Quadratic Programming

$$\min \; f(x)$$
$$\text{s.t.} \quad \left.\begin{aligned} AX &= b \\ X &\geq 0 \end{aligned}\right\} \text{ linear constraints}$$

$C \geq 0$, positive def.

$$f(x) = \underbrace{\frac{1}{2} x^T C x}_{\text{quadratic}} + \underbrace{p^T x}_{\text{linear}}$$

In terms of coordinates:

$$A^{m\times n}, \; X^{n\times 1}, \; C^{n\times n}, \; b^{m\times 1}, \; p^{n\times 1}$$

$C = (C_{ij})$ (WLOG: $C_{ij} = C_{ji}$, i.e. $C = C^T$, sym)

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \qquad p = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$$

$$f(x) = \frac{1}{2} \sum_{i,j} C_{ij} x_i x_j + \sum_i p_i x_i$$

pos. def.

$$\boxed{[\partial^2 f] = C \geq 0 \Rightarrow f \text{ is convex.}}$$

Lagrangian function

$$L(x,\lambda,\mu) = \frac{1}{2}x^T C x + p^T x + \lambda^T (Ax-b) - \mu^T x$$

$$\lambda = \begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_m\end{pmatrix} \qquad \mu = \begin{pmatrix}\mu_1\\ \vdots \\ \mu_n\end{pmatrix} \geq 0$$

KKT: find $x, \lambda, \mu \geq 0$ s.t.

$$D_x L = 0 \Rightarrow x^T C + p^T + \lambda^T A - \mu^T = 0$$

$$\Updownarrow$$

$$\begin{cases} Cx + p + A^T\lambda - \mu = 0 \\ Ax = b \\ x \geq 0 \\ \mu^T x = 0 \quad \longleftarrow \text{C.S.} \\ \mu \geq 0 \end{cases}$$

$$(*) \quad \boxed{\begin{array}{rl} Cx + A^T\lambda - \mu = -p & (1) \\ Ax = b & (2) \\ x \geq 0 & (3) \\ \mu \geq 0 & (4) \\ \mu^T x = 0 & (5) \end{array}}$$

(1)–(4): LP

(5): non lin.

Solution of (*) using LP/Simplex (Frank–Wolfe)

(1) WLOG, assume $p \leq 0$, $b \geq 0$
otherwise, multiply individual row by (-1)
eg

$$a_1 x_1 + a_2 x_2 = -3$$
$$\Longleftrightarrow (-a_1)x_1 + (-a_2)x_2 = 3$$

(2) Introduce auxiliary variables $\underline{\vec{W}_1, \vec{W}_2 \geq 0}$
s.t.

$$CX + A^T\lambda - \mu + W_1 = -p \qquad \begin{pmatrix} w_1^1 \\ \vdots \\ w_n^1 \end{pmatrix}, \begin{pmatrix} w_1^2 \\ \vdots \\ w_m^2 \end{pmatrix}$$
$$AX + W_2 = b$$

$$\Longleftrightarrow \left\{ \begin{array}{ll} W_1 = -p_1 - CX - A^T\lambda + \mu & \geq 0 \\ W_2 = \underbrace{b}_{\geq 0} - AX & \geq 0 \end{array} \right.$$

feasible dictionary

(3) Objective function (need: $W_1 = 0, W_2 = 0$)

$$\min \; w_1^1 + \cdots + w_n^1 + w_1^2 + \cdots + w_m^2$$
$$(\max \; -w_1^1 - \cdots - w_n^1 - w_1^2 - \cdots - w_m^2)$$

