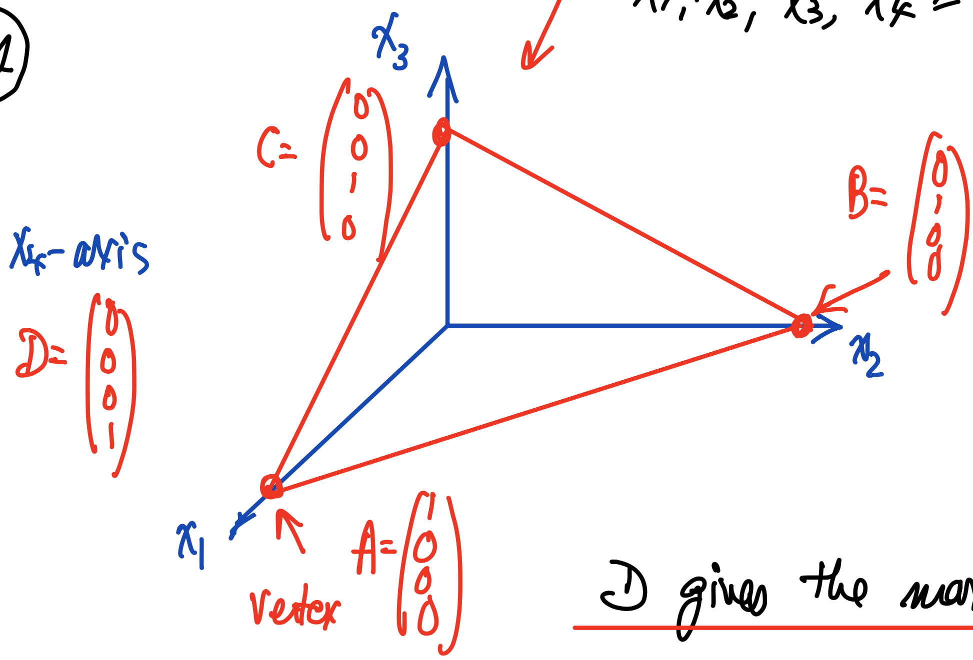
MA 421 Review for Midterm (Fall 2025)

[V] 2.10

$$\max \quad 6x_1 + 8x_2 + 5x_3 + 9x_4$$
$$\text{s.t.} \quad x_1 + x_2 + x_3 + x_4 = 1$$
$$x_1, x_2, x_3, x_4 \geq 0$$

M1



D gives the maximum value, 9.

M2

$$\max \quad 6x_1 + 8x_2 + 5x_3 + 9x_4$$
$$\text{s.t.} \quad x_1 + x_2 + x_3 + x_4 = 1$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Single out x_4: $\quad x_4 = 1 - x_1 - x_2 - x_3$

Objective fct becomes:

$$\zeta(x_1, x_2, x_3) = 6x_1 + 8x_2 + 5x_3 + 9(1 - x_1 - x_2 - x_3)$$
$$= \underline{9 - 3x_1 - x_2 - 4x_3}$$

$$\text{s.t.} \quad x_1, x_2, x_3 \geq 0$$

opt. dictionary

M3 Consider the dual problem:
(Note the primal problem has only 1 constraint.)

$$\left(\begin{array}{ll} \max & 6x_1 + 8x_2 + 5x_3 + 9x_4 \\ \text{s.t.} & x_1 + x_2 + x_3 + x_4 = 1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}\right)$$

$$\min_y \quad 1 \cdot y$$

$$\text{s.t.} \quad \left.\begin{array}{l} y \geq 6 \\ y \geq 8 \\ y \geq 5 \\ y \geq 9 \end{array}\right\} \Rightarrow y \geq 9$$

Hence $y = 9$ is the min.

By complementary slackness $\Rightarrow$

$$\boxed{\begin{array}{ll} x_1 = 0 & (\text{Since } 9 > 6) \\ x_2 = 0 & (\text{Since } 9 > 8) \\ x_3 = 0 & (\text{Since } 9 > 5) \\ x_4 = 1 & (\text{Since } x_1 + x_2 + x_3 + x_4 = 1) \end{array}}$$

[C] 1.2 min $-8x_1 + 9x_2 + 2x_3 - 6x_4 - 5x_5$

s.t. $6x_1 + 6x_2 - 10x_3 + 2x_4 - 8x_5 \geq 3$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

M1 From the coeff. of x_1, x_4

$\Rightarrow$ x_1, x_2 can be as big as possible

$\Rightarrow$ $\min = -\infty$ (unbounded)

M2 Consider dual problem:

($y \geq 0 \Rightarrow 3y \leq 6yx_1 + 6yx_2 - 10yx_3 + 2yx_4 - 8yx_5$

$\leq -8x_1 + 9x_2 + 2x_3 - 6x_4 - 5x_5$)

$$\max_{y \geq 0} \; 3y$$

s.t.

$6y \leq -8$ ← infeasible with $y \geq 0$

$6y \leq 9$

$-10y \leq 2$

$2y \leq -6$ ← infeasible with $y \geq 0$

$-8y \leq -5$

Hence primal problem is unbounded

(Since primal problem is feasibe to start with)

[V] **2.3**

$$\begin{array}{lrcl} \text{maximize} & 2x_1 - 6x_2 & & \\ \text{subject to} & -x_1 - x_2 - x_3 & \le & -2 \\ & 2x_1 - x_2 + x_3 & \le & 1 \\ & x_1,\ x_2,\ x_3 & \ge & 0. \end{array}$$

M) Phase I using auxiliary variable.

$$\max \ -x_0$$

$$-x_1 - x_2 - x_3 - x_0 \le -2$$
$$2x_1 - x_2 + x_3 - x_0 \le 1$$

$x_i \geq 0 \quad i = 0, 1, 2, 3$

$$w_1 = -2 + x_0 + x_1 + x_2 + x_3$$ ← least feasible
$$w_2 = 1 + x_0 - 2x_1 + x_2 - x_3$$

$$x_0 = 2 - x_1 - x_2 - x_3 + w_1$$

$$w_2 - w_1 = 3 - 3x_1 - 2x_3$$

i.e. $$w_2 = 3 - 3x_1 - 2x_3 + w_1$$

Initial dictionary:

$$\boxed{\begin{array}{ll} \max & -2 + x_1 + x_2 + x_3 - w_1 \\ \text{s.t.} & x_0 = 2 - x_1 - x_2 - x_3 + w_1 \\ & w_2 = 3 - 3x_1 - 2x_3 + w_1 \end{array}}$$

Opt. dict. (using simplex)

$$\boxed{\begin{array}{ll} \zeta = & -x_0 \quad = 0! \\ \text{s.t.} & x_2 = 1 + \frac{1}{3}w_2 - x_0 - \frac{1}{3}x_3 + \frac{2}{3}w_1 \\ & x_1 = 1 - \frac{1}{3}w_2 - \frac{2}{3}x_3 + \frac{1}{3}w_1 \end{array}}$$

End of phase I

Back to original problem:

$$\zeta = 2x_1 - 6x_2$$

$$= 2\left(1 - \tfrac{1}{3}w_2 \quad - \tfrac{2}{3}x_3 + \tfrac{1}{3}w_1\right)$$

$$-6\left(1 + \tfrac{1}{3}w_2 - \cancel{x_0} - \tfrac{1}{3}x_3 + \tfrac{2}{3}w_1\right)$$

$$\boxed{\begin{aligned} \zeta &= -4 - \tfrac{8}{3}w_2 + \tfrac{2x_3}{3} - \tfrac{10w_1}{3} \\ x_2 &= 1 + \tfrac{1}{3}w_2 - \tfrac{1}{3}x_3 + \tfrac{2}{3}w_1 \\ x_1 &= 1 - \tfrac{1}{3}w_2 - \tfrac{2}{3}x_3 + \tfrac{1}{3}w_1 \end{aligned}}$$

feasible dictionary for Phase II.

One step simplex

x_3 enters, x_1 leaves

$$\boxed{\begin{aligned} \zeta &= -3 - 3w_2 - x_1 - w_1 \\ x_2 &= \tfrac{1}{2} + \tfrac{w_2}{2} + \tfrac{x_1}{2} + \tfrac{w_1}{2} \\ x_3 &= \tfrac{3}{2} - \tfrac{w_2}{2} - \tfrac{3}{2}x_1 + \tfrac{w_1}{2} \end{aligned}}$$

Opt!

(If we would have known to solve for x_2, x_3 to start with ....)

