Farkas' Lemma

LEMMA 10.5. *The system $Ax \leq b$ has no solutions if and only if there is a y such that*

$$\begin{aligned} A^T y &= 0 \\ y &\geq 0 \\ b^T y &< 0. \end{aligned} \tag{10.8}$$

PROOF. Consider the linear program

$$(P) \qquad \begin{array}{ll} \text{maximize} & 0 \\ \text{subject to} & Ax \leq b \end{array}$$

and its dual

$$(D) \qquad \begin{array}{lrl} \text{minimize} & b^T y & \\ \text{subject to} & A^T y &= 0 \\ & y &\geq 0. \end{array}$$

(I) If $\exists\, y \geq 0$ s.t. $y^T A = 0$, $y^T b < 0$
then (P) is not feasible.
Otherwise, $\exists\, X$ s.t.

$$AX \leq b$$

$$0 = 0^T X \leq (y^T A) X \leq y^T b < 0$$

!!

② If (P) is not feasible

Consider (D): $\min\ \xi(y) = y^T b$

s.t. $y^T A = 0$, $y \geq 0$

Note that $y=0$ is feasible

$\Rightarrow \quad \xi^* = \min \xi(y) \leq \xi(0) = 0$

If $\xi^* = 0$, by strong duality Thm, [V, Thm 5.2]

then (P) is feasible !!

((P) is infeasible)

$\Uparrow$

Hence $\exists y$ (feasible) s.t. $\xi(y) < 0$

(In this case $\xi(ty) = t\xi(y) \to -\infty$ as $t \to +\infty$)

Farkas Lemma and Fredholm Alternatives

LEMMA 10.5. *The system $Ax \leq b$ has no solutions if and only if there is a y such that*

$$(10.8) \qquad \begin{aligned} A^T y &= 0 \\ y &\geq 0 \\ b^T y &< 0. \end{aligned}$$