(4) Set $\lambda = \lambda_+ - \lambda_-$, $\quad \lambda_+, \lambda_- \geq 0$

(5) C.S. $\mu^T x = 0$

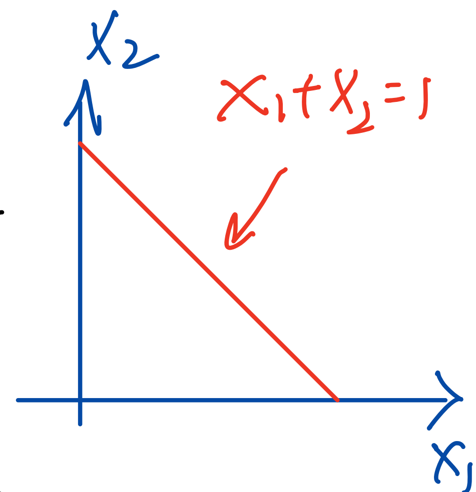
i.e. $\mu_1 x_1 + \cdots + \mu_n x_n = 0$

$x_i \geq 0, \ \mu_i \geq 0$

i.e. x_i, μ_i cannot be simultaneously positive.

During simplex, while choosing entering and leaving variables, never let x_i, μ_i be basic variables at the same time.

Example [F] p. 185



$$\min f(x) = x_1^2 + x_1 x_2 + x_2^2 - 5x_2$$

s.t. $x_1 + x_2 = 1$

$x_1, x_2 \geq 0$

$$f(x) = \frac{1}{2}(x_1 \ \ x_2) \underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{(0 \ \ -5)}_{p^T} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = (\lambda-2)^2 - 1$$

$$= (\lambda-3)(\lambda-1)$$

$\Rightarrow \lambda = 3, 1 > 0$

$\Rightarrow C$ is pos. def

$\Rightarrow f$ is convex.

$$L(x, \lambda, \mu) = x_1^2 + x_1x_2 + x_2^2 - 5x_2 + \lambda(x_1 + x_2 - 1) - \mu_1 x_1 - \mu_2 x_2$$

$$\text{KKT} \Rightarrow \begin{cases} 2x_1 + x_2 + \lambda - \mu_1 = 0 \\ x_1 + 2x_2 - 5 + \lambda - \mu_2 = 0 \\ x_1 + x_2 = 1 \\ \mu_1, \mu_2 \geq 0 \\ \mu_1 x_1 + \mu_2 x_2 = 0 \end{cases}$$

$$\begin{aligned} 2x_1 + x_2 + \lambda - \mu_1 &= 0 \\ x_1 + 2x_2 + \lambda - \mu_2 &= 5 \\ x_1 + x_2 &= 1 \end{aligned}$$

$\mu_1, \mu_2 \geq 0$

$\mu_1 x_1 + \mu_2 x_2 = 0$

(M1) Of course, solve for x_1: $x_1 = 1 - x_2$

$$f(x) = x_1^2 + x_1 x_2 + x_2^2 - 5x_2$$

$$= \underbrace{(1-x_2)^2 + (1-x_2)x_2 + x_2^2 - 5x_2}_{\text{1D problem: } 0 \leq x_2 \leq 1}$$

(M2) Using LP

$$\max \quad -w_1 - w_2 - w_3$$

$$\text{s.t.} \quad \begin{aligned} w_1 &= 0 - 2x_1 - x_2 - \lambda + \mu_1 \\ w_2 &= 5 - x_1 - 2x_2 - \mu + \mu_2 \\ w_3 &= 1 - x_1 - x_2 \end{aligned}$$

(initial feasible dictionary)

$$w_1, w_2, w_3, x_1, x_2, \mu_1, \mu_2 \geq 0$$

$$(\lambda = \lambda_+ - \lambda_-)$$

(M3) Solving KKT directly:

$$\begin{aligned} 2x_1 + x_2 + \lambda - \mu_1 &= 0 \\ x_1 + 2x_2 - 5 + \lambda - \mu_2 &= 0 \\ x_1 + x_2 &= 1 \\ \mu_1, \mu_2 &\geq 0 \\ \mu_1 x_1 + \mu_2 x_2 &= 0 \end{aligned}$$