M2

2.3 maximize ~~$2x_1 - 6x_2$~~ change to $-x_1-x_2-x_3$

(dual based Phase I)

$$\begin{aligned} \text{subject to} \quad -x_1 - x_2 - x_3 &\le -2 \\ 2x_1 - x_2 + x_3 &\le 1 \\ x_1,\ x_2,\ x_3 &\ge 0. \end{aligned}$$

$$\begin{aligned} \min \quad & -2y_1 + y_2 \\ \text{s.t.} \quad & -y_1 + 2y_2 \geq -1 \\ & -y_1 - y_2 \geq -1 \\ & -y_1 + y_2 \geq -1 \\ & y_1, y_2, y_3 \geq 0 \end{aligned} \iff \boxed{\begin{aligned} \max \quad & -\xi = 2y_1 - y_2 \\ \text{s.t.} \quad & y_1 - 2y_2 \leq 1 \\ & y_1 + y_2 \leq 1 \\ & y_1 - y_2 \leq 1 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}}$$

$$\left(\begin{aligned} z_1 &= 1 - y_1 + 2y_2 \\ z_2 &= 1 - y_1 - y_2 \\ z_3 &= 1 - y_1 + y_2 \end{aligned} \right)$$

$$\Downarrow$$

$$\boxed{\begin{aligned} -\xi &= 2 - z_1 - z_2 \\ y_1 &= 1 - \frac{z_1}{3} - \frac{2}{3} z_2 \\ y_2 &= \frac{z_1}{3} - \frac{1}{3} z_2 \\ z_3 &= \frac{2z_1}{3} + \frac{z_2}{3} \end{aligned}}$$

$z_1 \sim x_1$, $z_2 \sim x_2$, $z_3 \sim x_3$

$y_1 \sim w_1$, $y_2 \sim w_2$

$$\boxed{\begin{aligned} x_1 &= 1 + \frac{1}{3} w_1 - \frac{1}{3} w_2 - \frac{2}{3} x_3 \\ x_2 &= 1 + \frac{2}{3} w_1 + \frac{1}{3} w_2 - \frac{1}{3} x_3 \end{aligned}}$$

$$A^T = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \Longleftarrow A = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

Back to original problem:

$$\zeta(x) = 2x_1 - 6x_6$$

$$= 2\left(1 + \frac{1}{3}w_1 - \frac{1}{3}w_2 - \frac{2}{3}x_3\right)$$

$$-6\left(1 + \frac{2}{3}w_1 + \frac{1}{3}w_2 - \frac{1}{3}x_3\right)$$

$$\boxed{\begin{aligned} \zeta(x) &= -4 - \frac{10w_1}{3} - \frac{8}{3}w_2 + \frac{2}{3}x_3 \\ x_1 &= 1 + \frac{1}{3}w_1 - \frac{1}{3}w_2 - \frac{2}{3}x_3 \\ x_2 &= 1 + \frac{2}{3}w_1 + \frac{1}{3}w_2 - \frac{1}{3}x_3 \end{aligned}}$$

(same as M1)

[V] **2.4**

$$\begin{array}{ll} \text{maximize} & -x_1 - 3x_2 - x_3 \\ \text{subject to} & 2x_1 - 5x_2 + x_3 \leq -5 \\ & 2x_1 - x_2 + 2x_3 \leq 4 \\ & x_1,\ x_2,\ x_3 \geq 0. \end{array}$$

neg coeff.

Use dual

$$\begin{array}{ll} \min & \xi = -5y_1 + 4y_2 \\ \text{s.t.} & 2y_1 + 2y_2 \geq -1 \\ & -5y_1 - y_2 \geq -3 \\ & y_1 + 2y_2 \geq -1 \\ & y_1,\ y_2,\ y_3 \geq 0 \end{array} \iff \begin{array}{ll} \max & -\xi = 5y_1 - 4y_2 \\ \text{s.t.} & -2y_1 - 2y_2 \leq 1 \\ & 5y_1 + y_2 \leq 3 \\ & -y_1 - 2y_2 \leq 1 \\ & y_1,\ y_2,\ y_3 \geq 0 \end{array}$$

feasible dict. to start with

one step

Opt.!

$$\boxed{\begin{aligned} -\xi &= 3 - z_2 - y_2 \\ z_1 &= \tfrac{11}{5} - \tfrac{2}{5} z_2 + \tfrac{8}{5} y_2 \\ y_1 &= \tfrac{3}{5} - \tfrac{1}{5} z_2 - \tfrac{1}{5} y_2 \\ z_3 &= \tfrac{8}{5} - \tfrac{1}{5} z_2 + \tfrac{9}{5} y_2 \end{aligned}} \implies$$

$$\underbrace{\begin{aligned} &z_1 > 0 \Rightarrow \underline{x_1 = 0} \\ &y_1 > 0 \Rightarrow w_1 = 0 \\ &\qquad \Rightarrow 2x_1 - 5x_2 + x_3 = -5 \\ &z_3 > 0 \Rightarrow \underline{x_3 = 0} \end{aligned}}$$

$$\Downarrow$$

$$\underline{x_2 = 1}$$

$$\underline{\zeta^* = -x_1 - 3x_2 - x_3 = -3}$$

Same as ξ^*

[V] **2.6** maximize $\cancel{x_1 + 3x_2}$ change to e.g. $-x_1 - x_2$

subject to
$$\begin{aligned} -x_1 - x_2 &\le -3 \\ -x_1 + x_2 &\le -1 \\ x_1 + 2x_2 &\le 2 \\ x_1,\ x_2 &\ge 0. \end{aligned}$$

Consider dual:

$$\begin{aligned} \min\ & -3y_1 - y_2 + 2y_3 \\ \text{s.t.}\ & -y_1 - y_2 + y_3 \ge -1 \\ & -y_1 + y_2 + 2y_3 \ge -1 \\ & y_1, y_2 \ge 0 \end{aligned} \iff \begin{aligned} \max\ & 3y_1 + y_2 - 2y_3 \\ \text{s.t.}\ & y_1 + y_2 - y_3 \le 1 \\ & y_1 - y_2 - 2y_3 \le 1 \\ & y_1, y_2, y_3 \ge 0 \end{aligned}$$

$\Updownarrow$

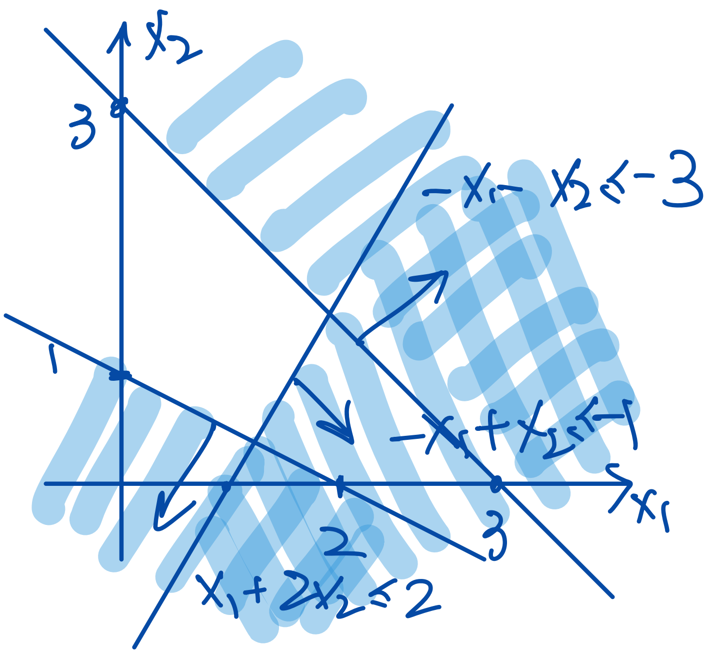
"Opt. dict"

$$\begin{aligned} \zeta &= 3 - 3z_1 - 2y_2 + \underline{y_3} \\ y_1 &= 1 - z_1 - y_2 + \underline{y_3} \\ z_2 &= 0 + z_1 + y_2 + \underline{y_3} \end{aligned}$$

$\Downarrow$

unbounded

$\Leftarrow$ initial problem not feasible



[V] #3.4 $\max f(X) = \sum_{j}^{n} c_j x_j$

(M1) s.t. $\sum_j a_{ij} x_j \le 0 \qquad i = 1, \dots, m$

$x_j \ge 0 \qquad j = 1, \dots, n$

Must be feasible (take: $x_1 = x_2 = \dots = x_n = 0$)

Note: $f(0) = 0$

Either $f(0) = 0$ is optimal,

i.e. $f(X) \le 0$ for all feasible X

Or there is $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ s.t. $f(X) > 0$

If X is feasible, so is tX, $t > 0$.

then $f(tX) = t f(X) \to +\infty$ as $t \to +\infty$

(M2) Consider dual problem

$\min g(Y) = 0y_1 + 0y_2 + \dots + 0y_m \ (= 0)$

s.t. $\sum_i a_{ij} y_i \ge c_j \qquad j = 1, \dots, n$

$y_i \ge 0 \qquad i = 1, \dots, m$

Note that the dual objective function is always 0

Hence

if the dual problem is feasible, then $\xi^* = 0$

$$\Rightarrow \zeta^* = 0$$

if the dual problem is not feasible, then $\zeta^* = +\infty$ (unbounded)

(Since the primal problem is always feasible.)

