Solution of Primal vs Dual Problems

(P) $\max \ \zeta(x) = c^T x$

s.t. $Ax \leq b$

$x \geq 0$

$w = b - Ax$ slacks

$$\begin{array}{c|c} \zeta & c^T \\ \hline b & -A \end{array}$$

(D) $\min \ \xi(y) = b^T y$

s.t. $A^T y \geq c$

$y \geq 0$

$z = A^T y - c$ slacks

Solution of Primal vs Dual Problems

(P) $\max \quad \zeta(x) = c^T x$

s.t. $\quad Ax \leq b$

$\quad x \geq 0$

$w = b - Ax$ slacks

ζ	c^T
b	$-A$

$(-ve)^T$

(D) $\max \quad -\xi(y) = -b^T y$

s.t. $\quad -A^T y \leq -c$

$\quad y \geq 0$

$z = A^T y - c$ slacks

$-\xi$	$-b^T$
$-c$	A^T

Solution of Primal vs Dual Problems

$$AX \leq b$$

$$\underbrace{y^T A}\, X \leq y^T b$$

$$c^T \leq y^T A$$

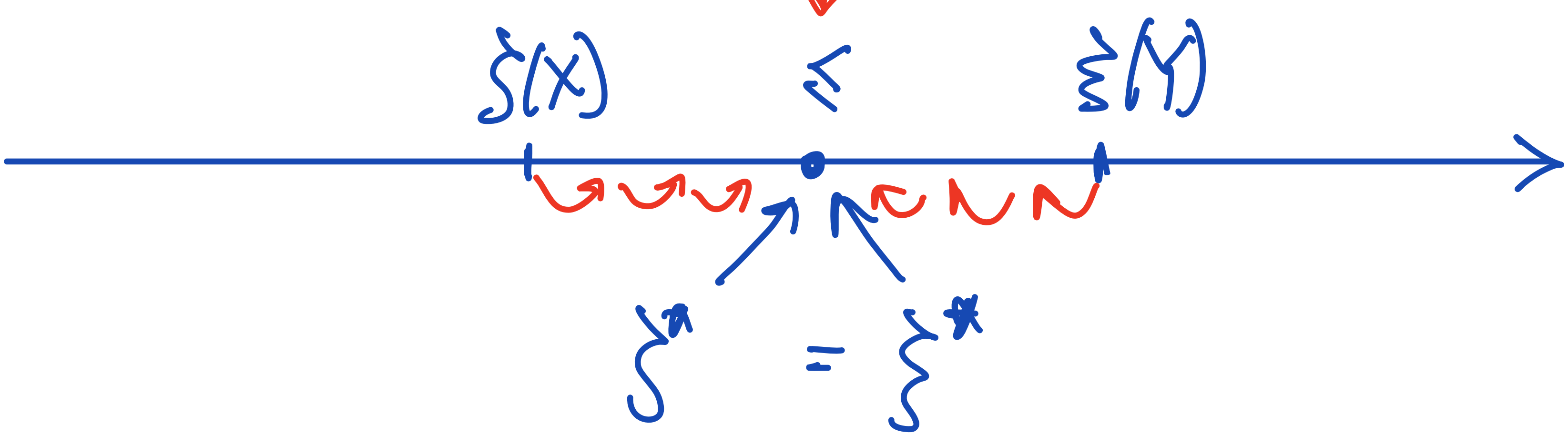
$$\zeta(x) = c^T X \leq y^T A X \leq y^T b = \xi(Y)$$

max

min

Solution of Primal vs Dual Problems

Weak Duality Thm 5.1 [V]



$$\zeta(X) \longrightarrow \zeta^* = \xi^* \longleftarrow \xi(Y)$$

Strong Duality Thm 5.2 [V]

Dual Simplex Method [V] p.71

We begin with an example:

Step 0

P

$$\begin{array}{lrcr} \text{maximize} & -x_1 - \ \ x_2 & & \\ \text{subject to} & -2x_1 - \ \ x_2 & \leq & 4 \\ & -2x_1 + 4x_2 & \leq & -8 \\ & -x_1 + 3x_2 & \leq & -7 \\ & x_1, \ x_2 & \geq & 0. \end{array}$$

all neg. coeffs.

