Nonlinear Programming [E, Ch. 4], [CZ, Ch. 21]

Unconstrained Opt:

[E, 1.1] [CZ, Ch. 6]

$$\min_{x \in \mathbb{R}^n} f(x)$$

$$\nabla f(x) = 0, \quad D^2 f \geq 0$$

Equality Constrained Opt

[E, 1.3] [CZ, Ch. 12]

$$\min f(x)$$

$$\text{s.t.} \quad \vec{h}(x) = 0$$

Lagrange multiplier $\vec{\lambda}$:

$$Df(x) + \lambda^T Dh(x) = 0$$

(+ 2nd der. test)

(Under regulairty assumption:

$[Dh(x)]$ has full rank.)

Inequality Constrained Opt [E, Ch. 4] [CZ Ch 21]

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad \vec{h}(x) = \vec{0}, \quad \vec{g}(x) \leq \vec{0}$$

[CZ]

Theorem 21.1 ***Karush-Kuhn-Tucker (KKT) Theorem.*** *Let $f, \boldsymbol{h}, \boldsymbol{g} \in \mathcal{C}^1$. Let $\boldsymbol{x}^*$ be a regular point and a local minimizer for the problem of minimizing f subject to $\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}$, $\boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0}$. Then, there exist $\boldsymbol{\lambda}^* \in \mathbb{R}^m$ and $\boldsymbol{\mu}^* \in \mathbb{R}^p$ such that:*

1. $\boldsymbol{\mu}^* \geq \boldsymbol{0}$.
2. $Df(\boldsymbol{x}^*) + \boldsymbol{\lambda}^{*\top} D\boldsymbol{h}(\boldsymbol{x}^*) + \boldsymbol{\mu}^{*\top} D\boldsymbol{g}(\boldsymbol{x}^*) = \boldsymbol{0}^\top$.
3. $\boldsymbol{\mu}^{*\top} \boldsymbol{g}(\boldsymbol{x}^*) = 0$.

$$\left(\begin{array}{c} h(x^*) = 0 \\ g(x^*) \leq 0 \end{array} \right)$$

Definition 21.2 Let $\boldsymbol{x}^*$ satisfy $\boldsymbol{h}(\boldsymbol{x}^*) = \boldsymbol{0}$, $\boldsymbol{g}(\boldsymbol{x}^*) \leq \boldsymbol{0}$, and let $J(\boldsymbol{x}^*)$ be the index set of active inequality constraints:

$$J(\boldsymbol{x}^*) \triangleq \{j : g_j(\boldsymbol{x}^*) = 0\}.$$

Then, we say that $\boldsymbol{x}^*$ is a *regular point* if the vectors

$$\nabla h_i(\boldsymbol{x}^*), \quad \nabla g_j(\boldsymbol{x}^*), \quad 1 \leq i \leq m, \quad j \in J(\boldsymbol{x}^*)$$