[C]

2.2 Use the simplex method to describe *all* the optimal solutions of the following problem:

$$\begin{array}{ll} \text{maximize} & 2x_1 + 3x_2 + 5x_3 + 4x_4 \\ \text{subject to} & x_1 + 2x_2 + 3x_3 + x_4 \leq 5 \quad \leftarrow w_1 \\ & x_1 + x_2 + 2x_3 + 3x_4 \leq 3 \quad \leftarrow w_2 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{array}$$

$$\begin{cases} w_1 = 5 - x_1 - 2x_2 - 3x_3 - x_4 \\ w_2 = 3 - x_1 - x_2 - 2x_3 - 3x_4 \end{cases}$$

Use simplex $\Longrightarrow$ optimal dictionary:

$$\begin{aligned} \zeta &= 8 - w_1 - w_2 \\ x_2 &= 1 + x_1 - 2w_1 + 3w_2 + 7x_4 \\ x_3 &= 1 - x_1 + w_1 - 2w_2 - 5x_4 \end{aligned}$$

Solution: $\zeta^* = 8$, $w_1 = 0$, $w_2 = 0$

$$\begin{aligned} x_2 &= 1 + x_1 + 7x_4 \\ x_3 &= 1 - x_1 - 5x_4 \end{aligned}$$

All solutions:

$$\boxed{\begin{aligned} & x_1, x_2, x_3, x_4 \geq 0 \\ & x_2 = 1 + x_1 + 7x_4 \\ & x_3 = 1 - x_1 - 5x_4 \end{aligned}}$$

[v] #5.1

$$\begin{aligned}
\max \quad & x_1 - 2x_2 \\
\text{s.t.} \quad & -x_1 - 2x_2 + x_3 - x_4 \leq 0 \\
& 4x_1 + 3x_2 + 4x_3 - 2x_4 \leq 3 \\
& -x_1 - x_2 + 2x_3 + x_4 = 1 \\
& x_2, x_3 \geq 0
\end{aligned}$$

$$\begin{aligned}
-y_1 x_1 - 2y_1 x_2 + y_1 x_3 - y_1 x_4 &\leq 0 & y_1 \geq 0 \\
4y_2 x_1 + 3y_2 x_2 + 4y_2 x_3 - 2y_2 x_4 &\leq 3y_2 & y_2 \geq 0 \\
-y_3 x_1 - y_3 x_2 + 2y_3 x_3 + y_3 x_4 &= y_3 &
\end{aligned}$$

$$\underbrace{(-y_1 + 4y_2 - y_3)x_1 + (-2y_1 + 3y_2 - y_3)x_2 + (y_1 + 4y_2 + 2y_3)x_3 + (-y_1 - 2y_2 + y_3)x_4}_{\geq \; x_1 - 2x_2} \leq 3y_2 + y_3$$

Hence

$$\begin{aligned}
x_1 &\leq (-y_1 + 4y_2 - y_3)x_1 \Rightarrow -y_1 + 4y_2 - y_3 = 1 \\
-2x_2 &\leq (-2y_1 + 3y_2 - y_3)x_2 \xRightarrow{(x_2 \geq 0)} -2y_1 + 3y_2 - y_3 \geq -2 \\
0 &\leq (y_1 + 4y_2 + 2y_3)x_3 \xRightarrow{(x_3 \geq 0)} y_1 + 4y_2 + 2y_3 \geq 0 \\
0 &\leq (-y_1 - 2y_2 + y_3)x_4 \Rightarrow -y_1 - 2y_2 + y_3 = 0
\end{aligned}$$

Dual Problem:

$$\begin{aligned} \min \quad & 3y_2 + y_3 \\ \text{s.t.} \quad & -y_1 + 4y_2 - y_3 = 1 \\ & -2y_1 + 3y_2 - y_3 \geq -2 \\ & y_1 + 4y_2 + 2y_3 \geq 0 \\ & -y_1 - 2y_2 + y_3 = 0 \\ & y_1, y_2 \geq 0 \end{aligned}$$

5.6 Solve the following linear program:

negative coeffs.

$$\begin{array}{lrcr} \text{maximize} & -x_1 - 2x_2 & & \\ \text{subject to} & -2x_1 + 7x_2 & \leq & 6 \\ & -3x_1 + x_2 & \leq & -1 \\ & 9x_1 - 4x_2 & \leq & 6 \\ & x_1 - x_2 & \leq & 1 \\ & 7x_1 - 3x_2 & \leq & 6 \\ & -5x_1 + 2x_2 & \leq & -3 \\ & x_1, x_2 & \geq & 0. \end{array}$$

Origin not feasible.

Consider dual problem:

$$\begin{array}{ll} \min & 6y_1 - y_2 + 6y_3 + y_4 + 6y_5 - 3y_6 \\ \text{s.t.} & -2y_1 - 3y_2 + 9y_3 + y_4 + 7y_5 - 5y_6 \geq -1 \\ & 7y_1 + y_2 - 4y_3 - y_4 - 3y_5 + 2y_6 \geq -2 \\ & y_1, \ldots, y_6 \geq 0 \end{array}$$

Now the origin is feasible for dual problem.

$$\left(\begin{array}{lll} \text{or} & \max & -6y_1 + y_2 - 6y_3 - y_4 - 6y_5 + 3y_6 \\ & \text{s.t.} & 2y_1 + 3y_2 - 9y_3 - y_4 - 7y_5 + 5y_6 \leq 1 \\ & & -7y_1 - y_2 + 4y_3 + y_4 + 3y_5 - 2y_6 \leq 2 \\ & & y_1, \ldots, y_6 \geq 0 \end{array} \right)$$

Init Dict.

$$
\begin{array}{llcrcrcrcrcrcrcr}
\text{maximize} & -\zeta & = & & & -6\,y_1 & + & 1\,y_2 & + & -6\,y_3 & + & -1\,y_4 & + & -6\,y_5 & + & 3\,y_6 \\
\text{subject to:} & z_1 & = & 1 & - & 2\,y_1 & - & 3\,y_2 & - & -9\,y_3 & - & -1\,y_4 & - & -7\,y_5 & - & 5\,y_6 \\
 & z_2 & = & 2 & - & -7\,y_1 & - & -1\,y_2 & - & 4\,y_3 & - & 1\,y_4 & - & 3\,y_5 & - & -2\,y_6
\end{array}
$$

$$y_1 \;\; y_2 \;\; y_3 \;\; y_4 \;\; y_5 \;\; y_6 \quad z_1 \;\; z_2 \geq 0$$

Opt. Dict. (after one step)

$$
\begin{array}{llcrcrcrcrcrcrcr}
\text{maximize} & -\zeta & = & 3/5 & + & -36/5\,y_1 & + & -4/5\,y_2 & + & -3/5\,y_3 & + & -2/5\,y_4 & + & -9/5\,y_5 & + & -3/5\,z_1 \\
\text{subject to:} & y_6 & = & 1/5 & - & 2/5\,y_1 & - & 3/5\,y_2 & - & -9/5\,y_3 & - & -1/5\,y_4 & - & -7/5\,y_5 & - & 1/5\,z_1 \\
 & z_2 & = & 12/5 & - & -31/5\,y_1 & - & 1/5\,y_2 & - & 2/5\,y_3 & - & 3/5\,y_4 & - & 1/5\,y_5 & - & 2/5\,z_1
\end{array}
$$

$$y_1 \;\; y_2 \;\; y_3 \;\; y_4 \;\; y_5 \;\; y_6 \quad z_1 \;\; z_2 \geq 0$$

$$
\left.\begin{array}{ll}
y_6 = \frac{1}{5} \Rightarrow & -5x_1 + 2x_2 = -3 \\
z_2 = \frac{12}{5} \Rightarrow & x_2 = 0
\end{array}\right\} \Rightarrow \left(\begin{array}{l} x_1 = 3/5 \\ x_2 = 0 \end{array} \right)
$$

$$\zeta^* = -\frac{3}{5}, \iff \xi^* = -\frac{3}{5} - 2(0) = -\frac{3}{5}$$

(Note that $\left(x_1 = \frac{3}{5},\ x_2 = 0\right)$ satisfies all the constraints:

$$
\begin{array}{rcrlcl}
-2x_1 + 7x_2 & \leq & 6 & & < & y_1 = 0 \\
-3x_1 + x_2 & \leq & -1 & & < & y_2 = 0 \\
9x_1 - 4x_2 & \leq & 6 & & < & y_3 = 0 \\
x_1 - x_2 & \leq & 1 & & < & y_4 = 0 \\
7x_1 - 3x_2 & \leq & 6 & & < & y_5 = 0 \\
-5x_1 + 2x_2 & \leq & -3 & & = & y_6 > 0 \\
x_1,\ x_2 & \geq & 0. & & &
\end{array}
$$

)

[C] p.69 #5.3

a. Maximize $7x_1 + 6x_2 + 5x_3 - 2x_4 + 3x_5$

subject to

$$
\begin{aligned}
x_1 + 3x_2 + 5x_3 - 2x_4 + 2x_5 &\le 4 \quad \text{— } y_1\\
4x_1 + 2x_2 - 2x_3 + x_4 + x_5 &\le 3 \quad \text{— } y_2\\
2x_1 + 4x_2 + 4x_3 - 2x_4 + 5x_5 &\le 5 \quad \text{— } y_3\\
3x_1 + x_2 + 2x_3 - x_4 - 2x_5 &\le 1 \quad \text{— } y_4\\
x_1, x_2, x_3, x_4, x_5 &\ge 0.
\end{aligned}
$$

