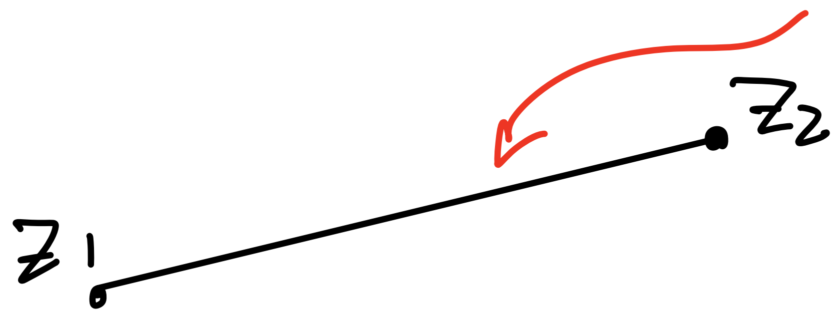
Convex Analysis [V] Chapter 10

Def 1 $S \subseteq \mathbb{R}^m$ is convex if

for any $z_1, z_2 \in S$, $\Rightarrow t z_1 + (1-t) z_2 \in S$

$0 \le t \le 1$

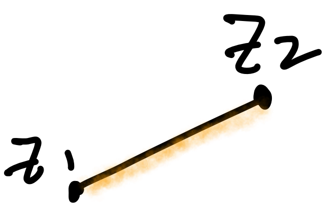


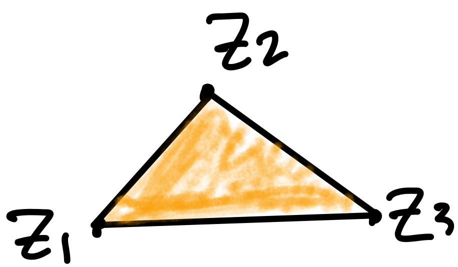
Def 2 Convex combination of $z_1, z_2, \cdots, z_n \in \mathbb{R}^m$

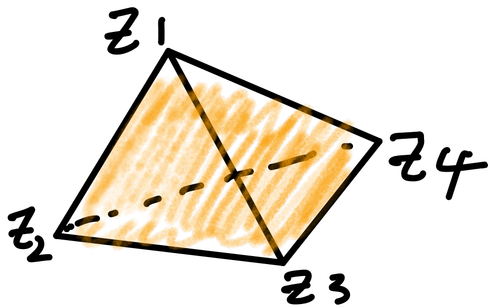
$t_1 z_1 + t_2 z_2 + \cdots + t_n z_n$

$t_i \geq 0$

$\sum_i t_i = 1$







Thm 10.1 S is convex iff it contains all conv. comb. of points in S

Pf "$\Longleftarrow$"

Suppose S contain all conv. comb. of pts in S,
then clearly, for any $z_1, z_2 \in S$

$$\underbrace{t z_1 + (1-t) z_2}_{\text{conv. comb. of } z_1,\, z_2} \in S$$

$\Rightarrow$ S is convex

Thm 10.1 S is convex iff it contains all conv. comb. of points in S

Pf "$\Longrightarrow$" Suppose S is convex.

$n=2$: $z_1, z_2 \in S \Longrightarrow t_1 z_1 + t_2 z_2 \in S$, $t_1, t_2 \geq 0$, $t_1+t_2=1$

$n=3$: $z_1, z_2, z_3 \in S \Longrightarrow$

$$t_1 z_1 + t_2 z_2 + t_3 z_3 = (t_1+t_2)\underbrace{\left(\frac{t_1}{t_1+t_2} z_1 + \frac{t_2}{t_1+t_2} z_2\right)}_{\in S}$$

$$+ t_3 z_3 \in S$$

$t_1+t_2+t_3=1$

$t_1+t_2 \geq 0$, $t_3 \geq 0$

Thm 10.1 S is convex iff it contains all conv. comb. of points in S

Pf "$\Longrightarrow$" Suppose S is convex.

$n=2$: $z_1, z_2 \in S \Rightarrow t_1 z_1 + t_2 z_2 \in S$, $t_1, t_2 \geq 0$, $t_1 + t_2 = 1$

$n=4$: $z_1, z_2, z_3, z_4 \in S$

$$t_1 z_1 + t_2 z_2 + t_3 z_3 + t_4 z_4 = (t_1 + t_2 + t_3)\left(\overbrace{\frac{t_1 z_1 + t_2 z_2 + t_3 z_3}{t_1 + t_2 + t_3}}^{\in S}\right) + t_4 z_3 \in S$$

$(t_1 + t_2 + t_3) + t_4 = 1$

$t_1 + t_2 + t_3 \geq 0$, $t_4 \geq 0$

Def $S \subseteq \mathbb{R}^m$, Convex Hull of S,

$$\mathrm{Conv}(S) = \bigcap_{\substack{G \text{ convex},\ S \subseteq G}} G$$

(= Smallest convex set containing S)

(Note intersection of convex set is convex.)