Fredholm Alternatives $\quad AX = b$

Either

(F1) $AX = b$ has a solution

or

(F2) There is y s.t. $y^T A = 0$ and $y^T b \neq 0$

Fredholm Alternatives $\quad AX = b$

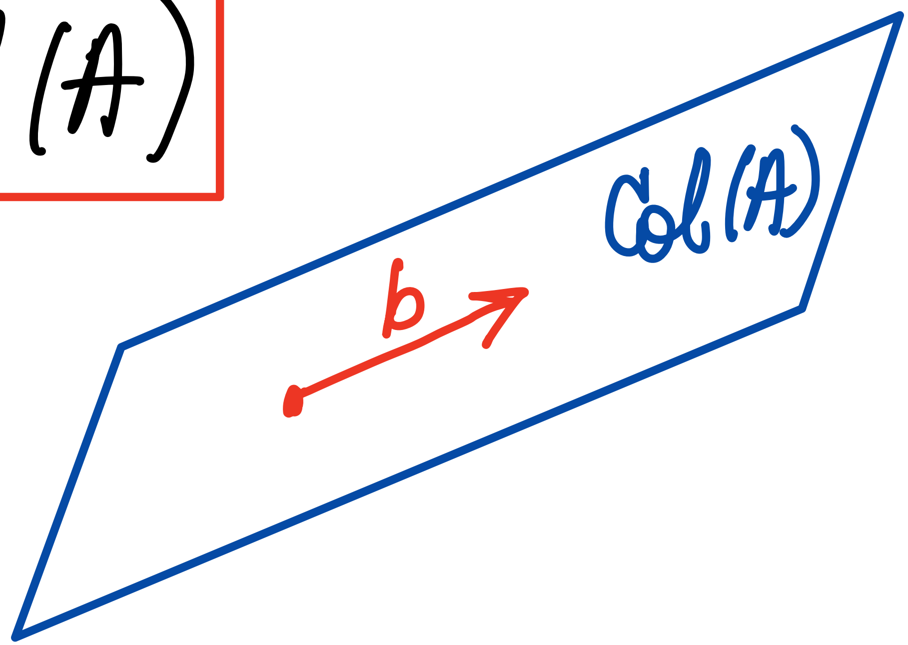
Either

(F1) $AX = b$ has a solution

or

(F2) There is y s.t. $y^T A = 0$ and $y^T b \neq 0$

$$\boxed{(F1) \iff b \in \mathrm{Col}(A)}$$



Fredholm Alternatives $\qquad AX = b$

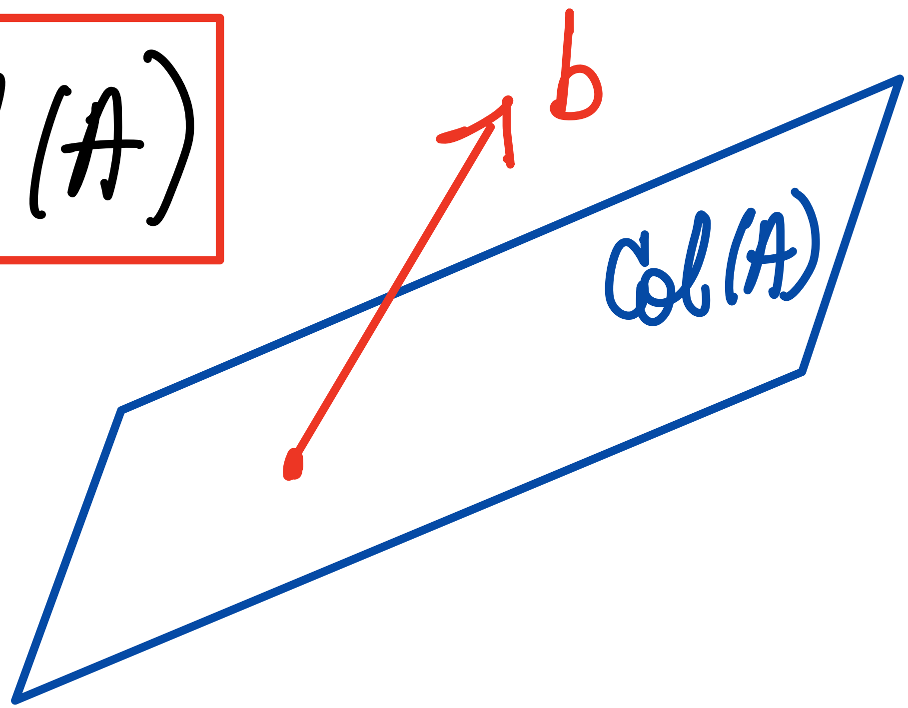
Either

(F1) $AX = b$ has a solution

or

(F2) There is y s.t. $y^T A = 0$ and $y^T b \neq 0$

$$\boxed{(F2) \iff b \notin \mathrm{Col}(A)}$$



Fredholm Alternatives $\quad AX = b$

Either

(F1) $AX = b$ has a solution

or

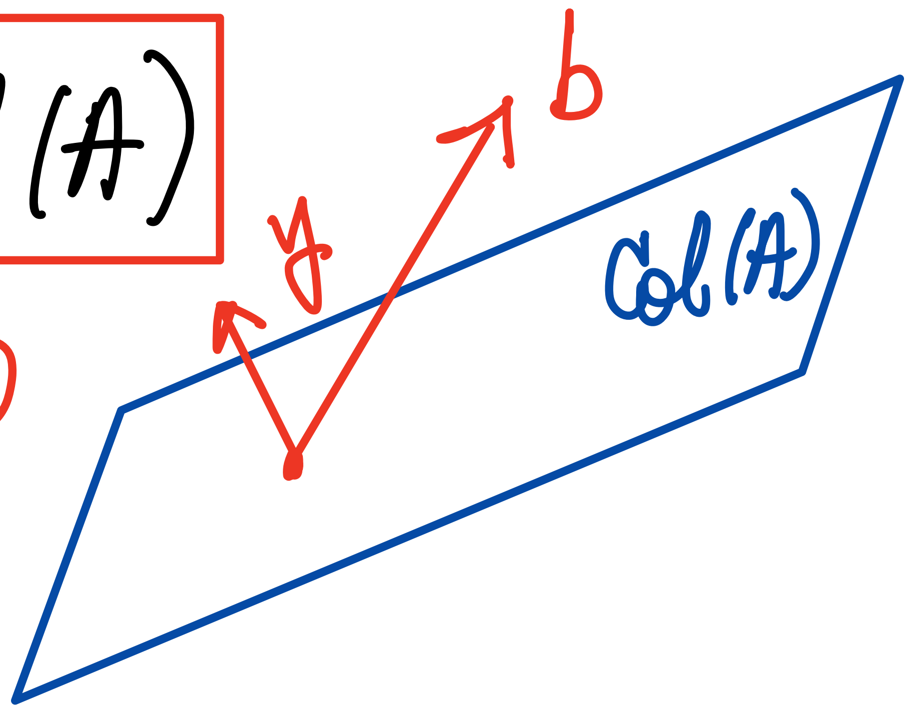
(F2) There is y s.t. $y^T A = 0$ and $y^T b \neq 0$

$$\boxed{(F2) \iff b \notin \mathrm{Col}(A)}$$

$y \perp \mathrm{Col}(A)$ & $y^T b \neq 0$

$\Updownarrow$

$y^T A = 0$



Farkas Lemma (Other Versions)

