Selected Solution for Practice Problems on nonlin. opt.

6.8 Consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f(\boldsymbol{x}) = \boldsymbol{x}^\top \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} \boldsymbol{x} + \boldsymbol{x}^\top \begin{bmatrix} 3 \\ 5 \end{bmatrix} + 6. \qquad \text{Let } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

a. Find the gradient and Hessian of f at the point $[1, 1]^\top$.

b. Find the directional derivative of f at $[1, 1]^\top$ with respect to a unit vector in the direction of maximal rate of increase.

c. Find a point that satisfies the FONC (interior case) for f. Does this point satisfy the SONC (for a minimizer)?

$$f(x_1, x_2) = (x_1 \;\; x_2) \begin{pmatrix} 1 & 2 \\ 4 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (x_1 \;\; x_2) \begin{pmatrix} 3 \\ 5 \end{pmatrix} + 6$$

$$= (x_1 \;\; x_2) \begin{pmatrix} x_1 + 2x_2 \\ 4x_1 + 7x_2 \end{pmatrix} + 3x_1 + 5x_2 + 6$$

$$= x_1^2 + 6x_1 x_2 + 7x_2^2 + 3x_1 + 5x_2 + 6$$

$$\nabla f = (\partial_{x_1} f, \partial_{x_2} f)^T$$

$$= (2x_1 + 6x_2 + 3, \; 6x_1 + 14x_2 + 5)^T$$

(a) At $(1,1)$, $\nabla f(1,1) = \begin{pmatrix} 11 \\ 25 \end{pmatrix}$

(b) $\|\nabla f\| = \sqrt{11^2 + 25^2} = \sqrt{746}$

(c) Set $Df = 0 \Rightarrow$ $2x_1 + 6x_2 + 3 = 0$
$6x_1 + 14x_2 + 5 = 0$

$$\Rightarrow \begin{pmatrix} x_1 = 3/2 \\ x_2 = -1 \end{pmatrix}$$

$$[D^2 f] = \begin{bmatrix} 2 & 6 \\ 6 & 14 \end{bmatrix}$$

Find eigenvalues of $[D^2 f]$:

$$\begin{vmatrix} 2-\lambda & 6 \\ 6 & 14-\lambda \end{vmatrix} = \lambda^2 - 16\lambda + 28 - 36 = 0$$

$$\lambda^2 - 16\lambda - 8 = 0$$

$$\lambda = \frac{16 \pm \sqrt{16^2 + 4(8)}}{2} = 8 \pm \sqrt{72}$$

$$\lambda_1 = 8 + \sqrt{72} > 0 > 8 - \sqrt{72} = \lambda_2$$

Saddle pt. $\Rightarrow$ not a min/max.

6.10 Consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f(\boldsymbol{x}) = \boldsymbol{x}^\top \begin{bmatrix} 2 & 5 \\ -1 & 1 \end{bmatrix} \boldsymbol{x} + \boldsymbol{x}^\top \begin{bmatrix} 3 \\ 4 \end{bmatrix} + 7.$$

a. Find the directional derivative of f at $[0,1]^\top$ in the direction $[1,0]^\top$.

b. Find all points that satisfy the first-order necessary condition for f. Does f have a minimizer? If it does, then find all minimizer(s); otherwise, explain why it does not.

$$f(x_1, x_2) = 2x_1^2 + 4x_1x_2 + x_2^2 + 3x_1 + 4x_2 + 7$$

$$\nabla f = (4x_1 + 4x_2 + 3,\ 4x_1 + 2x_2 + 4)^T$$

(a) $\nabla f(0,1) = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$

(b) $\nabla f = 0 \Longrightarrow \begin{matrix} 4x_1 + 4x_2 + 3 = 0 \\ 4x_1 + 2x_2 + 4 = 0 \end{matrix} \Rightarrow \begin{pmatrix} x_1 = -5/4 \\ x_2 = \frac{1}{2} \end{pmatrix}$

$$[D^2 f] = \begin{bmatrix} 4 & 4 \\ 4 & 2 \end{bmatrix}$$

$$\begin{vmatrix} 4-\lambda & 4 \\ 4 & 2-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 8 - 16 = \lambda^2 - 6\lambda - 8$$
$$= (\lambda - 2)(\lambda - 4)$$

$\lambda_1 = 2 > 0, \lambda_2 = 4 > 0 \Longrightarrow$ local min.

6.30 Suppose that we are given a set of vectors $\{\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(p)}\}$, $\boldsymbol{x}^{(i)} \in \mathbb{R}^n$, $i = 1, \ldots, p$. Find the vector $\bar{\boldsymbol{x}} \in \mathbb{R}^n$ such that the average squared distance (norm) between $\bar{\boldsymbol{x}}$ and $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(p)}$,

$$\frac{1}{p}\sum_{i=1}^{p} \|\bar{\boldsymbol{x}} - \boldsymbol{x}^{(i)}\|^2,$$

$$f(\bar{x}) = \frac{1}{p}\sum_{i=1}^{p} \langle x^{(i)} - \bar{x}, x^{(i)} - \bar{x} \rangle$$

$$\nabla f(\bar{x}) = \frac{2}{p}\sum_{i=1}^{p} (x^{(i)} - \bar{x})(-1) = 0$$

$$\sum_{i=1}^{p} x^{(i)} - p\bar{x} = 0$$

$$\boxed{\bar{x} = \frac{1}{p}\sum_{i=1}^{p} x^{(i)}} \leftarrow \text{average of } x^{(i)}\text{'s, center of mass.}$$