Thm 10.2 $S \subseteq \mathbb{R}^m$, $\mathrm{Conv}(S)$ = collection of all conv. comb. of finitely many pts from S

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Pf. Let H = collection of all conv. comb. of finitely many pts from S

Need to show $\mathrm{Conv}(S) = H$

(i) $\mathrm{Conv}(S) \subseteq H$ ($\because$ H is conv.)

Thm 10.2 $S \subseteq \mathbb{R}^m$, $Conv(S)$ = collection of all conv. comb. of finitely many pts from S

Pf. Let H = collection of all conv. comb. of finitely many pts from S

Need to show $Conv(S) = H$

(2) $H \subseteq Conv(S)$

Thm 10.1, contains all conv. comb. of pts from S

Caratheodory Theorem

THEOREM 10.3. *The convex hull* conv(S) *of a set* S *in* $\mathbb{R}^m$ *consists of all convex combinations of* $m+1$ *points from* S*:*

$$\operatorname{conv}(S) = \left\{ z = \sum_{j=1}^{m+1} t_j z_j : z_j \in S \textit{ and } t_j \geq 0 \textit{ for all } j, \textit{ and } \sum_j t_j = 1 \right\}.$$

Pf $\operatorname{Conv}(S) = \left\{ z = \sum_{j=1}^{n} x_j z_j ,\ z_j \in S,\ x_j \geq 0,\ \sum_j x_j = 1 \right\}$

Consider the LP: $\max \quad c^T X$

$\text{s.t.} \quad AX = z$

$x_1 + x_2 + \cdots + x_n = 1$

$x_1, x_2, \ldots, x_n$

(z is given, fixed, $A = [z_1\ z_2\ \cdots\ z_n]^{m \times n}$)

Caratheodory Theorem

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Pf $\operatorname{Conv}(S) = \left\{ z = \sum_{j=1}^{n} x_j z_j ,\ z_j \in S,\ x_j \geq 0,\ \sum_j x_j = 1 \right\}$

Consider the LP: max $c^T x$

s.t. $\left.\begin{array}{l} Ax = z \\ x_1 + x_2 + \dots + x_n = 1 \end{array}\right\}$ $m+1$ constraints.

$x_1, x_2, \dots, x_n$

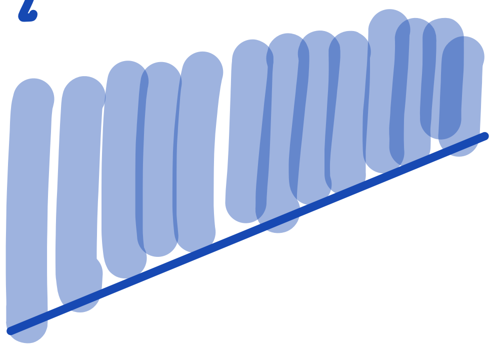
This is feasible, and hence has a basic opt. soln with $m+1$ basic variables.

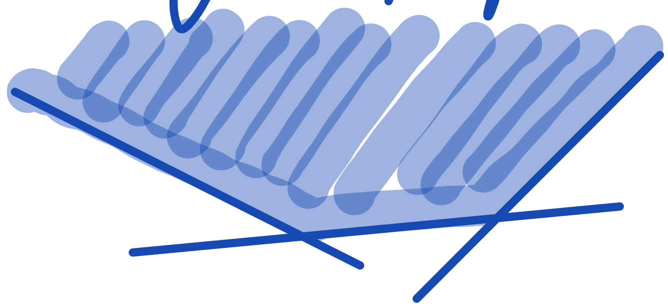
Separation Theorem

THEOREM 10.4. *Let P and $\tilde{P}$ be two disjoint nonempty polyhedra in $\mathbb{R}^n$. Then there exist disjoint half-spaces H and $\tilde{H}$ such that $P \subset H$ and $\tilde{P} \subset \tilde{H}$.*