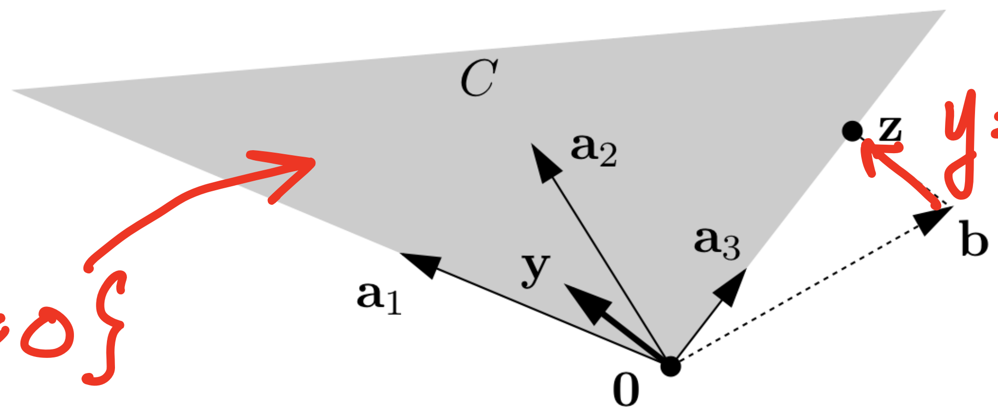
[MG] **6.4.1 Proposition (Farkas lemma).** *Let A be a real matrix with m rows and n columns, and let $\mathbf{b} \in \mathbb{R}^m$ be a vector. Then exactly one of the following two possibilities occurs:*

(F1) *There exists a vector $\mathbf{x} \in \mathbb{R}^n$ satisfying $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.*

(F2) *There exists a vector $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{y}^T A \geq \mathbf{0}^T$ and $\mathbf{y}^T\mathbf{b} < 0$.*

alternatives

The plan of the proof is straightforward: We let $\mathbf{z}$ be the point of C nearest to $\mathbf{b}$ (in the Euclidean distance), and we check that the vector $\mathbf{y} = \mathbf{z} - \mathbf{b}$ is as required; see the following illustration:



$y = z - b$

Cone: $\{AX \geq 0\}$

Farkas Lemma (Other Versions)

[V] $AX \leq b$ has no solution

$$\Updownarrow$$

There is $y \geq 0$ s.t. $y^T A = 0$ and $y^T b < 0$

$$\Updownarrow$$

[MG] $AX = b$, $X \geq 0$ has no solution

$$\Updownarrow$$

There is y s.t. $y^T A \geq 0$ and $y^T b < 0$

Farkas Lemma (Other Versions)

[V] $\Rightarrow$ [MG]

$$Ax = b,\ x \geq 0 \iff Ax \leq b,\ -Ax \leq -b,\ -x \leq 0$$

$$\iff \begin{bmatrix} A \\ -A \\ -I \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix} \quad (*)_{MG}$$

$(*)_{MG}$ has no solution

$\overset{[V]}{\iff}$ There is $y_1, y_2, y_3 \geq 0$ s.t. $y_1^T A - y_2^T A - y_3^T = 0$

and $y_1^T b - y_2^T b - y_3^T 0 < 0$

i.e. $\underbrace{(y_1^T - y_2^T)}_{y} A = y_3 \geq 0$, $\underbrace{(y_1^T - y_2^T)}_{y} b < 0$

Farkas Lemma (Other Versions)

[MG] $\Rightarrow$ [V]

$$Ax \leq b \iff A(x^+ - x^-) + w = b, \quad x^+, x^-, w \geq 0$$

$$\iff \begin{bmatrix} A & -A & I \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \\ w \end{bmatrix} = b \qquad (*)_V$$

$(*)_V$ has no solution

$\overset{[MG]}{\iff}$ There is y s.t. $y^T \begin{bmatrix} A & -A & I \end{bmatrix} \geq 0$, $y^T b < 0$

i.e. $\underbrace{y^T A \geq 0, \ -y^T A \geq 0}_{y^T A = 0}$, $y^T \geq 0$, $y^T b < 0$

[MG] Farkas Lemma (Other Versions)

6.4.3 Proposition (Farkas lemma in three variants). *Let A be a real matrix with m rows and n columns, and let $\mathbf{b} \in \mathbb{R}^m$ be a vector.*

(i) *The system $A\mathbf{x} = \mathbf{b}$ has a nonnegative solution if and only if every $\mathbf{y} \in \mathbb{R}^m$ with $\mathbf{y}^T A \geq \mathbf{0}^T$ also satisfies $\mathbf{y}^T\mathbf{b} \geq \mathbf{0}$.*

(ii) *The system $A\mathbf{x} \leq \mathbf{b}$ has a nonnegative solution if and only if every nonnegative $\mathbf{y} \in \mathbb{R}^m$ with $\mathbf{y}^T A \geq \mathbf{0}^T$ also satisfies $\mathbf{y}^T\mathbf{b} \geq 0$.*

