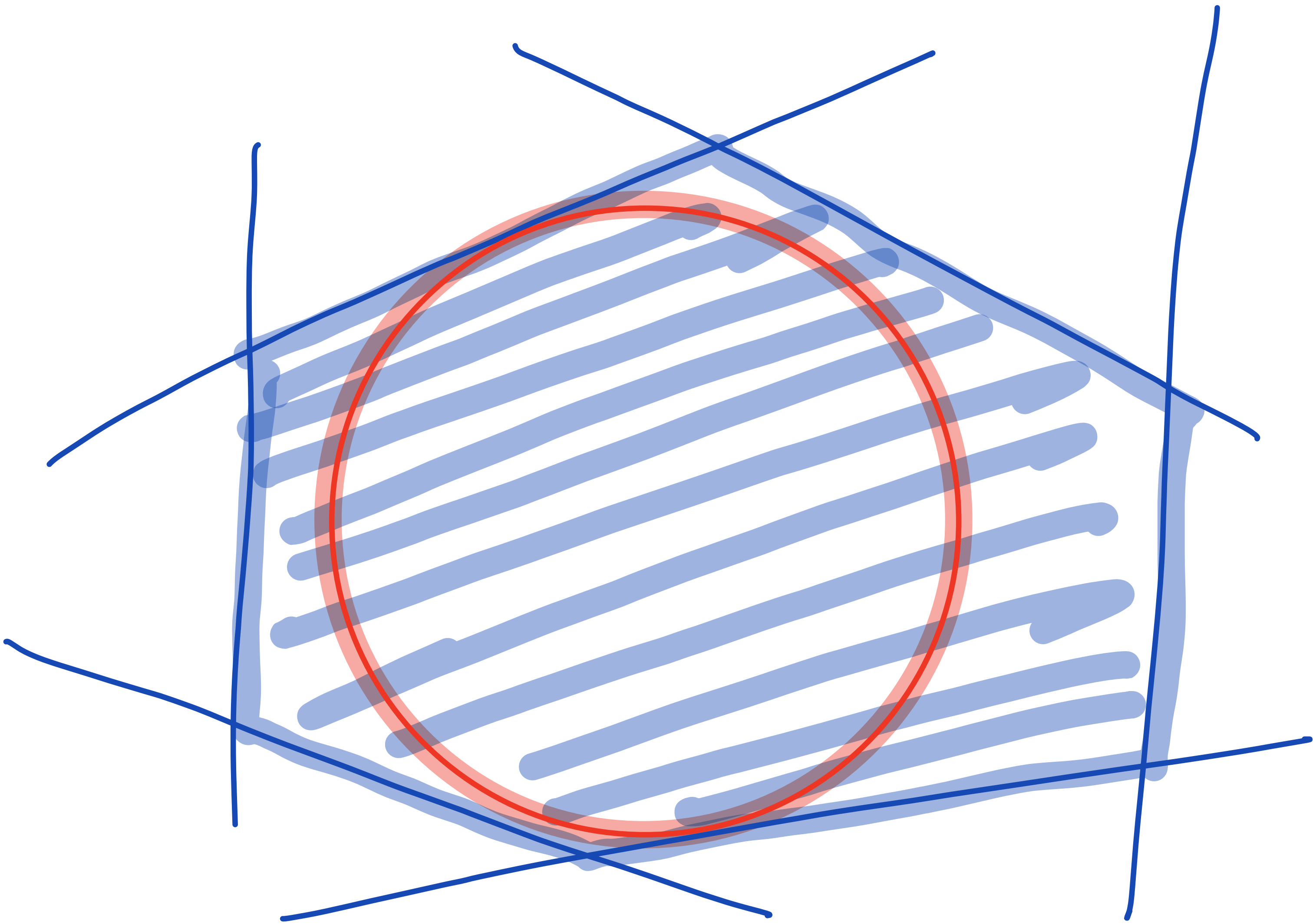
$$-1 \leq a_1 \leq 1, \quad -1 \leq a_2 \leq 1$$