$\{x : v^T x \le c\}$

intersection of finitely many half spaces





$\{x : Ax \le b\}$

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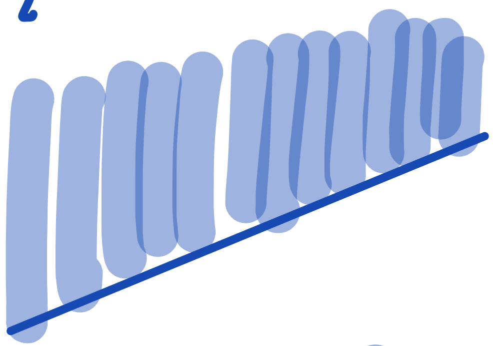

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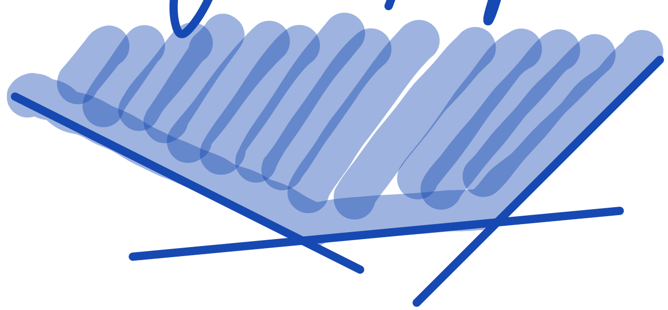

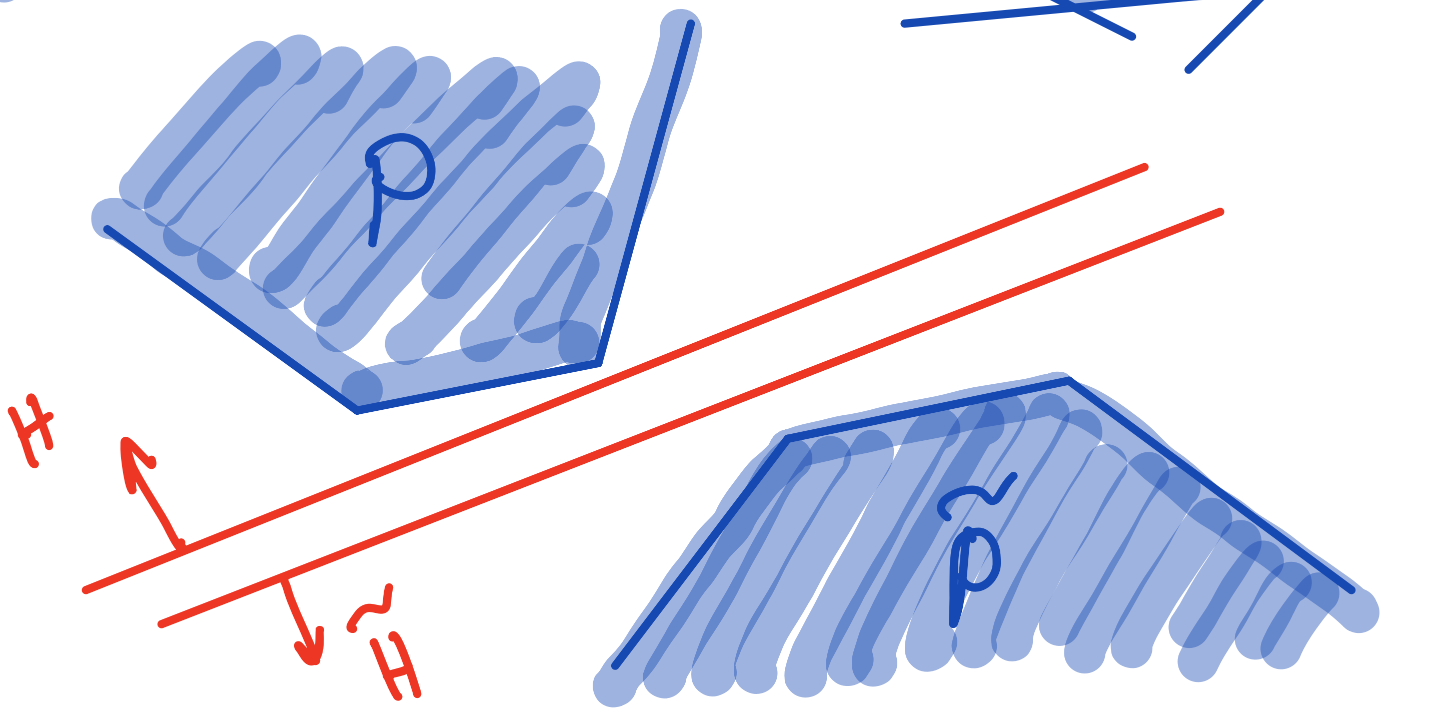

Separation Theorem

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Pf $P = \{X : AX \leq b\}$, $\tilde{P} = \{X : \tilde{A}X \leq \tilde{b}\}$

$$P \cap \tilde{P} = \phi \iff \begin{cases} AX \leq b \\ \tilde{A}\tilde{X} \leq \tilde{b} \end{cases} \text{ has no solution}$$