(iii) *The system $A\mathbf{x} \leq \mathbf{b}$ has a solution if and only if every nonnegative $\mathbf{y} \in \mathbb{R}^m$ with $\mathbf{y}^T A = \mathbf{0}^T$ also satisfies $\mathbf{y}^T\mathbf{b} \geq 0$.*

(in fact, (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii))

	The system $A\mathbf{x} \leq \mathbf{b}$	The system $A\mathbf{x} = \mathbf{b}$
has a solution $\mathbf{x} \geq \mathbf{0}$ iff	$\mathbf{y} \geq \mathbf{0}, \mathbf{y}^T A \geq \mathbf{0}$ $\Rightarrow \mathbf{y}^T\mathbf{b} \geq 0$	$\mathbf{y}^T A \geq \mathbf{0}^T$ $\Rightarrow \mathbf{y}^T\mathbf{b} \geq 0$
has a solution $\mathbf{x} \in \mathbb{R}^n$ iff	$\mathbf{y} \geq \mathbf{0}, \mathbf{y}^T A = \mathbf{0}$ $\Rightarrow \mathbf{y}^T\mathbf{b} \geq 0$	$\mathbf{y}^T A = \mathbf{0}^T$ $\Rightarrow \mathbf{y}^T\mathbf{b} = 0$

Fredholm Alt.

[MG] Farkas Lemma (Other Versions)

6.4.3 Proposition (Farkas lemma in three variants). *Let A be a real matrix with m rows and n columns, and let $\mathbf{b} \in \mathbb{R}^m$ be a vector.*

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(in fact, (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii))

$\sim$(i) $Ax = b,\ x \geq 0$ has no solution iff there is y s.t. $y^T A \geq 0^T,\ y^T b < 0$

[MG] Farkas Lemma (Other Versions)

6.4.3 Proposition (Farkas lemma in three variants). *Let A be a real matrix with m rows and n columns, and let $\mathbf{b} \in \mathbb{R}^m$ be a vector.*

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(in fact, (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii))

$\sim$(ii) $Ax \leq b$, $x \geq 0$ has no solution iff there is $y \geq 0$ s.t. $y^T A \geq 0^T$, $y^T b < 0$

[MG] Farkas Lemma (Other Versions)

6.4.3 Proposition (Farkas lemma in three variants). *Let A be a real matrix with m rows and n columns, and let $\mathbf{b} \in \mathbb{R}^m$ be a vector.*

(i) *The system $A\mathbf{x} = \mathbf{b}$ has a nonnegative solution if and only if every $\mathbf{y} \in \mathbb{R}^m$ with $\mathbf{y}^T A \geq \mathbf{0}^T$ also satisfies $\mathbf{y}^T\mathbf{b} \geq \mathbf{0}$.*

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(in fact, (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii))

$\sim$(iii) $Ax \leq b$ has no solution

iff there is y s.t. $y^TA = 0^T$, $y^Tb < 0$

Farkas Lemma (Other Versions)

[C] P. 248 #16.10

Derive the following theorems (with the vector inequality $\mathbf{v} > \mathbf{w}$ meaning, as usual, $v_k > w_k$ for all k) from the result of problem 16.9.

(i) P. Gordan (1873): The system $\mathbf{Ax} < \mathbf{0}$ is unsolvable if and only if the system $\mathbf{yA} = \mathbf{0}$, $\mathbf{y} \geq \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$ is solvable.

(ii) J. Farkas (1902): The system $\mathbf{Ax} \leq \mathbf{0}$, $\mathbf{bx} > 0$ is unsolvable if and only if the system $\mathbf{yA} = \mathbf{b}$, $\mathbf{y} \geq \mathbf{0}$ is solvable.

(iii) E. Stiemke (1915): The system $\mathbf{Ax} = \mathbf{0}$, $\mathbf{x} > \mathbf{0}$ is unsolvable if and only if the system $\mathbf{yA} \geq \mathbf{0}$, $\mathbf{yA} \neq \mathbf{0}$ is solvable.

(iv) J. A. Ville (1938): The system $\mathbf{Ax} < \mathbf{0}$, $\mathbf{x} \geq \mathbf{0}$ is unsolvable if and only if the system $\mathbf{yA} \geq \mathbf{0}$, $\mathbf{y} \geq \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$ is solvable.

(v) A. W. Tucker (1956): The system $\mathbf{Ax} \geq \mathbf{0}$, $\mathbf{x} \geq \mathbf{0}$ has no solution with $x_k > 0$ if and only if the system $\mathbf{yA} \leq \mathbf{0}$, $\mathbf{y} \geq \mathbf{0}$ has a solution with

$$\sum_{i=1}^{m} a_{ik} y_i < 0.$$