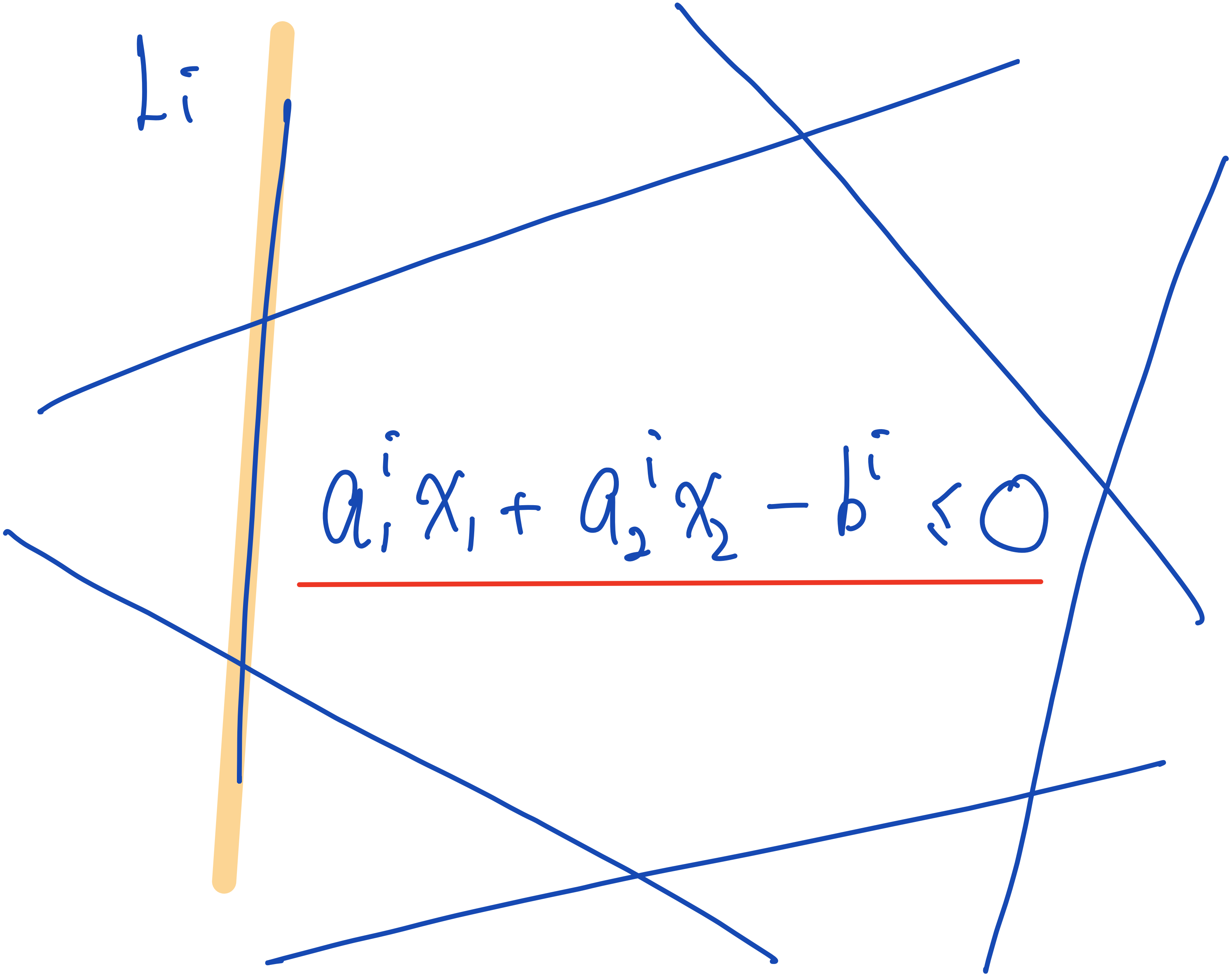
(normalization)