Proposed solution: $x_1^* = 0, \quad x_2^* = \frac{4}{3}, \quad x_3^* = \frac{2}{3}, \quad x_4^* = \frac{5}{3}, \quad x_5^* = 0.$

$x_2, x_3, x_4 > 0 \Rightarrow z_2 = z_3 = z_4 = 0$

y_1: $3\left(\frac{4}{3}\right) + 5\left(\frac{2}{3}\right) - 2\left(\frac{5}{3}\right) = 4$

y_2: $2\left(\frac{4}{3}\right) - 2\left(\frac{2}{3}\right) + \left(\frac{5}{3}\right) = 3$

y_3: $4\left(\frac{4}{3}\right) + 4\left(\frac{2}{3}\right) - 2\left(\frac{5}{3}\right) = \frac{14}{3} < 5 \Rightarrow y_3 = 0$

y_4: $\left(\frac{4}{3}\right) + 2\left(\frac{2}{3}\right) - \frac{5}{3} = 1$

Dual:

$$
\begin{aligned}
y_1 + 4y_2 + 2y_3 + 3y_4 &\ge 7 && z_1\\
3y_1 + 2y_2 + 4y_3 + y_4 &= 6 && z_2 = 0\\
5y_1 - 2y_2 + 4y_3 + 2y_4 &= 5 && z_3 = 0\\
-2y_1 + y_2 - 2y_3 - y_4 &= -2 && z_4 = 0\\
2y_1 + y_2 + 5y_3 - 2y_4 &\ge 3 && z_5
\end{aligned}
$$

$$3y_1+2y_2+y_4=6$$
$$5y_1-2y_2+2y_4=5 \qquad (y_3=0)$$
$$-2y_1+y_2-y_4=-2$$

$$\Rightarrow \begin{cases} y_1=1,\ y_2=1,\ y_4=1 \geq 0 \ \checkmark \\ y_1+4y_2+2y_3+3y_4 \geq 7 \ \checkmark \\ 2y_1+y_2+5y_3-2y_4 \geq 3 \ \checkmark \end{cases} \Bigg\} \ \underline{\underline{\text{Opt!}}}$$

(Note:
$$\zeta = 7x_1+6x_2+5x_3-2x_4+3x_5$$
$$= 7(0)+6\left(\frac{4}{3}\right)+5\left(\frac{2}{3}\right)-2\left(\frac{5}{3}\right)+3(0)$$
$$=8$$

$$\xi = 4y_1+3y_2+5y_3+y_4$$
$$=4(1)+3(1)+5(0)+(1)$$
$$=8.$$
)

$$\#$$

[C] p.69 #5.3

b. Maximize $4x_1 + 5x_2 + x_3 + 3x_4 - 5x_5 + 8x_6$

subject to

$$\begin{array}{rl}
x_1 - 4x_3 + 3x_4 + x_5 + x_6 \le 1 & y_1 \\
5x_1 + 3x_2 + x_3 - 5x_5 + 3x_6 \le 4 & y_2 \\
4x_1 + 5x_2 - 3x_3 + 3x_4 - 4x_5 + x_6 \le 4 & y_3 \\
-x_2 + 2x_4 + x_5 - 5x_6 \le 5 & y_4 \\
-2x_1 + x_2 + x_3 + x_4 + 2x_5 + 2x_6 \le 7 & y_5 \\
2x_1 - 3x_2 + 2x_3 - x_4 + 4x_5 + 5x_6 \le 5 & y_6 \\
x_1, x_2, x_3, x_4, x_5, x_6 \ge 0. &
\end{array}$$

Proposed solution: $x_1 = 0,\quad x_2 = 0,\quad x_3 = \frac{5}{2},\quad x_4 = \frac{7}{2},\quad x_5 = 0,\quad x_6 = \frac{1}{2}.$

$X_3, X_4, X_6 > 0 \Rightarrow Z_3, Z_4, Z_6 = 0$

y_1: $-4\left(\frac{5}{2}\right) + 3\left(\frac{7}{2}\right) + \frac{1}{2} = 1$

y_2: $\left(\frac{5}{2}\right) + 3\left(\frac{1}{2}\right) = 4$

y_3: $-3\left(\frac{5}{2}\right) + 3\left(\frac{7}{2}\right) + \frac{1}{2} = \frac{7}{2} < 4 \Rightarrow y_3 = 0$

y_4: $2\left(\frac{7}{2}\right) - 5\left(\frac{1}{2}\right) = \frac{9}{2} < 5 \Rightarrow y_4 = 0$

y_5: $\left(\frac{5}{2}\right) + \left(\frac{7}{2}\right) + 2\left(\frac{1}{2}\right) = 7$

y_6: $2\left(\frac{5}{2}\right) - \frac{7}{2} + 5\left(\frac{1}{2}\right) = 4 < 5 \Rightarrow y_6 = 0$

$Z_3 = 0 \Rightarrow -4y_1 + y_2 - 3y_3 + y_5 + 2y_6 = 1$

$Z_4 = 0 \Rightarrow 3y_1 + 3y_3 + 2y_4 + y_5 - y_6 = 3$

$Z_6 = 0 \Rightarrow y_1 + 3y_2 + \underline{y_3} - \underline{5y_4} + 2y_5 + \underline{5y_6} = 8$

$$y_3 = y_4 = y_6 = 0 \Rightarrow \begin{cases} -4y_1 + y_2 + y_5 = 1 \\ 3y_1 + y_5 = 3 \\ y_1 + 3y_2 + 2y_5 = 8 \end{cases}$$

$$y_1 = \frac{1}{2}, \quad y_2 = \frac{3}{2}, \quad y_5 = \frac{3}{2} \geq 0$$

z_1: $y_1 + 5y_2 + 4y_3 - 2y_5 + 2y_6 \geq 4$

$$\left(\frac{1}{2} + 5\left(\frac{3}{2}\right) - 2\left(\frac{3}{2}\right) = 5 > 4\right)$$

z_2: $3y_2 + 5y_3 - y_4 + y_5 - 3y_6 \geq 5$

$$\left(3\left(\frac{3}{2}\right) + \frac{3}{2} = 6 > 5\right)$$

z_5: $y_1 - 5y_2 - 4y_3 + y_4 + 2y_5 + 4y_6 \geq -5$

$$\left(\frac{1}{2} - 5\left(\frac{3}{2}\right) + 2\left(\frac{3}{2}\right) = -4 \geq -5\right)$$

Opt.!

Note:

$$\zeta = 4x_1 + 5x_2 + x_3 + 3x_4 - 5x_5 + 8x_6$$
$$= \frac{5}{2} + 3\left(\frac{7}{2}\right) + 8\left(\frac{1}{2}\right) = 17$$

$$\xi = y_1 + 4y_2 + 4y_3 + 5y_4 + 7y_5 + 5y_6$$
$$= \left(\frac{1}{2}\right) + 4\left(\frac{3}{2}\right) + 7\left(\frac{3}{2}\right) = 17$$

6.1 Consider the following linear programming problem:

$$
\begin{aligned}
\text{maximize} \quad & -6x_1 + 32x_2 - 9x_3 \\
\text{subject to} \quad & -2x_1 + 10x_2 - 3x_3 \leq -6 \\
& x_1 - 7x_2 + 2x_3 \leq 4 \\
& x_1,\ x_2,\ x_3 \geq 0.
\end{aligned}
$$

Suppose that, in solving this problem, you have arrived at the following dictionary:

$$
\begin{array}{rcrcrcrcr}
\zeta & = & -18 & - & 3\,x_4 & + & 2\,x_2 & & \\
\hline
x_3 & = & 2 & - & x_4 & + & 4\,x_2 & - & 2\,x_5 \\
x_1 & = & 0 & + & 2\,x_4 & - & x_2 & + & 3\,x_5
\end{array}
$$

(a) Which variables are basic? Which are nonbasic?
(b) Write down the vector, $x_{\mathcal{B}}^*$, of current primal basic solution values.
(c) Write down the vector, $z_{\mathcal{N}}^*$, of current dual nonbasic solution values.
(d) Write down $B^{-1}N$.
(e) Is the primal solution associated with this dictionary feasible?
(f) Is it optimal?
(g) Is it degenerate?

(a) Basic vars: X_3, X_1; Non-basic vars: X_4, X_2, X_5

(b) $X_B^* = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ (c) $Z_N^* = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$

(d) $B^{-1}N = \begin{pmatrix} 1 & -4 & 2 \\ -2 & 1 & -3 \end{pmatrix}$

(e) Yes as $X_B^* \geq 0$

(f) No as $Z_N^* \ngeq 0$

(g) Yes as 2nd entry of $X_B^* = 0$

6.2 Consider the following linear programming problem:

$$\begin{array}{lrl} \text{maximize} & x_1 + 2x_2 + 4x_3 + 8x_4 + 16x_5 & \\ \text{subject to} & x_1 + 2x_2 + 3x_3 + 4x_4 + \ 5x_5 & \leq 2 \\ & 7x_1 + 5x_2 - 3x_3 - 2x_4 \qquad & \leq 0 \\ & x_1,\ x_2,\ x_3,\ x_4,\ x_5 & \geq 0. \end{array}$$

Consider the situation in which x_3 and x_5 are basic and all other variables are nonbasic. Write down:

(a) B,
(b) N,
(c) b,
(d) $c_{\mathcal{B}}$,
(e) $c_{\mathcal{N}}$,
(f) $B^{-1}N$,
(g) $x^*_{\mathcal{B}} = B^{-1}b$,
(h) $\zeta^* = c^T_{\mathcal{B}}B^{-1}b$,
(i) $z^*_{\mathcal{N}} = (B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}}$,
(j) the dictionary corresponding to this basis.