① $x_1, x_2 > 0 \Rightarrow \mu_1 = \mu_2 = 0$

$$2x_1 + x_2 = -\lambda$$
$$x_1 + 2x_2 = 5 - \lambda \Rightarrow x_1 = -1 - \lambda$$
$$x_1 + x_2 = 1 \qquad x_2 = 4 - \lambda$$
$$x_1 + x_2 = 1$$

$$\Rightarrow 1 = 3 - 2\lambda \Rightarrow \lambda = 1$$
$$\Rightarrow x_1 = -2 \; !!$$

② $x_1 > 0, \; x_2 = 0 \Rightarrow \mu_1 = 0, \; x_1 = 1$

$$\Rightarrow 2 + \lambda = 0$$
$$1 - 5 + \lambda - \mu_2 = 0$$
$$\Rightarrow \lambda = -2$$
$$\mu_2 = 1 - 5 - 2 = -6 \; !!$$

③ $x_1 = 0, \; x_2 > 0 \Rightarrow \mu_2 = 0, \; x_2 = 1$

$$\Rightarrow 1 + \lambda - \mu_1 = 0$$
$$2 - 5 + \lambda = 0$$
$$\Rightarrow \lambda = 3, \; \mu_1 = 4$$

$$\boxed{x_1 = 0, \; x_2 = 1, \; \lambda = 3, \; \mu_1 = 4, \; \mu_2 = 0}$$

[F, p.187]

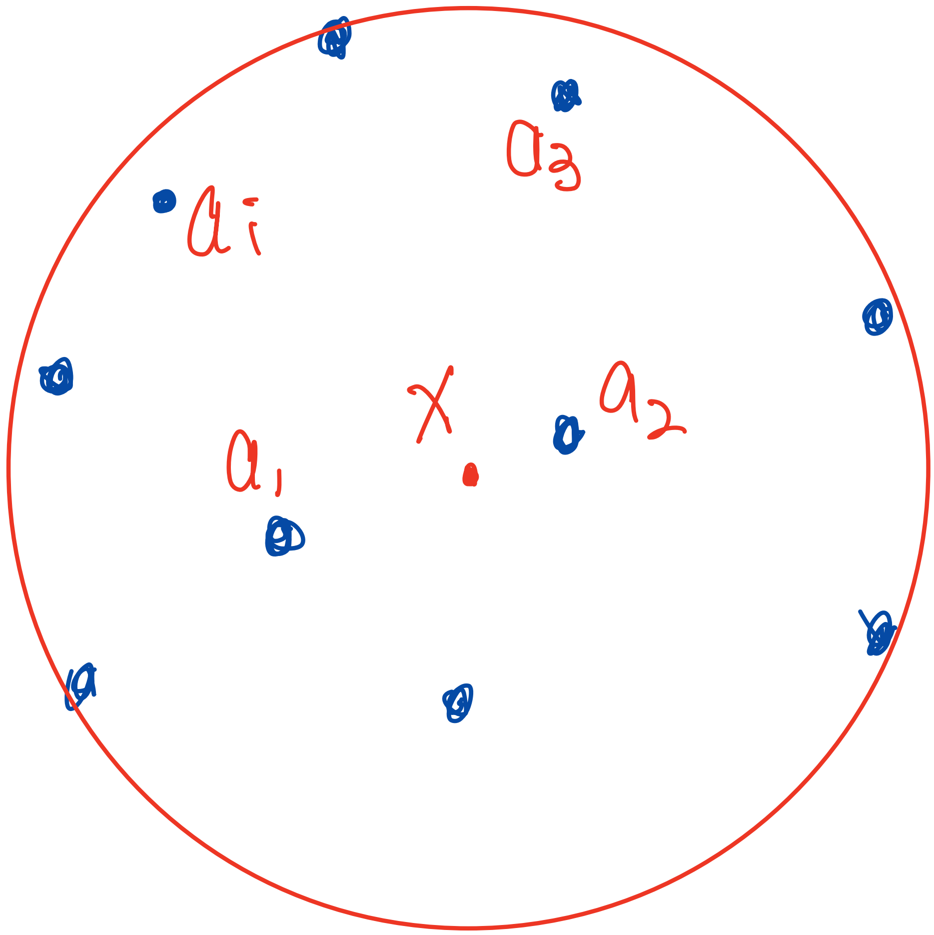
J. J. Sylvester, "A Question in the Geometry of Situation," Quarterly J. Pure and Appl. Math, vol. 1 (1857) p. 79.