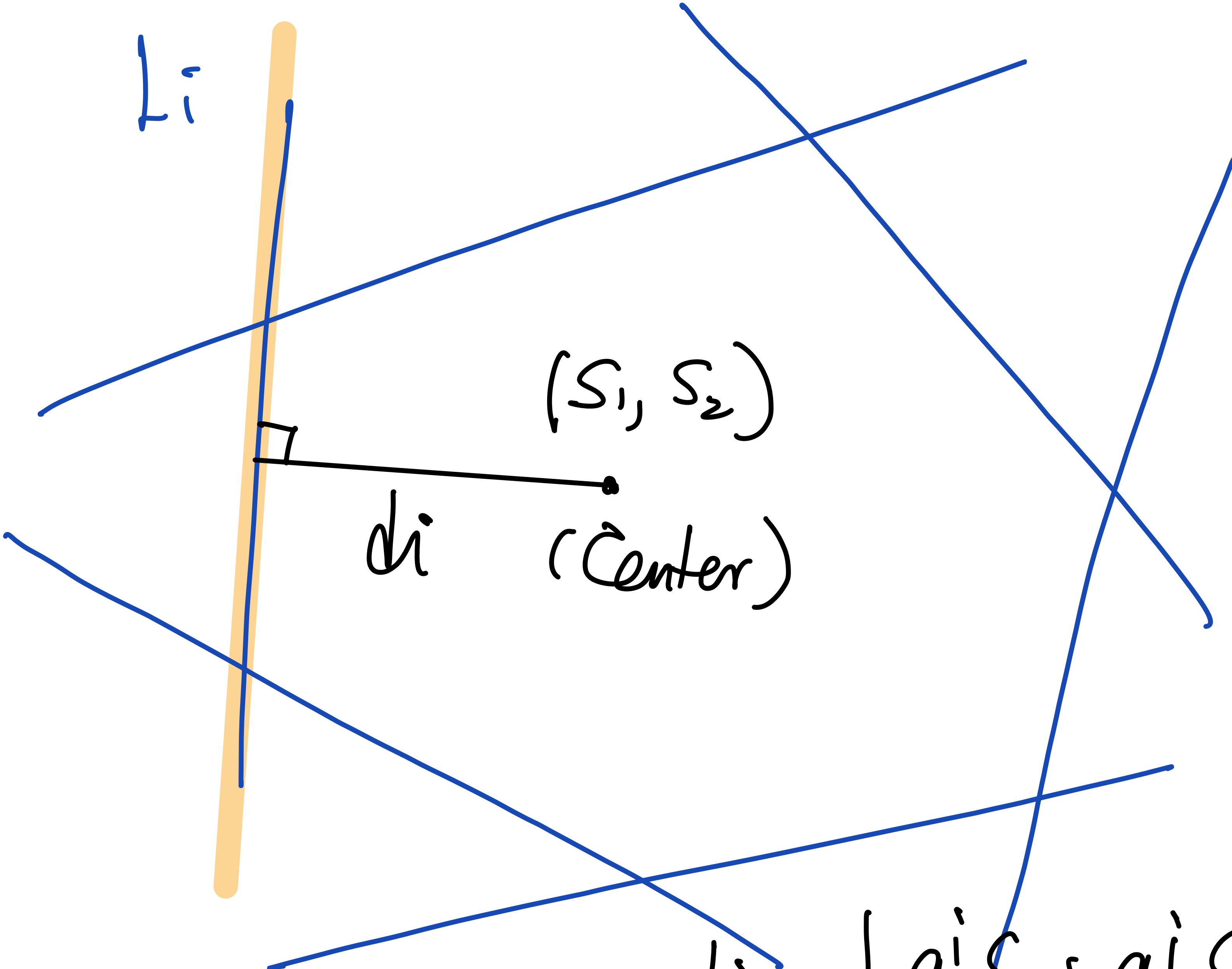
Maximum Inscribed Circle in a Convex Polyhedron [MG, Sec 2.6]



Maximum Inscribed Circle in a Convex Polyhedron [MG, Sec 2.6]




L_i
$a_1^i x_1 + a_2^i x_2 - b^i \le 0$



$$d_i = \left| \frac{a^i_1 s_1 + a^i_2 s_2 - b^i}{\sqrt{(a^i_1)^2 + (a^i_2)^2}} \right|$$

$$\underset{(s_1, s_2, d)}{\text{maximize}} \quad d$$

$$\text{s.t.} \quad d \leq d_i = \left| \frac{a_1^i s_1 + a_2^i s_2 - b^i}{\sqrt{(a_1^i)^2 + (a_2^i)^2}} \right|$$