original not P-feasible

The dual of this problem is

D

$$\begin{array}{lrcr} \text{minimize} & 4y_1 - 8y_2 - 7y_3 & & \\ \text{subject to} & -2y_1 - 2y_2 - \ \ y_3 & \geq & -1 \\ & -y_1 + 4y_2 + 3y_3 & \geq & -1 \\ & y_1, \ y_2, \ y_3 & \geq & 0. \end{array}$$

D-feasible

Introducing variables w_i, $i = 1, 2, 3$, for the primal slacks and z_j, $j = 1, 2$, for the dual slacks, we can write down the initial primal and dual dictionaries:

Dual Simplex Method [V] p.71

Step 0

(P)

$$
\begin{array}{rcrrrrrrr}
\zeta & = & & - & 1 & x_1 & - & 1 & x_2 \\
\hline
w_1 & = & 4 & + & 2 & x_1 & + & & x_2 \\
w_2 & = & -8 & + & 2 & x_1 & - & 4 & x_2 \\
w_3 & = & -7 & + & & x_1 & - & 3 & x_2
\end{array}
$$

$-(\quad)^T$

(D)

$$
\begin{array}{rcrrrrrrrrr}
-\xi & = & & - & 4 & y_1 & + & 8 & y_2 & + & 7 & y_3 \\
\hline
z_1 & = & 1 & - & 2 & y_1 & - & 2 & y_2 & - & & y_3 \\
z_2 & = & 1 & - & & y_1 & + & 4 & y_2 & + & 3 & y_3
\end{array}
$$

origin is D - feasible

Dual Simplex Method [V] p.71

Step 1

(P)

$$
\begin{array}{rcrcrcr}
\zeta & = & -4 & - & 0.5\ w_2 & - & 3\ x_2 \\
\hline
w_1 & = & 12 & + & w_2 & + & 5\ x_2 \\
x_1 & = & 4 & + & 0.5\ w_2 & + & 2\ x_2 \\
w_3 & = & -3 & + & 0.5\ w_2 & - & x_2
\end{array}
$$

$-(\quad)^T$

(D)

$$
\begin{array}{rcrcrcrcr}
-\xi & = & 4 & - & 12\ y_1 & - & 4\ z_1 & + & 3\ y_3 \\
\hline
y_2 & = & 0.5 & - & 1\ y_1 & - & 0.5\ z_1 & - & 0.5\ y_3 \\
z_2 & = & 3 & - & 5\ y_1 & - & 2\ z_1 & + & 1\ y_3
\end{array}
$$

Step 2

(P)

$$
\begin{array}{rcrcrcr}
\zeta & = & -7 & - & 1\ w_3 & - & 4\ x_2 \\
\hline
w_1 & = & 18 & + & 2\ w_3 & + & 7\ x_2 \\
x_1 & = & 7 & + & w_3 & + & 3\ x_2 \\
w_2 & = & 6 & + & 2\ w_3 & + & 2\ x_2
\end{array}
$$

$-(\quad)^T$

Opt. for both →

(D)

$$
\begin{array}{rcrcrcrcr}
-\xi & = & 7 & - & 18\ y_1 & - & 7\ z_1 & - & 6\ y_2 \\
\hline
y_3 & = & 1 & - & 2\ y_1 & - & z_1 & - & 2\ y_2 \\
z_2 & = & 4 & - & 7\ y_1 & - & 3\ z_1 & - & 2\ y_2
\end{array}
$$

Dual Based Phase I Algorithm [V] p.73

$$\begin{array}{lrcr}
\text{maximize} & -x_1 + 4x_2 & & \\
\text{subject to} & -2x_1 - x_2 & \leq & 4 \\
& -2x_1 + 4x_2 & \leq & -8 \\
& -x_1 + 3x_2 & \leq & -7 \\
& x_1,\ x_2 & \geq & 0.
\end{array}$$