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\ \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 1 & 0 \\ 7 & 5 & -3 & -2 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \left.\begin{matrix} \\ \end{matrix}\right\} \text{slacks}$$

$\mathcal{B} = \{3, 5\}$ $\quad$ $\mathcal{N} = \{1, 2, 4, 6, 7\}$

(a) $B = \begin{bmatrix} 3 & 5 \\ -3 & 0 \end{bmatrix}$

(b) $N = \begin{bmatrix} 1 & 2 & 4 & 1 & 0 \\ 7 & 5 & -2 & 0 & 1 \end{bmatrix}$

(c) $b = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

(d) $\zeta = x_1 + 2x_2 + 4x_3 + 8x_4 + 16x_5 + 0x_6 + 0x_7$

$C_B = [4 \quad 16]^T$

(e) $C_N = [1 \quad 2 \quad 8 \quad 0 \quad 0]^T$

(f) $B^{-1}N = \dfrac{\begin{bmatrix} 0 & -5 \\ 3 & 3 \end{bmatrix}}{15} \begin{bmatrix} 1 & 2 & 4 & 1 & 0 \\ 7 & 5 & -2 & 0 & 1 \end{bmatrix}$

$= \dfrac{1}{15} \begin{bmatrix} -35 & -25 & 10 & 0 & -5 \\ 24 & 21 & 6 & 3 & 3 \end{bmatrix}$

(g) $B^{-1}b = \dfrac{1}{15} \begin{bmatrix} 0 & -5 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \dfrac{1}{15} \begin{bmatrix} 0 \\ 6 \end{bmatrix}$

(h) $\zeta^* = C_B^T B^{-1} b$

$= [4 \quad 16] \begin{bmatrix} 0 \\ 6 \end{bmatrix} \dfrac{1}{15} = \dfrac{96}{15} = \dfrac{32}{5}$

(i) $(Z_N^*) = (B^{-1}N)^T C_B - C_N$

$$= \frac{1}{15}\begin{bmatrix} -35 & 24 \\ -25 & 21 \\ 10 & 6 \\ 0 & 3 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 16 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 8 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{15}\begin{bmatrix} 244 \\ 236 \\ 136 \\ 48 \\ 28 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 8 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 229 \\ 206 \\ 16 \\ 48 \\ 28 \end{bmatrix}$$

(j)

ζ^*

$$\begin{array}{c|c|ccccc}
 & & x_1 & x_2 & x_4 & x_6 & x_7 \\
 & \frac{32}{5} & -\frac{229}{15} & -\frac{206}{15} & -\frac{16}{15} & -\frac{48}{15} & -\frac{28}{15} \\
\hline
x_3 & 0 & \frac{7}{3} & \frac{5}{3} & -\frac{2}{3} & 0 & \frac{1}{3} \\
x_5 & \frac{2}{5} & -\frac{8}{5} & -\frac{7}{5} & -\frac{2}{5} & -\frac{1}{5} & -\frac{1}{5}
\end{array}$$

$-(Z_N^*)^T$

$B^{-1}b$

$\underbrace{\qquad\qquad\qquad}_{-B^{-1}N}$

(Opt!)

7.1 The final dictionary for

$$
\begin{array}{ll}
\text{maximize} & x_1 + 2x_2 + x_3 + x_4 \\
\text{subject to} & 2x_1 + x_2 + 5x_3 + x_4 \le 8 \\
& 2x_1 + 2x_2 + 4x_4 \le 12 \\
& 3x_1 + x_2 + 2x_3 \le 18 \\
& x_1,\ x_2,\ x_3,\ x_4 \ge 0
\end{array}
$$

is

$$
\begin{array}{rcl}
\zeta &=& 12.4 - 1.2\,x_1 - 0.2\,x_5 - 0.9\,x_6 - 2.8\,x_4 \\
\hline
x_2 &=& 6 - x_1 - 0.5\,x_6 - 2\,x_4 \\
x_3 &=& 0.4 - 0.2\,x_1 - 0.2\,x_5 + 0.1\,x_6 + 0.2\,x_4 \\
x_7 &=& 11.2 - 1.6\,x_1 + 0.4\,x_5 + 0.3\,x_6 + 1.6\,x_4
\end{array}
$$

(the last three variables are the slack variables).

(a) What will be an optimal solution to the problem if the objective function is changed to

$$3x_1 + 2x_2 + x_3 + x_4?$$

$$(c_1\ c_2\ c_3\ c_4\ c_5\ c_6\ c_7) = (1\ 2\ 1\ 1\ 0\ 0\ 0)$$

$$\mathcal{B} = \{2, 3, 7\}, \qquad N = \{1, 5, 6, 4\}$$

$$c_{\mathcal{B}} = (2, 1, 0)^T, \qquad c_N = (1, 0, 0, 0)^T$$

$$x_{\mathcal{B}}^* = B^{-1} b = \begin{pmatrix} 6 \\ 0.4 \\ 11.2 \end{pmatrix}$$

$$z_N^* = (B^{-1}N)^T c_{\mathcal{B}} - c_N = (1.2, 0.2, 0.9, 2.8)^T$$

$$B^{-1}N = \begin{pmatrix} 1 & 0 & 0.5 & 2 \\ 0.2 & 0.2 & -0.1 & -0.2 \\ 1.6 & -0.4 & -0.3 & -1.6 \end{pmatrix}$$

C_1 changes from 1 to 3

$\Delta C_1 = 2 \Rightarrow \Delta C_N = (2, 0, 0, 0)^T$

C_B no change, $(\Delta C_B = 0)$

$$\Delta Z_N^* = -\Delta C_N = (-2, 0, 0, 0)^T$$

$$Z_N^*(\text{new}) = Z_N^*(\text{old}) + \Delta Z_N^*$$

$$= (1.2,\ 0.2,\ 0.9,\ 2.8)^T + (-2,\ 0,\ 0,\ 0)^T$$

$$= (\underline{-0.8},\ 0.2,\ 0.9,\ 2.8)^T \not\geq 0$$

Hence dictionary is not opt. for the new C.

But we can still start from here:

$$\begin{array}{rl}
\zeta = & 12.4 \cancel{-\ 1.2}^{\,+0.8}\ x_1 - 0.2\ x_5 - 0.9\ x_6 - 2.8\ x_4 \\
\hline
x_2 = & 6 - x_1 - 0.5\ x_6 - 2\ x_4 \\
x_3 = & 0.4 - 0.2\ x_1 - 0.2\ x_5 + 0.1\ x_6 + 0.2\ x_4 \\
x_7 = & 11.2 - 1.6\ x_1 + 0.4\ x_5 + 0.3\ x_6 + 1.6\ x_4
\end{array}$$

X_1 enters and X_3 leaves (use Simplex tool to continue)

(b) What will be an optimal solution to the problem if the objective function is changed to

$$x_1 + 2x_2 + 0.5x_3 + x_4?$$

C_3 changes from 1 to 0.5

$$\Delta C_3 = -0.5 \Longrightarrow \Delta C_B = (0, -0.5, 0)^T$$

$$Z_N^* = (B^{-1}N)^T C_B - C_N = (1.2, 0.2, 0.9, 2.8)^T$$

$(C_2, C_3, C_4)^T$ no change

$$\Delta Z_N^* = (B^{-1}N)^T \Delta C_B = \begin{bmatrix} 1 & 0 & 0.5 & 2 \\ 0.2 & 0.2 & -0.1 & -0.2 \\ 1.6 & -0.4 & -0.3 & -1.6 \end{bmatrix}^T \begin{bmatrix} 0 \\ -0.5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -0.1 & -0.1 & 0.05 & 0.1 \end{bmatrix}^T$$

Hence $Z_N^*(\text{new}) = Z_N^*(\text{old}) + \Delta Z_N^*$

$$= (1.2 \ 0.2 \ 0.9 \ 2.8)^T + (-0.1 \ -0.1 \ 0.05 \ 0.1)^T$$

$$= (1.1 \ 0.1 \ 0.95 \ 2.9)^T > 0$$ still opt.

$$\zeta^* = C_B^T B^{-1} b$$

$$\zeta^*(\text{new}) = \zeta^*(\text{old}) + \Delta C_B^T B^{-1} b$$

$$= 12.4 + (0 \ -0.5 \ 0)\begin{pmatrix} 6 \\ 0.4 \\ 11.2 \end{pmatrix}$$

$$= 12.4 - 0.2 = 12.2$$

Still, $(x_1, x_2, x_3, x_4) = (0, 6, 0.4, 0)$

(c) What will be an optimal solution to the problem if the second constraint's right-hand side is changed to 26?

b_2 changes from $\underline{12 \text{ to } 26}$

The only thing needs to be checked is X_B^*

$\underline{Z_N^* = (B^{-1}N)^T C_B - C_N \text{ is not changed}}$

$X_B^* = B^{-1}b, \qquad \Delta X_B^* = B^{-1}\Delta b$

$$B = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}, \quad B^{-1}\Delta b = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 14 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ -1.4 \\ -4.2 \end{bmatrix}$$

$\mathcal{B} = (?, 3, 7)$

$$X_B^*(\text{new}) = X_B^*(\text{old}) + \Delta X_B^*$$

$$= \begin{pmatrix} 6 \\ 8.4 \\ 11.2 \end{pmatrix} + \begin{pmatrix} 7 \\ -1.4 \\ -4.2 \end{pmatrix} = \begin{pmatrix} 13 \\ -1 \\ 7 \end{pmatrix} \not\geq 0$$

not opt.