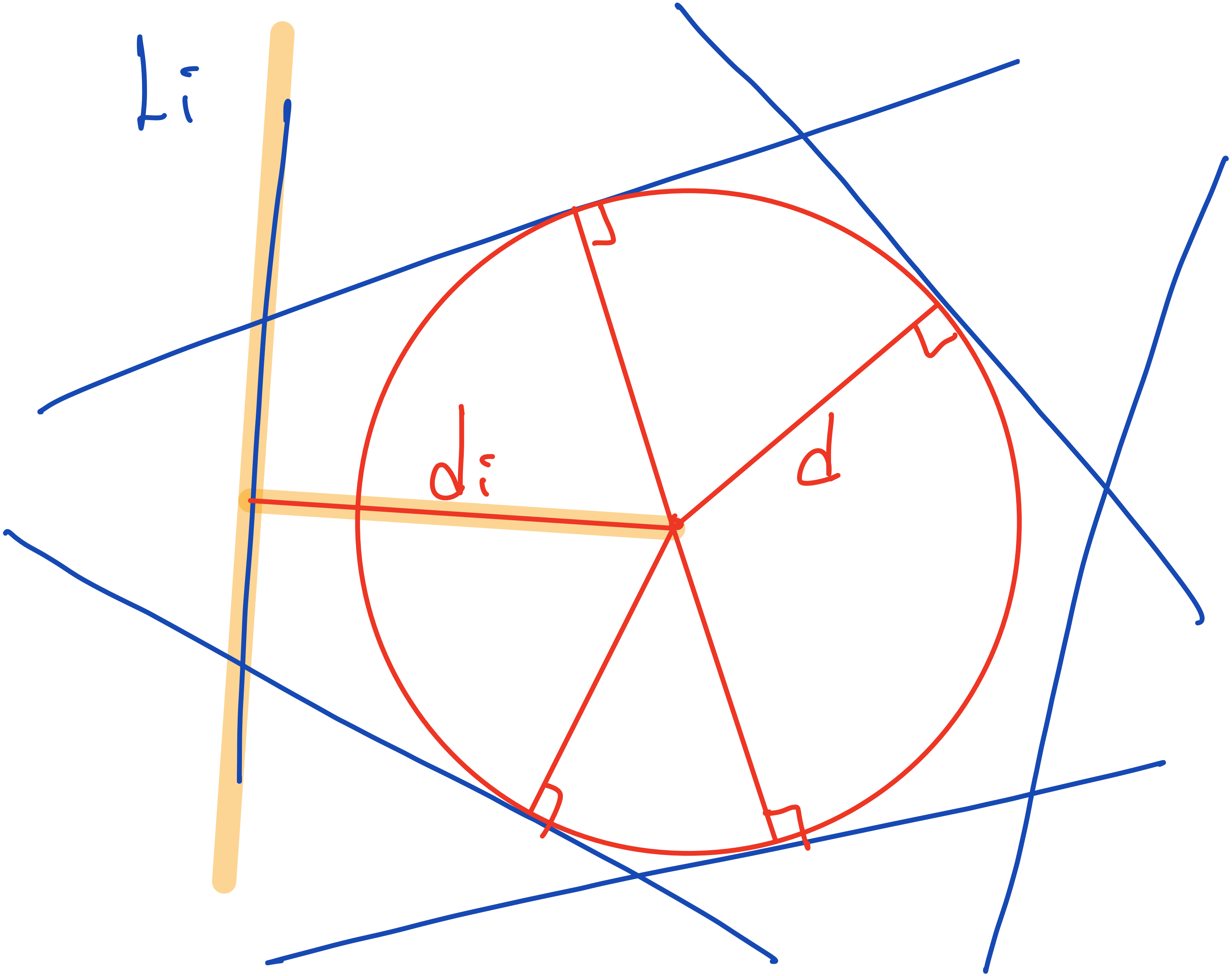
$$a_1^i s_1 + a_2^i s_2 - b_i \leq 0$$

$$\underset{(s_1, s_2, d)}{\text{maximize}} \quad d$$

$$\text{s.t.} \quad d \le \frac{b^i - a_1^i s_1 - a_2^i s_2}{\sqrt{(a_1^i)^2 + (a_2^i)^2}}$$