Sylvester's problem (1857): "It is required to find the least circle which shall contain a given set of points in the plane." Show that Sylvester's problem for finite point sets in R^n is equivalent to the following quadratic program: *If the points* $\mathbf{a}_1, \ldots, \mathbf{a}_m$ *are given, find a point* $\mathbf{x}$ *and a number* λ *solving*

$$\mathbf{a}_i \cdot \mathbf{x} + \lambda \geqslant \tfrac{1}{2}|\mathbf{a}_i|^2 \qquad (i = 1, \ldots, m),$$
$$\tfrac{1}{2}\mathbf{x} \cdot \mathbf{x} + \lambda = \min.$$

Then the required least sphere has center $\mathbf{x}$ and has radius $r = \sqrt{(\mathbf{x} \cdot \mathbf{x} + 2\lambda)}$.

(See also [MG] p.188)



$$\min_{x,\lambda} \ \lambda \quad \text{s.t. } \|a_i - x\|^2 \leq \lambda, \ i=1,2\ldots$$

$$\|a_i - x\|^2 = \|a_i\|^2 - 2\langle a_i, x\rangle + \|x\|^2 \le \lambda$$

$$\Rightarrow \|a_i\|^2 - 2\langle a_i, x\rangle \le \underbrace{\lambda - \|x\|^2}_{\substack{\text{new var.} \\ \tilde{\lambda}}}$$

$$\lambda = \lambda - \|x\|^2 + \|x\|^2$$
$$= \tilde{\lambda} + \|x\|^2$$

New problem:

quadratic

$$\min_{x, \tilde{\lambda}} \tilde{\lambda} + \|x\|^2$$

$$\text{s.t.} \quad \|a_i\|^2 - 2\langle a_i, x\rangle \le \tilde{\lambda}, \quad i = 1, 2, \dots$$

$$\Leftrightarrow \quad 2\langle a_i, x\rangle + \tilde{\lambda} \ge \|a_i\|^2$$

lin constraints

Markowitz Portfolio Selection Model [V, p. 415]

$\{R_j\}_{j=1,\dots}$ price of portfolio j

$$\max \quad R = \sum_j x_j R_j$$

$$\text{s.t.} \quad x_j \geq 0, \quad \sum_j x_j = 1$$

In general, R_j is random

$\Longrightarrow$ max average of R

& min fluctuation of R

$$ER = E\left[\sum_j x_j R_j\right] = \sum x_j \underbrace{ER_j}_{r_j}$$

r_j mean of R_j

$$\begin{aligned} Var(R) &= E\left[(R - ER)^2\right] \\ &= E\left[\left(\sum_j x_j R_j - \sum_j x_j ER_j\right)^2\right] \end{aligned}$$

$$= E\left[\left(\sum_j x_j (R_j - ER_j)\right)^2\right]$$

$$= E\left[\sum_{i,j} x_i x_j (R_i - ER_i)(R_j - ER_j)\right]$$

$$= \sum_{i,j} x_i x_j \underbrace{E\left[(R_i - ER_i)(R_j - ER_j)\right]}_{C_{ij} \text{ covariance between } R_i, R_j}$$

$$\boxed{\begin{aligned} \max \quad & \sum_j x_j r_j - \mu \sum_{i,j} x_i x_j C_{ij} \\ \text{s.t.} \quad & \sum_j x_j = 1, \quad x_j \geq 0 \end{aligned}}$$

linear

quadratic