(P)

$$\begin{array}{rcrrr}
\zeta & = & & -\ 1\ x_1 & \boxed{+\ 4}\ x_2 \\
\hline
w_1 & = & 4 & +\ 2\ x_1 & +\ \ x_2 \\
w_2 & = & \boxed{-8} & +\ 2\ x_1 & -\ 4\ x_2 \\
w_3 & = & \boxed{-7} & +\ \ x_1 & -\ 3\ x_2
\end{array}$$

(D)

$$\begin{array}{rcrrrr}
-\xi & = & & -\ 4\ y_1 & \boxed{+\ 8}\ y_2 & \boxed{+\ 7}\ y_3 \\
\hline
z_1 & = & 1 & -\ 2\ y_1 & -\ 2\ y_2 & -\ \ y_3 \\
z_2 & = & \boxed{-4} & -\ \ y_1 & +\ 4\ y_2 & +\ 3\ y_3
\end{array}$$

origin not feasible for both

Dual Based Phase I Algorithm [V] p.73

$$
\begin{array}{ll}
\text{maximize} & \underline{-x_1 + 4x_2} \\
\text{subject to} & -2x_1 - x_2 \le 4 \\
& -2x_1 + 4x_2 \le -8 \\
& -x_1 + 3x_2 \le -7 \\
& x_1,\ x_2 \ge 0.
\end{array}
$$

change obj fct to $-x_1 - x_2$

(P)

$$
\begin{array}{rcl}
\zeta & = & -\,1\,x_1 + \overset{-1}{\cancel{4}}\,x_2 \\
\hline
w_1 & = & 4 + 2x_1 + x_2 \\
w_2 & = & -8 + 2x_1 - 4x_2 \\
w_3 & = & -7 + x_1 - 3x_2
\end{array}
$$

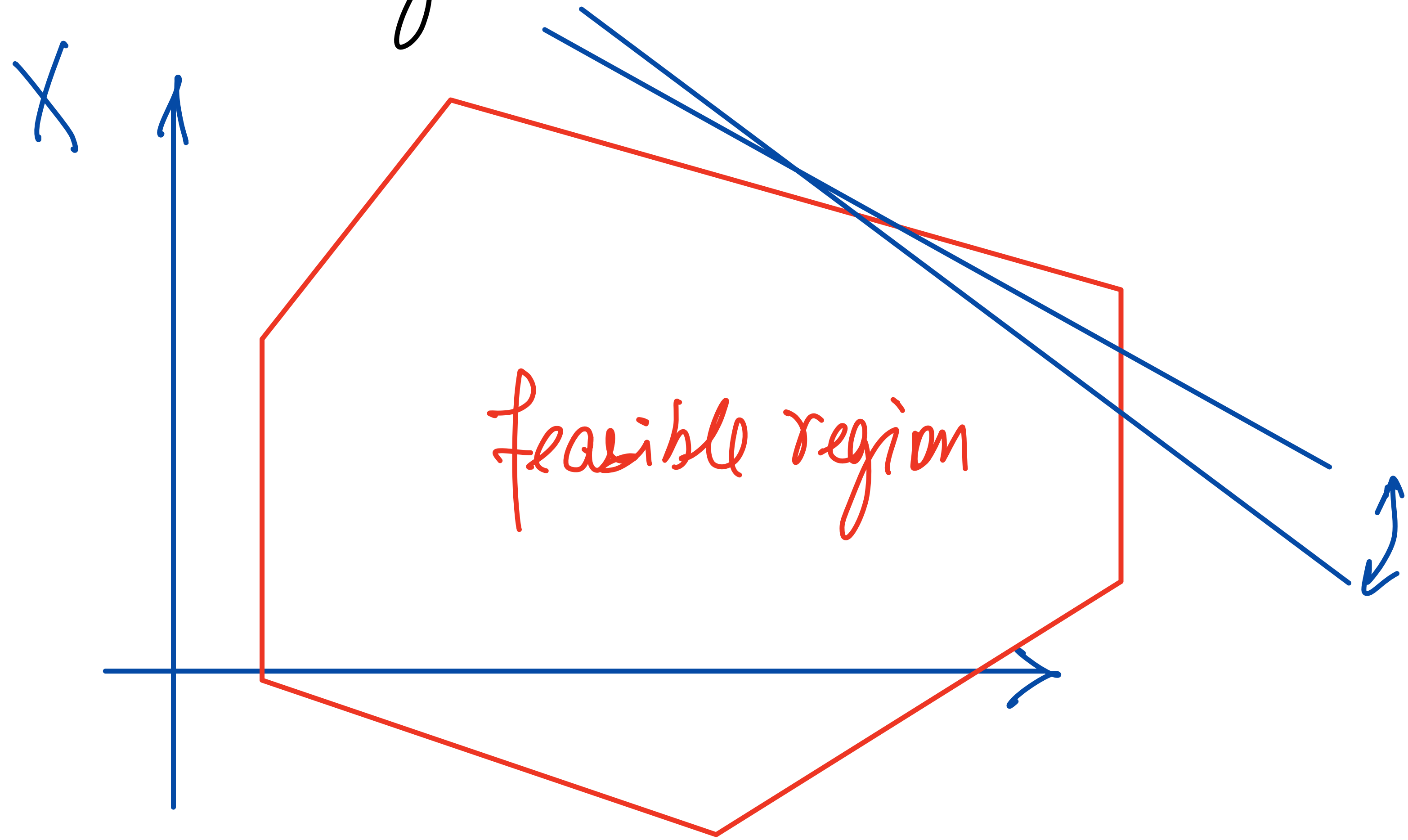
(D)