$$\cancel{a_1^i s_1 + a_2^i s_2 - b_i \le 0}$$

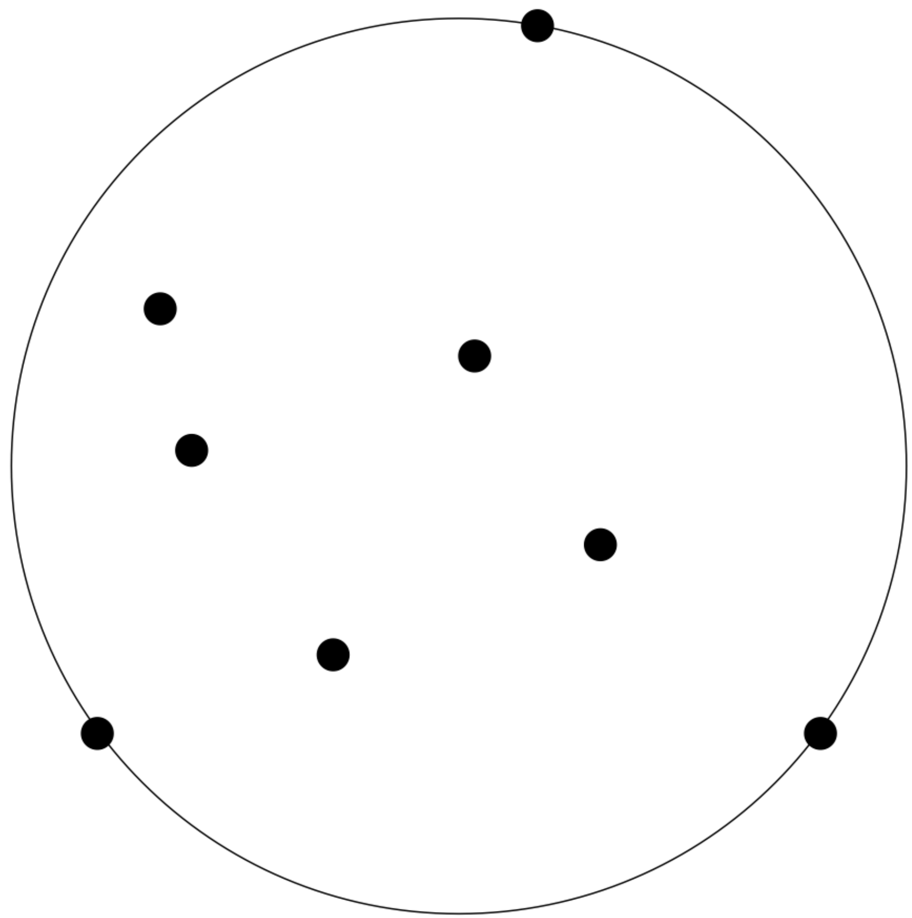
redundant


L_i
d_i
d

[MG]

8.7 Smallest Balls and Convex Programming

The smallest ball problem. We are given points $\mathbf{p}_1, \ldots, \mathbf{p}_n \in \mathbb{R}^d$, and we want to find a ball of the smallest radius that contains all the points.[8]



Sparse Solutions of Linear System [MG, p.167]

Sparse solution of underdetermined system of linear equations

Given an $m \times n$ matrix A with $m < n$, a vector $\mathbf{b} \in \mathbb{R}^m$, and an integer r, find an $\mathbf{x} \in \mathbb{R}^n$ such that

$$A\mathbf{x} = \mathbf{b} \quad \text{and} \quad \underbrace{|\mathrm{supp}(\mathbf{x})|}_{\#\{i : x_i \neq 0\}} \leq r \tag{8.10}$$

if one exists.