Start from:

$$
\begin{array}{rcl}
\zeta & = & 12.4 - 1.2\,x_1 - 0.2\,x_5 - 0.9\,x_6 - 2.8\,x_4 \\
\hline
x_2 & = & \overset{13}{\cancel{3}} - x_1 - 0.5\,x_6 - 2\,x_4 \\
x_3 & = & \overset{-1}{\cancel{0.4}} - 0.2\,x_1 - 0.2\,x_5 + 0.1\,x_6 + 0.2\,x_4 \\
x_7 & = & \underset{7}{\cancel{11.2}} - 1.6\,x_1 + 0.4\,x_5 + 0.3\,x_6 + 1.6\,x_4
\end{array}
$$

dictionary not feasible anymore but dual feasible

So start from its dual version:

$$
\begin{array}{c|cccc}
-\xi & -12.4 & -13 & 1 & -7 \\
\hline
 & 1.2 & 1 & 0.2 & 1.6 \\
 & 0.2 & 0 & 0.2 & -0.4 \\
 & 0.9 & 0.5 & -0.1 & -0.3 \\
 & 2.8 & 2 & -0.2 & -1.6
\end{array}
$$

use simplex to continue ....

9.5 Generalize Theorems 5.2 and 5.3 to the context of general LP problems. Use your result to find out if

$$x_1^* = 3, \quad x_2^* = -1, \quad x_3^* = 0, \quad x_4^* = 2$$

is an optimal solution of the problem

$$\begin{array}{ll} \text{maximize} & 6x_1 + x_2 - x_3 - x_4 \\ \text{subject to} & x_1 + 2x_2 + x_3 + x_4 \le 5 \\ & 3x_1 + x_2 - x_3 \le 8 \\ & x_2 + x_3 + x_4 = 1 \\ & x_3, x_4 \ge 0. \end{array}$$

$6x_1 + x_2 - x_3 - x_4$ $\qquad x_3, x_4 \geq 0$

(*) $\leq (y_1+3y_2)x_1 + (2y_1+y_2+y_3)x_2 + (y_1-y_2+y_3)x_3 + (y_1 \quad +y_3)x_4$

$\leq 5y_1 + 8y_2 + y_3$ $\qquad y_1 \geq 0, y_2 \geq 0.$

$$\boxed{\begin{array}{lll} (D) & \min & 5y_1 + 8y_2 + y_3 \\ & \text{s.t.} & y_1 + 3y_2 = 6 \\ & & 2y_1 + y_2 + y_3 = 1 \\ (**) \rightarrow & & y_1 - y_2 + y_3 \geq -1 \\ & & y_1 \quad + y_3 \geq -1 \\ & & y_1, y_2 \geq 0 \end{array}}$$

(x_1 has no sign)
(x_2 has no sign)
($x_3 \geq 0$)
($x_4 \geq 0$)

$x_1^* = 3_{\neq 0}$, $x_2^* = -1_{\neq 0}$, $x_3^* = 0$, $x_4^* = 2 > 0$

$\boxed{\text{To find } y^*, \text{ we need tightness of } (*):}$

$$6x_1 + x_2 - x_3 - x_4 \qquad x_3, x_4 \geq 0$$

$$(*) \leq (y_1+3y_2)x_1 + (2y_1+y_2+y_3)x_2 + (y_1-y_2+y_3)x_3 + (y_1 + y_3)x_4 \neq$$

$$\leq 5y_1 + 8y_2 + y_3 \qquad y_1 \geq 0, y_2 \geq 0.$$

i.e.

$$\begin{cases} y_1 + 3y_2 = 6 \\ 2y_1 + y_2 + y_3 = 1 \\ y_1 + y_3 = -1 \end{cases} \Rightarrow \begin{array}{l} y_1 + 3y_2 = 6 \\ y_1 + y_2 = 2 \end{array}$$

$$\Rightarrow \quad y_2 = 2,\ y_1 = 0,\ y_3 = -1$$

check $(**)$ $\quad y_1 - y_2 + y_3 = 0-2-1 = -3 \cancel{\geq} -1$

$\boxed{\text{Hence } x^* \text{ is not optimal}}$

HW3 3. Consider the following LP problem:

$$\begin{array}{lll} \max & 4x_1 + 2x_2 - x_3 + 2x_4 & \\ \text{s.t.} & x_1 + x_2 + 3x_3 + 4x_4 & = 8, \\ & x_1 + x_2 + x_3 + x_4 & = 4, \\ & x_1, x_2, x_3, x_4 \geq 0. & \end{array}$$

In the following, you will solve this problem using three methods.

M1

Choose x_2 & x_3 as basic variable:

$$\begin{cases} x_2 + 3x_3 = 8 - x_1 - 4x_4 \\ x_2 + x_3 = 4 - x_1 - x_4 \end{cases}$$

$$\begin{cases} x_2 = 2 - x_1 + \frac{1}{2}x_4 \\ x_3 = 2 \quad - \frac{3}{2}x_4 \end{cases}$$

$$\zeta(x) = 4x_1 + 2\left(2 - x_1 + \frac{1}{2}x_4\right) - \left(2 - \frac{3}{2}x_4\right) + 2x_4$$

$$\boxed{\begin{aligned} \zeta(x) &= 2 + 2x_1 + \frac{9}{2}x_4 \\ x_2 &= 2 - x_1 + \frac{1}{2}x_4 \\ x_3 &= 2 \quad - \frac{3}{2}x_4 \end{aligned}}$$

$\Rightarrow$ feasible dict.

$\Rightarrow$ Solve using simplex

M2

$$\begin{aligned} x_1 + x_2 + 3x_3 + 4x_4 &\leq 8 \\ -x_1 - x_2 - 3x_3 - 4x_4 &\leq -8 \\ x_1 + x_2 + x_3 + x_4 &\leq 4 \\ -x_1 - x_2 - x_3 - x_4 &\leq -4 \end{aligned}$$

x_0 auxiliary var.

$$\begin{aligned} x_1 + x_2 + 3x_3 + 4x_4 - x_0 &\leq 8 \\ -x_1 - x_2 - 3x_3 - 4x_4 - x_0 &\leq -8 \\ x_1 + x_2 + x_3 + x_4 - x_0 &\leq 4 \\ -x_1 - x_2 - x_3 - x_4 - x_0 &\leq -4 \end{aligned}$$

$$w_1 = 8 - x_1 - x_2 - 3x_3 - 4x_4 + x_0$$
$$w_2 = -8 + x_1 + x_2 + 3x_3 + 4x_4 + x_0$$
$$w_3 = 4 - x_1 - x_2 - x_3 - x_4 + x_0$$
$$w_4 = -4 + x_1 + x_2 + x_3 + x_4 + x_0$$

Most infeasible:

Phase I

$$\boxed{\begin{aligned} \max\ \underline{\zeta = -x_0} &= -8 + x_1 + x_2 + 3x_3 + 4x_4 - w_2 \\ x_0 &= 8 - x_1 - x_2 - 3x_3 - 4x_4 + w_2 \\ w_1 &= 16 - 2x_1 - 2x_2 - 6x_3 - 8x_4 + w_2 \\ w_3 &= 12 - 2x_1 - 2x_2 - 4x_3 - 5x_4 + w_2 \\ w_4 &= 4 \qquad\qquad\quad - 2x_3 - 3x_4 + w_2 \end{aligned}}$$

feasible dictionary.

M3 Using auxiliary var.:

Phase I

$$\begin{cases} x_1 + x_2 + 3x_3 + 4x_4 + y_1 = 8 \\ x_1 + x_2 + x_3 + x_4 + y_2 = 4 \\ \underline{\max\ -y_1 - y_2} \end{cases}$$

$$\Downarrow$$

s.t.
$$\underbrace{\begin{aligned} y_1 &= 8 - x_1 - x_2 - 3x_3 - 4x_4 \\ y_2 &= 4 - x_1 - x_2 - x_3 - x_4 \end{aligned}}_{\text{feasible dict.}}$$

$$\begin{aligned} \zeta(x) &= -y_1 - y_2 \\ &= -12 + 2x_1 + 2x_2 + 4x_3 + 5x_4 \end{aligned}$$

HW5 3. As a newly hired data scientist, your very first assignment was to solve problem (2.1) in [V, p.11].