$$
\begin{array}{rcl}
-\xi & = & -\,4\,y_1 + 8\,y_2 + 7\,y_3 \\
\hline
z_1 & = & 1 - 2y_1 - 2y_2 - y_3 \\
z_2 & = & \underset{1}{\cancel{-4}} - y_1 + 4y_2 + 3y_3
\end{array}
$$

Now origin is D-feasible

Dual Based Phase I Algorithm [V] p.73

Changing the objective function does not change the feasible region:



Dual Based Phase I Algorithm [V] p.73

(P)

$$
\begin{array}{rcrl}
\zeta & = & & -\,1\,x_1 + \overset{-1}{\cancel{4}}\,x_2 \\
\hline
w_1 & = & 4 & +\,2\,x_1 + x_2 \\
w_2 & = & -8 & +\,2\,x_1 - 4\,x_2 \\
w_3 & = & -7 & +\,x_1 - 3\,x_2
\end{array}
$$

(D)

$$
\begin{array}{rcrl}
-\xi & = & & -\,4\,y_1 + 8\,y_2 + 7\,y_3 \\
\hline
z_1 & = & 1 & -\,2\,y_1 - 2\,y_2 - y_3 \\
z_2 & = & \underset{1}{\cancel{-4}} & -\,y_1 + 4\,y_2 + 3\,y_3
\end{array}
$$

opt

(P)

$$
\begin{array}{rcrl}
\zeta & = & -7 & -\,1\,w_3 - 4\,x_2 \\
\hline
w_1 & = & 18 & +\,2\,w_3 + 7\,x_2 \\
x_1 & = & 7 & +\,w_3 + 3\,x_2 \\
w_2 & = & 6 & +\,2\,w_3 + 2\,x_2
\end{array}
$$

(D)

$$
\begin{array}{rcrl}
-\xi & = & 7 & -\,18\,y_1 - 7\,z_1 - 6\,y_2 \\
\hline
y_3 & = & 1 & -\,2\,y_1 - z_1 - 2\,y_2 \\
z_2 & = & 4 & -\,7\,y_1 - 3\,z_1 - 2\,y_2
\end{array}
$$

Original objective fct

$$
\begin{aligned}
\zeta(x) &= -x_1 + 4x_2 \\
&= -7 - w_3 - 3x_2 + 4x_2 \\
&= -7 - w_3 + x_2
\end{aligned}
$$

Dual Based Phase II Algorithm [V] p.73

$$\max: \ \zeta(x) = -7 - w_3 + x_2$$

$$\text{s.t.} \quad w_1 = 18 + 2w_3 + 7x_2$$

$$x_1 = 7 + w_3 + 3x_2$$

$$w_2 = 6 + 2w_3 + 2x_2$$

(unbounded LP, for this example)