Sparse Solutions of Linear System [MG, p.167]

Sparse solution of underdetermined system of linear equations

Given an $m \times n$ matrix A with $m < n$, a vector $\mathbf{b} \in \mathbb{R}^m$, and an integer r, find an $\mathbf{x} \in \mathbb{R}^n$ such that

$$A\mathbf{x} = \mathbf{b} \quad \text{and} \quad \underbrace{|\mathrm{supp}(\mathbf{x})|}_{\#\{i:\, x_i \neq 0\}} \leq r \tag{8.10}$$

if one exists.

Recovery of signals with noise:

$w \in \mathbb{R}^k \Longrightarrow$ encode $z = Qw$

$\Longrightarrow$ transmit $\tilde{z} = z + x$

noise: only a few zeros

Given $\tilde{z}$, how to recover z?

$$\tilde{z} = Qw + x$$

$$A\tilde{z} = \cancel{AQw} + Ax$$

Sparse Solutions of Linear System [MG, p.167]

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Given an $m \times n$ matrix A with $m < n$, a vector $\mathbf{b} \in \mathbb{R}^m$, and an integer r, find an $\mathbf{x} \in \mathbb{R}^n$ such that

$$A\mathbf{x} = \mathbf{b} \quad \text{and} \quad \underbrace{|\mathrm{supp}(\mathbf{x})|}_{\#\{i \,:\, x_i \neq 0\}} \leq r \tag{8.10}$$

if one exists.

$$\underbrace{A\tilde{z}}_{b \text{ given}} = \cancel{AQW} + AX \implies AX = b$$

find X with as few non-zeros as possible

$$\tilde{z} = z + X \implies z = \tilde{z} - X$$

Sparse Solutions of Linear System [MG, p.167]

Sparse solution of underdetermined system of linear equations

Given an $m \times n$ matrix A with $\underline{m < n}$, a vector $\mathbf{b} \in \mathbb{R}^m$, and an integer r, find an $\mathbf{x} \in \mathbb{R}^n$ such that

$$A\mathbf{x} = \mathbf{b} \quad \text{and} \quad \underbrace{|\text{supp}(\mathbf{x})|}_{\#\{i \,:\, x_i \neq 0\}} \leq r \tag{8.10}$$

if one exists.

Data Compression:

$$\underbrace{A}_{} \underset{\text{representation}}{X} = \underset{\text{data}}{b} \qquad A^{m\times n},\ b \in \mathbb{R}^m$$

$$\underline{x_1} u_1 + \underline{x_2} u_2 + \cdots + \underline{x_n} u_n = b$$

coordinates of b w.r.t. cols of A

Sparse Solutions of Linear System [MG, p.167]

Sparse solution of underdetermined system of linear equations

Given an $m \times n$ matrix A with $m < n$, a vector $\mathbf{b} \in \mathbb{R}^m$, and an integer r, find an $\mathbf{x} \in \mathbb{R}^n$ such that

$$A\mathbf{x} = \mathbf{b} \quad \text{and} \quad |\text{supp}(\mathbf{x})| \le r \tag{8.10}$$

$\#\{i : x_i \neq 0\}$

if one exists.

$$\min \|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$

$$\text{s.t. } Ax = b$$

Sparse Solutions of Linear System [MG, p.167]

Sparse solution of underdetermined system of linear equations

Given an $m \times n$ matrix A with $\underline{m < n}$, a vector $\mathbf{b} \in \mathbb{R}^m$, and an integer r, find an $\mathbf{x} \in \mathbb{R}^n$ such that

$$A\mathbf{x} = \mathbf{b} \quad \text{and} \quad \underbrace{|\mathrm{supp}(\mathbf{x})|}_{\#\{i \,:\, x_i \neq 0\}} \leq r \tag{8.10}$$