This was such a difficult problem that you needed to use one week to solve it. You submitted its solution (2.4) in [V, p.13] to your boss.

Your boss said, "Oops. I forgot to tell you an extra constraint, $3x_1 + x_2 - x_3 \le 3$. Can you tell me the new solution, *real soon*?"

Your inner-self tells you that "*real soon*" cannot mean another week of work. What would you do[1]?

$$
\begin{array}{llrlrlrlr}
\text{maximize} & \zeta & = & & +5x_1 & +4x_2 & +3x_3 \\
\text{subject to} & w_1 & = & 5 & -2x_1 & -3x_2 & -x_3 \\
 & w_2 & = & 11 & -4x_1 & -x_2 & -2x_3 \\
 & w_3 & = & 8 & -3x_1 & -4x_2 & -2x_3 \\
\end{array}
$$

$$x_1,\ x_2,\ x_3,\ w_1,\ w_2,\ w_3 \ge 0.$$

$\Rightarrow$

$$
\begin{array}{rcrrrr}
\zeta & = & 13 & -1w_1 & -3x_2 & -1w_3 \\
\hline
x_1 & = & 2 & -2w_1 & -2x_2 & +w_3 \\
w_2 & = & 1 & +2w_1 & +5x_2 & \\
x_3 & = & 1 & +3w_1 & +x_2 & -2w_3 \\
\end{array}
$$

Add $w_4 = 3 - 3x_1 - x_2 + x_3$

$= 3 - 3(2 - 2w_1 - 2x_2 + w_3) - x_2$

$+ (1 + 3w_1 + x_2 - 2w_3)$

$= -2 + 9w_1 + 6x_2 - 5w_3$

← all neg coeff

$$
\begin{array}{rcrrrr}
\zeta & = & 13 & -1w_1 & -3x_2 & -1w_3 \\
\hline
x_1 & = & 2 & -2w_1 & -2x_2 & +w_3 \\
w_2 & = & 1 & +2w_1 & +5x_2 & \\
x_3 & = & 1 & +3w_1 & +x_2 & -2w_3 \\
\end{array}
$$

$w_4 = \underline{-3} + 9w_1 + 6x_2 - 5w_3$

not feasible

$\Rightarrow$ dual

$$
\begin{array}{r|rrrr}
-13 & -2 & -1 & -1 & 3 \\
\hline
1 & 2 & -2 & -3 & -9 \\
3 & 2 & -5 & -1 & -6 \\
1 & -1 & 0 & 2 & 5 \\
\end{array}
$$

HW6 #1 [C] p.116, #7.5

7.5 Solve the systems $\mathbf{y}\mathbf{E}_1\mathbf{E}_2\mathbf{E}_3\mathbf{E}_4 = [1, 2, 3]$ and $\mathbf{E}_1\mathbf{E}_2\mathbf{E}_3\mathbf{E}_4\mathbf{d} = [1, 2, 3]^T$ with

$$\mathbf{E}_1 = \begin{bmatrix} 1 & 3 & \\ & 0.5 & \\ & 4 & 1 \end{bmatrix}, \quad \mathbf{E}_2 = \begin{bmatrix} 2 & & \\ 1 & 1 & \\ 4 & & 1 \end{bmatrix}, \quad \mathbf{E}_3 = \begin{bmatrix} 1 & & 1 \\ & 1 & 3 \\ & & 1 \end{bmatrix}, \quad \mathbf{E}_4 = \begin{bmatrix} 0.5 & & \\ 3 & 1 & \\ 1 & & 1 \end{bmatrix}.$$

$$y = [1 \;\; 2 \;\; 3]\, E_4^{-1} E_3^{-1} E_2^{-1} E_1^{-1}$$

$$= [1 \;\; 2 \;\; 3] \begin{bmatrix} -2 & & \\ 6 & 1 & \\ 2 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & -1 \\ & 1 & -3 \\ & & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & & \\ -\frac{1}{2} & 1 & \\ -2 & & 1 \end{bmatrix} \begin{bmatrix} 1 & -6 & \\ & 2 & \\ & -8 & 1 \end{bmatrix}$$

$$= [16 \;\; 2 \;\; 3]$$

$$[16 \;\; 2 \;\; -19]$$

$$[45 \;\; 2 \;\; -19]$$

$$[45 \;\; -114 \;\; -19]$$

$$d = E_4^{-1} E_3^{-1} E_2^{-1} E_1^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & & \\ 6 & 1 & \\ 2 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & -1 \\ & 1 & -3 \\ & & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & & \\ -\frac{1}{2} & 1 & \\ -2 & & 1 \end{bmatrix} \begin{bmatrix} 1 & -6 & \\ & 2 & \\ & -8 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 29 \\ -\frac{209}{2} \\ -20 \end{bmatrix} \leftarrow \begin{bmatrix} -\frac{29}{2} \\ -\frac{35}{2} \\ 9 \end{bmatrix} \leftarrow \begin{bmatrix} -\frac{11}{2} \\ \frac{19}{2} \\ 9 \end{bmatrix} \leftarrow \begin{bmatrix} -11 \\ 4 \\ -13 \end{bmatrix}$$

Hw6 #2

$$\begin{array}{ll} \max & -2x_1 - x_2 - My_1 - My_2 \\ \text{s.t.} & 3x_1 + 4x_2 + y_1 = 12, \\ & -x_1 + 2x_2 + w_1 = 2, \\ & x_1 + 4x_2 - w_2 + y_2 = 6, \\ & x_1, x_2, w_1, w_2, y_1, y_2 \geq 0, \end{array}$$

$$\begin{cases} y_1 = 12 - 3x_1 - 4x_2 \\ w_1 = 2 + x_1 - 2x_2 \\ y_2 = 6 - x_1 - 4x_2 + w_2 \end{cases}$$

$$\zeta = -2x_1 - x_2 - M(12 - 3x_1 - 4x_2) - M(6 - x_1 - 4x_2 + w_2)$$

$$\zeta = -18M + (4M - 2)x_1 + (8M - 1)x_2 - Mw_2$$

Set $M = 1000$

maximize	ζ	=	-18000	+	3998	x_1	+	7999	x_2	+	-1000	w_2
subject to:	y_1	=	12	-	3	x_1	-	4	x_2	-	0	w_2
	w_1	=	2	-	-1	x_1	-	2	x_2	-	0	w_2
	y_2	=	6	-	1	x_1	-	4	x_2	-	-1	w_2

maximize	ζ	=	-10001	+	15995/2	x_1	+	-7999/2	w_1	+	-1000	w_2
subject to:	y_1	=	8	-	5	x_1	-	-2	w_1	-	0	w_2
	x_2	=	1	-	-1/2	x_1	-	1/2	w_1	-	0	w_2
	y_2	=	2	-	3	x_1	-	-2	w_1	-	-1	w_2

$$\text{maximize} \quad \zeta = -14008/3 + (-15995/6)\, y_2 + 7993/6\, w_1 + 9995/6\, w_2$$

$$\begin{aligned}
\text{subject to:} \quad y_1 &= 14/3 - (-5/3)\, y_2 - 4/3\, w_1 - 5/3\, w_2 \\
x_2 &= 4/3 - 1/6\, y_2 - 1/6\, w_1 - (-1/6)\, w_2 \\
x_1 &= 2/3 - 1/3\, y_2 - (-2/3)\, w_1 - (-1/3)\, w_2
\end{aligned}$$

$$\text{maximize} \quad \zeta = -5 + (-1000)\, y_2 + (-1/2)\, w_1 + (-1999/2)\, y_1$$

$$\begin{aligned}
\text{subject to:} \quad w_2 &= 14/5 - (-1)\, y_2 - 4/5\, w_1 - 3/5\, y_1 \\
x_2 &= 9/5 - 0\, y_2 - 3/10\, w_1 - 1/10\, y_1 \\
x_1 &= 8/5 - 0\, y_2 - (-2/5)\, w_1 - 1/5\, y_1
\end{aligned}$$

Opt: $\quad y_2 = 0 \quad y_1 = 0$

$$x_1 = \frac{8}{5}, \quad x_2 = \frac{9}{5}, \quad \zeta = -5$$

$$\left(= -2\left(\frac{8}{5}\right) - \left(\frac{9}{5}\right) \right)$$

Hw6

#3

Find (all) the Pareto solution(s) for the following problems.

(a) $\min\{\zeta_1(x) = x,\ \zeta_2(x) = -x\}$
s.t. $1 \le x \le 2$. (This is a one-dimensional problem!)

(b) $\min\{\zeta_1(x_1, x_2) = x_1 + 2x_2,\ \zeta_2(x_1, x_2) = 2x_1 + x_2\}$
s.t. $1 \le x_1 \le 2,\ 1 \le x_2 \le 2$.