Complementary Slackness Condition

(P) $\max \quad \zeta(x) = c^T x$

s.t. $\quad Ax \le b$

$(x \ge 0)$

$A^{m\times n}$

$\underbrace{w}_{\mathbb{R}^m} = b - A\underbrace{x}_{\mathbb{R}^n} \ge 0$

(D) $\min \quad \xi(x) = b^T y$

s.t. $\quad A^T y \ge c$

$y \ge 0$

$(y^T b)$

$(y^T A \ge c^T)$

$\underbrace{z}_{\mathbb{R}^n} = -c + A^T \underbrace{y}_{\mathbb{R}^m} \ge 0$

Complementary Slackness Condition

THEOREM 5.3. *Suppose that $x = (x_1, x_2, \ldots, x_n)$ is primal feasible and that $y = (y_1, y_2, \ldots, y_m)$ is dual feasible. Let $(w_1, w_2, \ldots, w_m)$ denote the corresponding primal slack variables, and let $(z_1, z_2, \ldots, z_n)$ denote the corresponding dual slack variables. Then x and y are optimal for their respective problems if and only if*

$$\begin{aligned} x_j z_j &= 0, \qquad \text{for } j = 1, 2, \ldots, n, \\ w_i y_i &= 0, \qquad \text{for } i = 1, 2, \ldots, m. \end{aligned} \tag{5.7}$$

PROOF. We begin by revisiting the chain of inequalities used to prove the weak duality theorem:

$$\sum_j c_j x_j \le \sum_j \left(\sum_i y_i a_{ij} \right) x_j \tag{5.8}$$

$$= \sum_i \left(\sum_j a_{ij} x_j \right) y_i$$

$$\le \sum_i b_i y_i. \tag{5.9}$$

Complementary Slackness Condition

$$\zeta(x) = c^T x \le y^T A x \le y^T b = \xi(y)$$

$$=$$

$$c^T x = y^T A x$$
$$0 = (y^T A - c^T)\, x$$
$$= \underbrace{z^T}_{\geq 0} \underbrace{x}_{\geq 0}$$

$$y^T A x = y^T b$$
$$0 = y^T (b - Ax)$$
$$= \underbrace{y^T}_{\geq 0} \underbrace{w}_{\geq 0}$$

Certificate of Optimality

[V, p.69]

Duality theory is often useful in that it provides a *certificate of optimality*. For example, suppose that you were asked to solve a really huge and difficult linear program. After spending weeks or months at the computer, you are finally able to get the simplex method to solve the problem, producing as it does an optimal dual solution y^* in addition to the optimal primal solution x^*. Now, how are you going to convince your boss that your solution is correct? Do you really want to ask her to verify the correctness of your computer programs? The answer is probably not. And in fact it is not necessary. All you need to do is supply the primal and the dual solution, and she only has to check that the primal solution is feasible for the primal problem (that is easy), the dual solution is feasible for the dual problem (that is just as easy), and the primal and dual objective values agree (and that is even easier). Certificates of optimality have also been known to dramatically reduce the amount of time certain underpaid professors have to devote to grading homework assignments!

[C, p.62]

COMPLEMENTARY SLACKNESS

Now we shall show how the supervisor can often recover the certificate of optimality $y_1^*, y_2^*, \ldots, y_m^*$ from the optimal solution $x_1^*, x_2^*, \ldots, x_n^*$ alone. The key to the procedure is a convenient way of breaking down equation (5.16) into simple constituents.