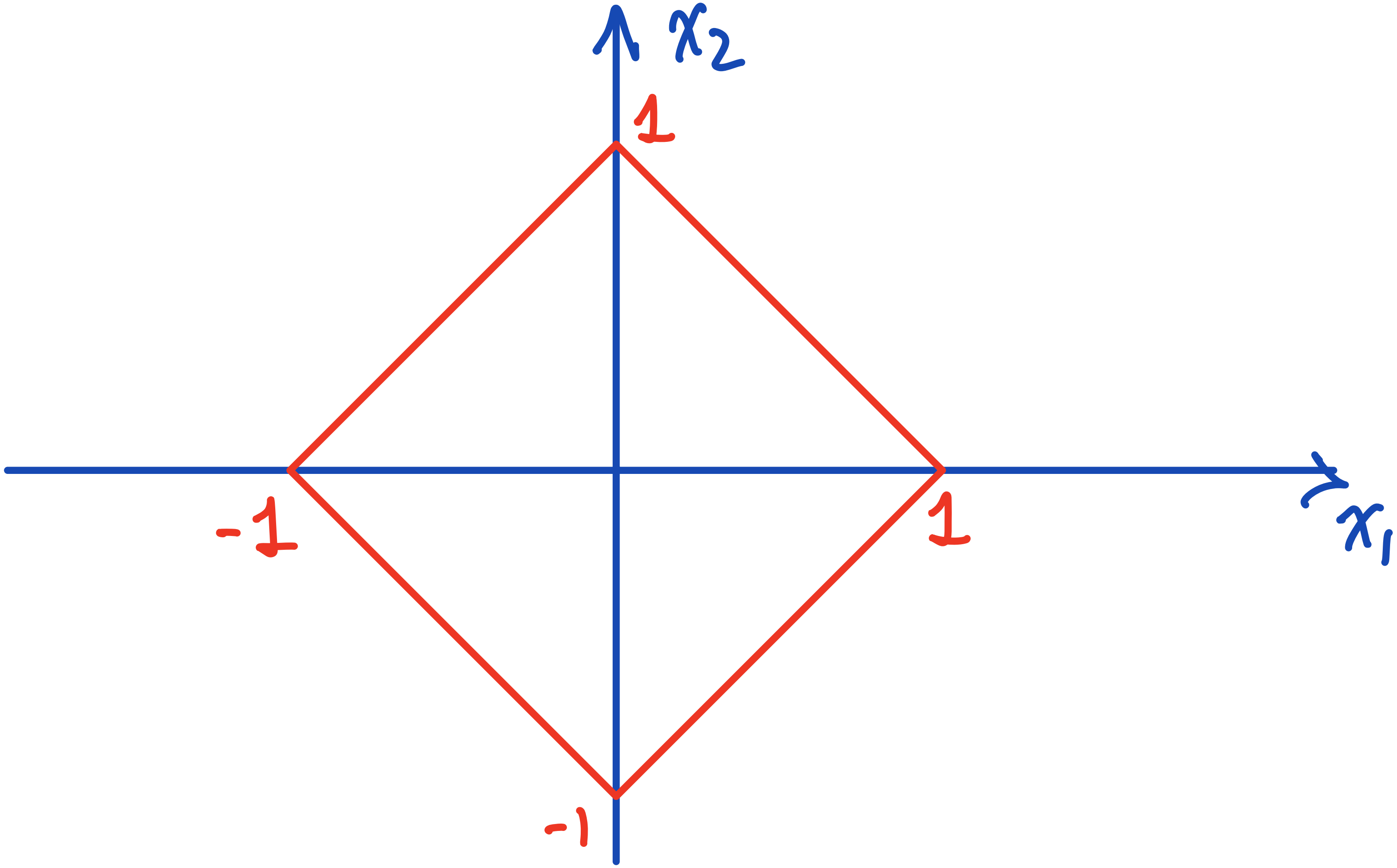
if one exists.

$$\min \; \|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$
$$\text{s.t.} \quad Ax = b$$

$$\left(\begin{array}{ll} \min & u_1 + u_2 + \cdots + u_n \\ \text{s.t.} & -u_i \leq x_i \leq u_i, \quad i = 1, 2, \cdots, n \\ & Ax = b \end{array} \right)$$

Sparse Solutions of Linear System [MG, p.167]

$n=2 \qquad \|x\|_1 = |x_1| + |x_2| = 1$



Sparse Solutions of Linear System [MG, p.167]

$$\min \quad \|x\|_1 \quad \text{s.t.} \quad Ax = b$$