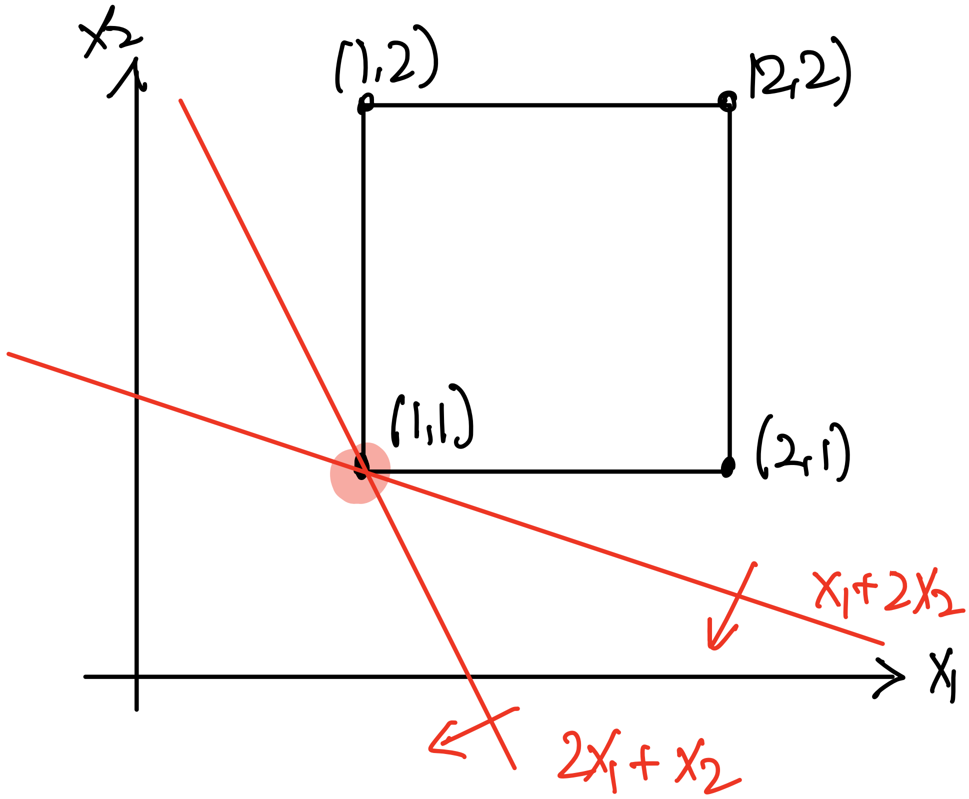
(c) $\min\{\zeta_1(x_1, x_2) = x_1 + x_2,\ \zeta_2(x_1, x_2) = -x_1 + x_2\}$
s.t. $1 \le x_1 \le 2,\ 1 \le x_2 \le 2$.

(d) $\min\{\zeta_1(x_1, x_2) = x_1 + x_2,\ \zeta_2(x_1, x_2) = -x_1 + x_2,\ \zeta_3(x_1, x_2) = -x_2\}$
s.t. $1 \le x_1 \le 2,\ 1 \le x_2 \le 2$.

(a)

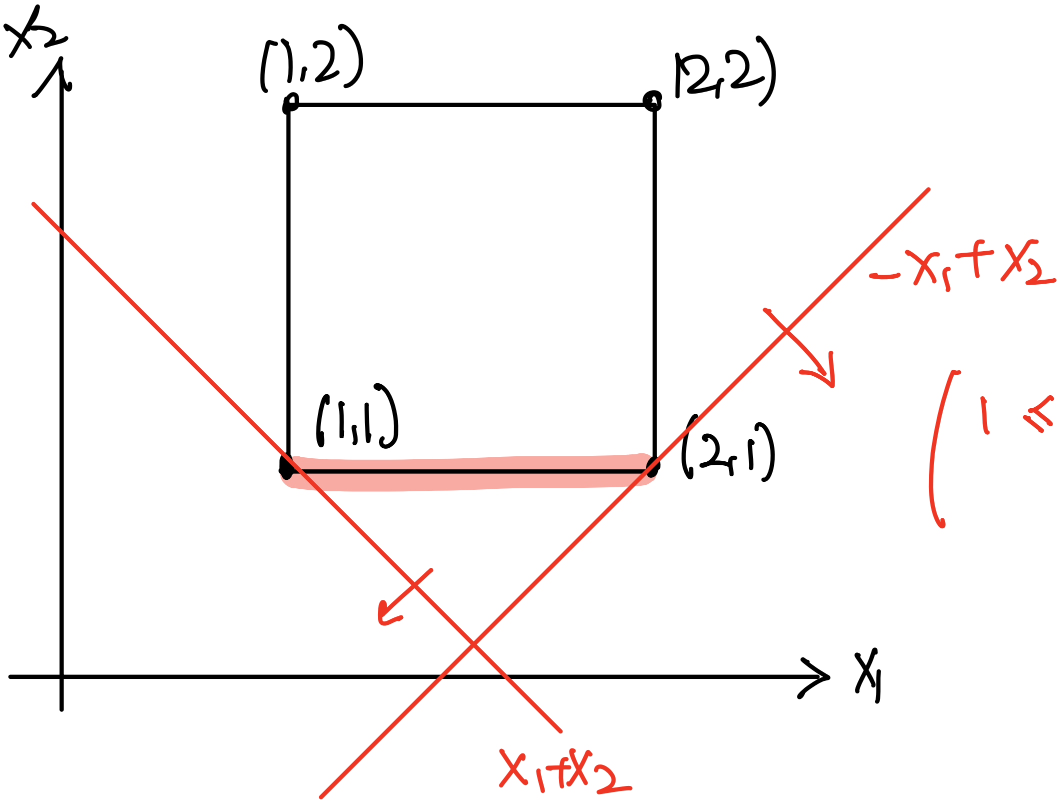
$1 \le x \le 2$

(b)



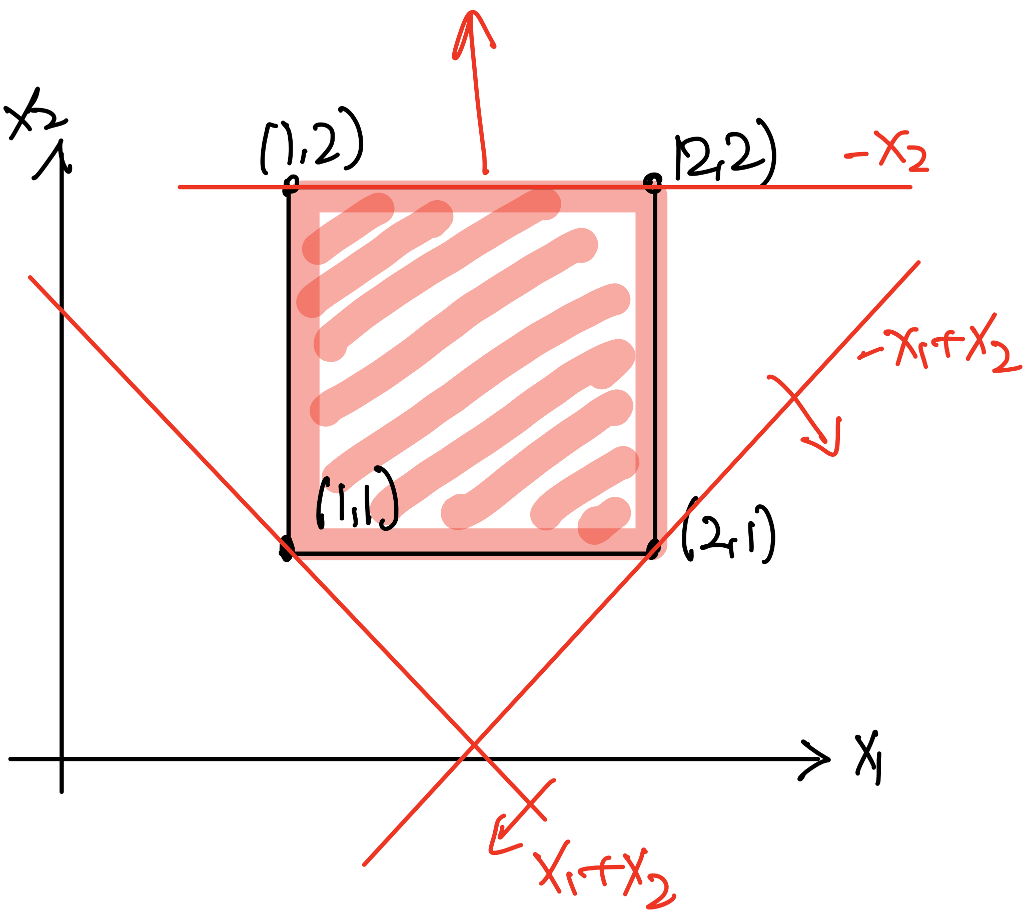
(1,1)

(c)



$$\begin{pmatrix} 1 \le x_1 \le 2 \\ x_2 = 1 \end{pmatrix}$$

(d)



$$\begin{pmatrix} 1 \le x_1 \le 2 \\ 1 \le x_2 \le 2 \end{pmatrix}$$