are linearly independent. ∎

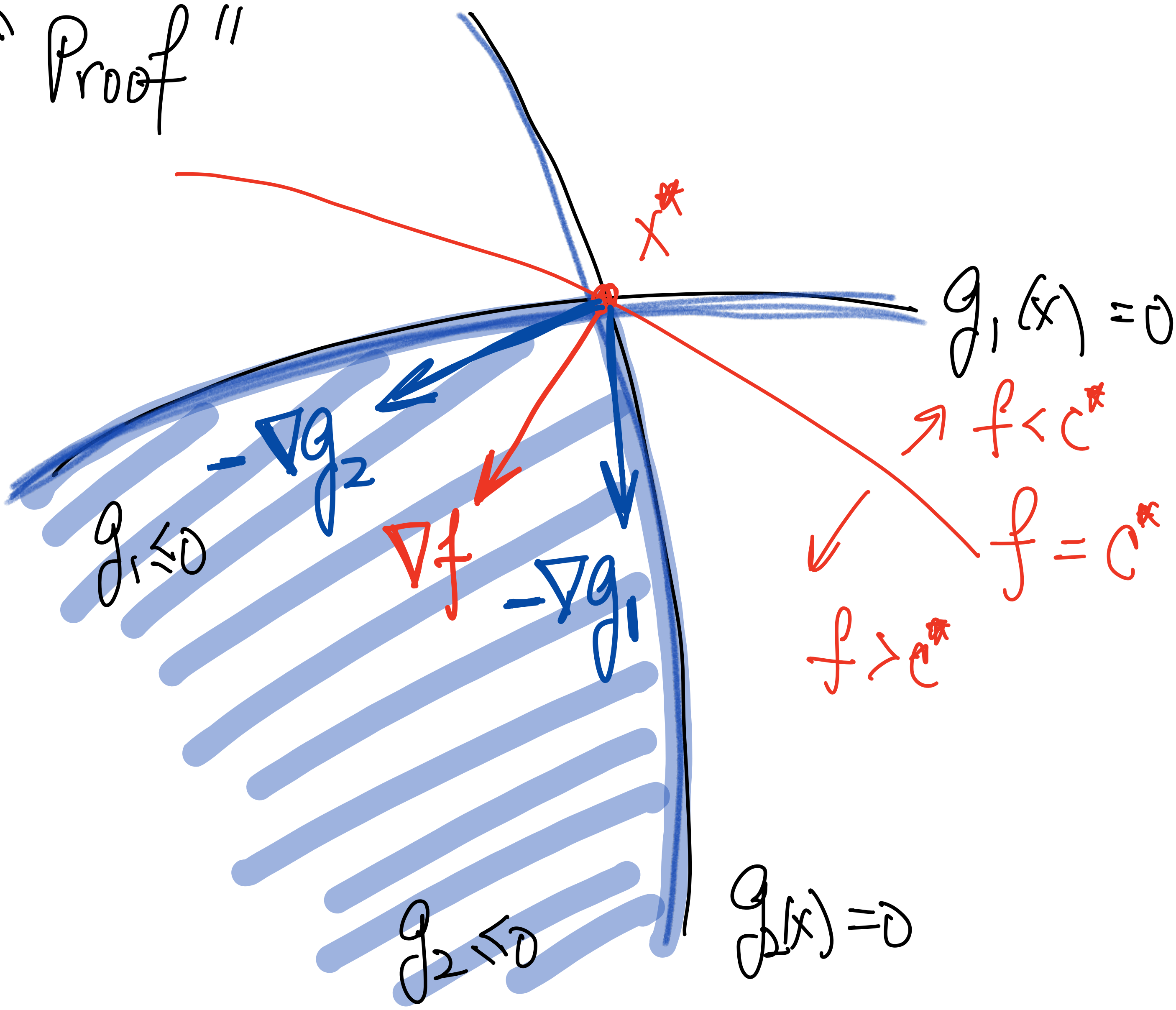
(3) : $\quad \mu^\top \vec{g}(x) = 0$

$$0 = \underset{\geq 0}{\mu_1}\, \underset{\leq 0}{g_1(x)} + \mu_2 g_2(x) + \cdots + \mu_p g_p(x) \leq 0$$

$\mu_i = 0$ if $g_i(x) < 0$ (not active)

(complementary slackness)

"Proof"



$$\nabla f = \mu_1(-\nabla g_1) + \mu_2(-\nabla g_2)$$

$$\mu_1, \mu_2 \geq 0$$

i.e. $\nabla f + \mu_1 \nabla g_1 + \mu_2 \nabla g_2 = 0$

$$\mu_1 g_1 + \mu_2 g_2 = 0$$

Example 21.4 Consider the problem

[C2]

$$\begin{aligned} \text{minimize} \quad & f(x_1, x_2) \\ \text{subject to} \quad & x_1, x_2 \geq 0, \end{aligned}$$