(Farka's Lemma) $\iff$ There is a $y, \tilde{y}$ s.t.

$$y^T A + \tilde{y}^T \tilde{A} = 0$$
$$y^T b + \tilde{y}^T \tilde{b} < 0$$
$$y, \tilde{y} \geq 0$$

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Farkas' Lemma

LEMMA 10.5. *The system $Ax \leq b$ has no solutions if and only if there is a y such that*

$$(10.8) \qquad \begin{aligned} A^T y &= 0 \\ y &\geq 0 \\ b^T y &< 0. \end{aligned} \qquad \left(\iff \begin{aligned} y^T A &= 0 \\ y &\geq 0 \\ y^T b &< 0 \end{aligned} \right)$$

Separation Theorem

THEOREM 10.4. *Let P and $\tilde{P}$ be two disjoint nonempty polyhedra in $\mathbb{R}^n$. Then there exist disjoint half-spaces H and $\tilde{H}$ such that $P \subset H$ and $\tilde{P} \subset \tilde{H}$.*

Pf $P = \{X : AX \le b\}$, $\tilde{P} = \{X : \tilde{A}X \le \tilde{b}\}$

There is a $y, \tilde{y}$ s.t. $\begin{cases} y^T A + \tilde{y}^T \tilde{A} = 0 \\ y^T b + \tilde{y}^T \tilde{b} < 0 \\ y, \tilde{y} \ge 0 \end{cases}$

Define

$H := \{X : y^T A X \le y^T b\}$, $\tilde{H} := \{X : \tilde{y}^T \tilde{A} X \le \tilde{y}^T \tilde{b}\}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

(textbook) $\{X : y^T A X \ge -\tilde{y}^T \tilde{b}\}$

Separation Theorem

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Pf $P = \{x : Ax \leq b\}$, $\tilde{P} = \{x : \tilde{A}x \leq \tilde{b}\}$

There is a $y, \tilde{y}$ s.t. $\begin{cases} y^T A + \tilde{y}^T \tilde{A} = 0 \\ y^T b + \tilde{y}^T \tilde{b} < 0 \\ y, \tilde{y} \geq 0 \end{cases}$

Define

$H := \{x : y^T A x \leq y^T b\}$, $\tilde{H} := \{x : \tilde{y}^T \tilde{A} x \leq \tilde{y}^T \tilde{b}\}$

$H \cap \tilde{H} = \phi$, $P \subseteq H$, $\tilde{P} \subseteq \tilde{H}$

Separation Theorem

$$P = \{x : Ax \le b\}, \quad \tilde{P} = \{x : \tilde{A}x \le \tilde{b}\}$$

$$H := \{x : y^T A x \le y^T b\}, \quad \tilde{H} := \{x : \tilde{y}^T \tilde{A} x \le \tilde{y}^T \tilde{b}\}$$

$P \subseteq H$: $\quad Ax \le b \Rightarrow y^T A x \le y^T b \quad (\because y \ge 0)$

$\tilde{P} \subseteq \tilde{H}$: $\quad \tilde{A}x \le \tilde{b} \Rightarrow \tilde{y}^T \tilde{A} x \le \tilde{y}^T b \quad (\because \tilde{y} \ge 0)$

$H \cap \tilde{H} = \phi$:

$$
\begin{array}{rl}
 & y^T A x \le y^T b \\
+ & \tilde{y}^T \tilde{A} x \le \tilde{y}^T b \\
\hline
\end{array}
$$

$$0 = \underbrace{(y^T A + \tilde{y}^T \tilde{A})}_{0} x \le y^T b + \tilde{y}^T b < 0 \quad !!$$

Farkas' Lemma

LEMMA 10.5. *The system $Ax \leq b$ has no solutions if and only if there is a y such that*

$$A^T y = 0$$
$$y \geq 0 \tag{10.8}$$
$$b^T y < 0.$$

PROOF. Consider the linear program

(P)
$$\begin{array}{ll} \text{maximize} & 0 \\ \text{subject to} & Ax \leq b \end{array}$$

and its dual

(D)
$$\begin{array}{lrcl} \text{minimize} & b^T y & & \\ \text{subject to} & A^T y & = & 0 \\ & y & \geq & 0. \end{array}$$