THEOREM 5.2. Let $x_1^*, x_2^*, \ldots, x_n^*$ be a feasible solution of (5.12) and let $y_1^*, y_2^*, \ldots, y_m^*$ be a feasible solution of (5.13). Necessary and sufficient conditions for simultaneous optimality of $x_1^*, x_2^*, \ldots, x_n^*$ and $y_1^*, y_2^*, \ldots, y_m^*$ are

$$\sum_{i=1}^{m} a_{ij} y_i^* = c_j \quad \text{or} \quad x_j^* = 0 \quad \text{(or both)} \quad \text{for every} \quad j = 1, 2, \ldots, n \qquad (5.17)$$

$x_j^* z_j^* = 0$

and

$$\sum_{j=1}^{n} a_{ij} x_j^* = b_i \quad \text{or} \quad y_i^* = 0 \quad \text{(or both)} \quad \text{for every} \quad i = 1, 2, \ldots, m. \qquad (5.18)$$

$y_i^* w_i^* = 0$

[C, p.65]

THEOREM 5.3. A feasible solution $x_1^*, x_2^*, \ldots, x_n^*$ of (5.12) is optimal if and only if there are numbers $y_1^*, y_2^*, \ldots, y_m^*$ such that

$$\begin{aligned} \sum_{i=1}^{m} a_{ij} y_i^* &= c_j \quad \text{whenever} \quad x_j^* > 0 \\ y_i^* &= 0 \quad \text{whenever} \quad \sum_{j=1}^{n} a_{ij} x_j^* < b_i \end{aligned} \tag{5.22}$$

and such that

$$\begin{aligned} \sum_{i=1}^{m} a_{ij} y_i^* &\geq c_j \quad \text{for all} \quad j = 1, 2, \ldots, n \\ y_i^* &\geq 0 \quad \text{for all} \quad i = 1, 2, \ldots, m. \end{aligned} \tag{5.23}$$

THEOREM 5.4. If $x_1^*, x_2^*, \ldots, x_n^*$ is a nondegenerate basic feasible solution of (5.12), then (5.22) has a unique solution.

[C, p. 67]

maximize $40x_1 + 70x_2$

subject to

$$x_1 + x_2 \le 100$$
$$10x_1 + 50x_2 \le 4{,}000$$
$$x_1, x_2 \ge 0.$$

true? → Its optimal solution is $x_1^* = 25$ and $x_2^* = 75$.

$$\left.\begin{array}{l} x_1^* = 25 > 0 \implies z_1 = 0 \\ x_2^* = 75 > 0 \implies z_2 = 0 \end{array}\right\} \leftarrow \text{slacks for dual}$$

$$(D):\ A^T y \geq c, \quad z = A^T y - c = \begin{bmatrix} 1 & 10 \\ 1 & 50 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} 40 \\ 70 \end{bmatrix}$$

$$\left.\begin{array}{l} z_1 = 0 \implies y_1 + 10y_2 = 40 \\ z_2 = 0 \implies y_1 + 50y_2 = 70 \end{array}\right\} \implies \begin{array}{l} y_1 = \frac{130}{4} \\ y_2 = \frac{3}{4} \end{array}$$

[C, p. 67]

true? →

$$\begin{aligned} \text{maximize} \quad & 40x_1 + 70x_2 \\ \text{subject to} \quad & x_1 + x_2 \le 100 \\ & 10x_1 + 50x_2 \le 4{,}000 \\ & x_1, x_2 \ge 0. \end{aligned}$$

Its optimal solution is $x_1^* = 25$ and $x_2^* = 75$.

$$\zeta(x^*) = 40x_1^* + 70x_2^* = 40(25) + 70(75)$$
$$= 6250$$

the same

$$\xi(y^*) = 100y_1^* + 4000y_2^* = 100\left(\frac{130}{4}\right) + 4000\left(\frac{3}{4}\right)$$
$$= 6250$$

Hence opt.

[C, p.64]

First, let us consider the claim that

$$x_1^* = 2, \quad x_2^* = 4, \quad x_3^* = 0, \quad x_4^* = 0, \quad x_5^* = 7, \quad x_6^* = 0$$

is an optimal solution of the problem

$$\begin{array}{lrcl}
\text{maximize} & 18x_1 - 7x_2 + 12x_3 + 5x_4 + 8x_6 & & \\
\text{subject to} & 2x_1 - 6x_2 + 2x_3 + 7x_4 + 3x_5 + 8x_6 & \leq & 1 \quad = \\
& -3x_1 - x_2 + 4x_3 - 3x_4 + x_5 + 2x_6 & \leq & -2 \quad < \\
& 8x_1 - 3x_2 + 5x_3 - 2x_4 + 2x_6 & \leq & 4 \quad = \\
& 4x_1 + 8x_3 + 7x_4 - x_5 + 3x_6 & \leq & 1 \quad = \\
& 5x_1 + 2x_2 - 3x_3 + 6x_4 - 2x_5 - x_6 & \leq & 5 \quad < \\
& x_1, x_2, x_3, x_4, x_5, x_6 & \geq & 0.
\end{array}$$