$-x_1, -x_2 \leq 0$

where

$$f(x_1, x_2) = x_1^2 + x_2^2 + x_1 x_2 - 3x_1.$$

Consider $L(x, \mu) = f(x_1, x_2) + \mu_1(-x_1) + \mu_2(-x_2)$

$\partial_{x_1} L \Rightarrow \quad \partial_{x_1} f - \mu_1 = 0$

$\partial_{x_2} L \Rightarrow \quad \partial_{x_2} f - \mu_2 = 0$

$$\mu_1, \mu_2 \geq 0$$

$$\mu_1(-x_1) + \mu_2(-x_2) = 0$$

$$-x_1 \leq 0, \quad -x_2 \leq 0$$

i.e.
$$\begin{aligned} 2x_1 + x_2 - 3 - \mu_1 &= 0 \\ x_1 + 2x_2 - \mu_2 &= 0 \\ \mu_1 x_1 + \mu_2 x_2 &= 0 \\ x_1, x_2 \geq 0, \quad \mu_1, \mu_2 &\geq 0 \end{aligned}$$

(1) $x_1 = 0,\ x_2 > 0 \implies \mu_2 = 0$ $\qquad \mu_1 = -3$!!

$$\left.\begin{aligned} \cancel{2x_1} + x_2 - 3 - \mu_1 &= 0 \\ \cancel{x_1} + 2x_2 \quad - \cancel{\mu_2} &= 0 \end{aligned}\right\} \implies \begin{aligned} x_2 - 3 - \mu_1 &= 0 \\ x_2 &= 0 \end{aligned}$$

(2) $x_1 > 0,\ x_2 = 0 \implies \mu_1 = 0$

$$\left.\begin{aligned} 2x_1 + \cancel{x_2} - 3 - \cancel{\mu_1} &= 0 \\ x_1 + \cancel{2x_2} \quad - \mu_2 &= 0 \end{aligned}\right\} \implies \begin{aligned} x_1 &= 3/2 \\ \mu_2 &= 3/2 \end{aligned}$$ ✓

(3) $x_1 = 0,\ x_2 = 0$

$$\left.\begin{aligned} \cancel{2x_1} + \cancel{x_2} - 3 - \mu_1 &= 0 \\ \cancel{x_1} + \cancel{2x_2} \quad - \mu_2 &= 0 \end{aligned}\right\} \implies \begin{aligned} \mu_1 &= -3 \\ \mu_2 &= 0 \end{aligned}$$!!

(4) $x_1 > 0,\ x_2 > 0 \implies \mu_1 = 0,\ \mu_2 = 0$

$$\left.\begin{aligned} 2x_1 + x_2 - 3 - \cancel{\mu_1} &= 0 \\ x_1 + 2x_2 \quad - \cancel{\mu_2} &= 0 \end{aligned}\right\} \implies \begin{aligned} x_1 &= 2 \\ x_2 &= -1 \end{aligned}$$!!

$\boxed{\text{"Candidate" for min: } (x_1 = 3/2,\ x_2 = 0)}$

Example 21.5 Consider the following problem:

$$\begin{aligned} \text{minimize} \quad & x_1 x_2 \\ \text{subject to} \quad & x_1 + x_2 \geq 2 \\ & x_2 \geq x_1. \end{aligned}$$

$-x_1 - x_2 + 2 \leq 0$
$x_1 - x_2 \leq 0$
[CZ]

$$L(x,\mu) = x_1x_2 + \mu_1(-x_1 - x_2 + 2) + \mu_2(x_1 - x_2)$$

$$\partial_{x_1} L \Longrightarrow \quad x_2 - \mu_1 + \mu_2 = 0$$

$$\partial_{x_2} L \Longrightarrow \quad x_1 - \mu_1 - \mu_2 = 0$$

$$\mu_1(-x_1 - x_2 + 2) + \mu_2(x_1 - x_2) = 0$$

$$x_1 + x_2 \geq 2, \quad x_1 \leq x_2$$

$$\mu_1, \mu_2 \geq 0$$

(1) $x_1 + x_2 = 2$, $\underline{x_1 < x_2} \Rightarrow \mu_2 = 0$

$\Rightarrow \quad x_2 = \mu_1, \ x_1 = \mu_1 \Rightarrow \mu_1 = 1$

$\underline{x_1 = x_2 = 1}$!!