$z_1 = 0, \; z_2 = 0, \; z_5 = 0, \quad y_2 = 0, \; y_5 = 0$

[C, p.64]

In this case, (5.22) reads

$$\begin{aligned}
2y_1^* - 3y_2^* + 8y_3^* + 4y_4^* + 5y_5^* &= 18 \\
-6y_1^* - y_2^* - 3y_3^* \qquad\qquad + 2y_5^* &= -7 \\
3y_1^* + y_2^* \qquad\qquad - y_4^* - 2y_5^* &= 0 \\
y_2^* &= 0 \\
y_5^* &= 0.
\end{aligned}$$

$A^T y \geq c$

Since its solution $(\frac{1}{3}, 0, \frac{5}{3}, 1, 0)$ satisfies (5.23), the proposed solution $x_1^*, x_2^*, \ldots, x_6^*$ is optimal.

Second, let us consider the claim that

$$x_1^* = 0, \quad x_2^* = 2, \quad x_3^* = 0, \quad x_4^* = 7, \quad x_5^* = 0$$

is an optimal solution of the problem

[C, p.64] $A^T y \geq C$

$$\begin{bmatrix} 2 & -3 & 8 & 4 & 5 \\ -6 & -1 & -3 & 0 & 2 \\ 2 & 4 & 5 & 8 & -3 \\ 7 & -3 & -2 & 7 & 6 \\ 3 & 1 & 0 & -1 & -2 \\ 8 & 2 & 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1/3 \\ 0 \\ 5/3 \\ 1 \\ 0 \end{bmatrix} \overset{\checkmark}{\geq} \begin{bmatrix} 18 \\ -7 \\ 12 \\ 5 \\ 0 \\ 8 \end{bmatrix}$$

[C, p.65]

Second, let us consider the claim that

$$x_1^* = 0, \quad \underline{x_2^* = 2}, \quad x_3^* = 0, \quad \underline{x_4^* = 7}, \quad x_5^* = 0$$

is an optimal solution of the problem

$$\begin{array}{lrl}
\text{maximize} & 8x_1 - 9x_2 + 12x_3 + 4x_4 + 11x_5 & \\
\text{subject to} & 2x_1 - 3x_2 + 4x_3 + x_4 + 3x_5 \le 1 & = \\
& x_1 + 7x_2 + 3x_3 - 2x_4 + x_5 \le 1 & < \\
& 5x_1 + 4x_2 - 6x_3 + 2x_4 + 3x_5 \le 22 & = \\
& x_1, x_2, x_3, x_4, x_5 \ge 0. &
\end{array}$$

Here (5.22) becomes

$z_2 = 0, \quad z_4 = 0, \quad y_2 = 0$

$$\begin{aligned}
-3y_1^* + 7y_2^* + 4y_3^* &= -9 \\
y_1^* - 2y_2^* + 2y_3^* &= 4 \\
y_2^* &= 0.
\end{aligned}$$

$A^T y \ge c$

Since its unique solution (3.4, 0, 0.3) <u>violates (5.23)</u>, the proposed solution $x_1^*, x_2^*, \ldots, x_5^*$ is not optimal.

$$\begin{bmatrix} 2 & 1 & 5 \\ -3 & 7 & 4 \\ 4 & 3 & -6 \\ 1 & -2 & 2 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3.4 \\ 0 \\ 0.3 \end{bmatrix} \geqslant \begin{array}{cl} \begin{bmatrix} 8 \\ -9 \\ 12 \\ 4 \\ 11 \end{bmatrix} & \begin{array}{l} \\ \\ \leftarrow x \\ \\ \\ \end{array} \end{array}$$