(2) $x_1 + x_2 > 2$, $x_1 = x_2 \Rightarrow \mu_1 = 0$

$$\left.\begin{aligned} x_2 + \mu_2 = 0 \\ x_1 - \mu_2 = 0 \end{aligned}\right\} \Longrightarrow x_1 + x_2 = 0 \quad !!$$

(3) $x_1 + x_2 = 2, \quad x_1 = x_2 \implies x_1 = x_2 = 1$ ✓

$$\left.\begin{aligned} \mu_1 - \mu_2 &= 1 \\ \mu_1 + \mu_2 &= 1 \end{aligned}\right\} \implies \begin{aligned} \mu_1 &= 1 \\ \mu_2 &= 0 \end{aligned}$$ ✓

(4) $\underline{x_1 + x_2 > 2,} \quad x_1 < x_2 \implies \mu_1 = \mu_2 = 0$

$\implies \underline{x_1 = x_2 = 0}$!!

$$\boxed{\text{"Candidate" for min is } (x_1 = x_2 = 1)}$$

Example 21.6 We wish to minimize $f(\boldsymbol{x}) = (x_1 - 1)^2 + x_2 - 2$ subject to

$$h(\boldsymbol{x}) = x_2 - x_1 - 1 = 0,$$
$$g(\boldsymbol{x}) = x_1 + x_2 - 2 \leq 0.$$

$$2(x_1 - 1) - \lambda + \mu = 0$$
$$1 + \lambda + \mu = 0$$
$$\mu(x_1 + x_2 - 2) = 0$$
$$\mu \geq 0$$
$$-x_1 + x_2 - 1 = 0$$
$$x_1 + x_2 - 2 \leq 0$$

$$\underline{\mu > 0} \Rightarrow \left.\begin{array}{l} x_1 + x_2 = 2 \\ -x_1 + x_2 = 1 \end{array}\right\} \Rightarrow \begin{array}{l} x_1 = 1/2 \\ x_2 = 3/2 \end{array}$$

$$\begin{array}{l} 2(x_1 - 1) - \lambda + \mu = 0 \\ 1 + \lambda + \mu = 0 \end{array} \Rightarrow \begin{array}{l} -\lambda + \mu = 1 \\ \lambda + \mu = -1 \end{array}$$

$$\Rightarrow \underline{\mu = 0} \quad !!$$

$$\boxed{\mu = 0 \Rightarrow \left.\begin{array}{l} 2(x_1 - 1) - \lambda = 0 \\ \lambda = -1 \end{array}\right\} \Rightarrow \begin{array}{l} x_1 = 1/2 \\ x_2 = 3/2 \end{array} \quad (x_1 + x_2 = 2 \leq 2)}$$

$$(LP): \quad \boxed{\begin{aligned} \min \quad & -c^T x \\ \text{s.t.} \quad & Ax \le b \\ & x \ge 0 \end{aligned}}$$

$$L(x, y, z) = -c^T x + y^T (Ax - b) - z^T x$$

$$\partial_x L \Rightarrow \quad -c^T + y^T A - z^T = 0 \qquad (1)$$

$$y^T (Ax - b) - z^T x = 0 \qquad (2)$$

$$y, z \ge 0 \qquad (3)$$

$$Ax \le b, \qquad (4)$$

$$x \ge 0 \qquad (5)$$

$$(4) \,\&\, (5) \iff (P)$$

$$(1), (3) \iff \left.\begin{aligned} & y^T A - z^T = c^T \\ & (\text{or } y^T A \ge c^T) \\ & y, z \ge 0 \end{aligned}\right\} (D)$$

$(2) \iff$ Complementary slackness

(1), (2), (3), (4), (5) $\Longrightarrow$ Opt.

Proof again $c^T x \underset{(1)}{=} y^T A x - z^T x \underset{(2)}{=} y^T b$

$$\underset{\text{(D) is feasible}}{\Uparrow} \qquad \underset{\text{C.S}}{\Uparrow}$$