Example 12.7 Find the point closest to the origin of $\mathbb{R}^3$ on the line of intersection of the two planes defined by the following two equations:

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1, \\ 4x_1 + x_2 + 3x_3 &= 0. \end{aligned}$$

Note that this problem is equivalent to the problem

$$\begin{array}{ll} \text{minimize} & \|\boldsymbol{x}\|^{2} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \end{array}$$

where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Thus, the solution to the problem is

$$\boxed{\boldsymbol{x}^* = \boldsymbol{A}^\top(\boldsymbol{A}\boldsymbol{A}^\top)^{-1}\boldsymbol{b} = \begin{bmatrix} 0.0952 \\ 0.3333 \\ -0.2381 \end{bmatrix}}.$$

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$$

$$L(x, \lambda) = x_1^2 + x_2^2 + x_3^2 + \lambda_1(x_1 + 2x_2 - x_3 - 1) + \lambda_2(4x_1 + x_2 + 3x_3)$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

$$\nabla_x L = 0 \implies \begin{aligned} 2x_1 + \lambda_1 + 4\lambda_2 &= 0 \\ 2x_2 + 2\lambda_1 + \lambda_2 &= 0 \\ 2x_3 - \lambda_1 + 3\lambda_2 &= 0 \end{aligned}$$

$$\Longleftrightarrow \begin{bmatrix} 1 & 4 \\ 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\nabla_\lambda L = 0 \Longrightarrow \begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ 4x_1 + x_2 + 3x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$-\frac{1}{2} \underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 4 \\ 2 & 1 \\ -1 & 3 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}}_{\lambda} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{b}$$

$$AA^T\lambda = -2b$$

$$\lambda = -2(AA^T)^{-1} b$$

$$x = -\frac{1}{2} A^T \lambda$$

$$x = A^T (AA^T)^{-1} b$$

12.19 Solve the problem

$$\begin{aligned} &\text{minimize} && \|\boldsymbol{x} - \boldsymbol{x}_0\|^2 \\ &\text{subject to} && \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \boldsymbol{x} = 1, \end{aligned}$$

where $\boldsymbol{x}_0 = [0, -3, 0]^\top$.

$$f(x) = x_1^2 + (x_2 + 3)^2 + x_3^2$$

$$g(x) = (x_1 + x_2 + x_3 - 1)$$

$$\nabla f + \lambda \nabla g = 0, \quad g = 0$$

$$\begin{aligned} 2x_1 &= \lambda \\ 2(x_2 + 3) &= \lambda \\ 2x_3 &= \lambda \end{aligned}$$

$$x_1 + x_2 + x_3 = 1$$

$$\frac{\lambda}{2} + \frac{\lambda}{2} - 3 + \frac{\lambda}{2} = 1$$

$$\lambda = \frac{8}{3}$$

Hence $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4/3 \\ -5/3 \\ 4/3 \end{pmatrix}$$

20.2 Find local extremizers for the following optimization problems:

a.
$$\begin{aligned} \text{Minimize} \quad & x_1^2 + 2x_1x_2 + 3x_2^2 + 4x_1 + 5x_2 + 6x_3 \\ \text{subject to} \quad & x_1 + 2x_2 = 3 \\ & 4x_1 + 5x_3 = 6. \end{aligned}$$

b.
$$\begin{aligned} \text{Maximize} \quad & 4x_1 + x_2^2 \\ \text{subject to} \quad & x_1^2 + x_2^2 = 9. \end{aligned}$$

c.
$$\begin{aligned} \text{Maximize} \quad & x_1x_2 \\ \text{subject to} \quad & x_1^2 + 4x_2^2 = 1. \end{aligned}$$

(a) $x_1^2 + 2x_1x_2 + 3x_2^2 + 4x_1 + 5x_2 + 6x_3$
$+ \lambda_1(x_1 + 2x_2 - 3)$
$+ \lambda_2(4x_1 + 5x_3 - 6)$

$\partial x_1, \partial x_2 \Rightarrow$
$$\begin{aligned} 2x_1 + 2x_2 + 4 + \lambda_1 + 4\lambda_2 &= 0 \\ 2x_1 + 6x_2 + 5 + 2\lambda_1 + 5\lambda_2 &= 0 \end{aligned}$$

$$\Leftrightarrow \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}^{-1} \left(-\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \end{bmatrix} \right)$$

$$\partial \lambda_1, \partial \lambda_2 \Rightarrow \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}\left(\begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}^{-1}\left(-\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \end{bmatrix}\right)\right) = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$$= -\begin{bmatrix} 3 \\ 6 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}^{-1}\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

(Note: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad-bc}$)

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}\frac{\begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix}}{8}\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$$= -\begin{bmatrix} 3 \\ 6 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}\frac{\begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix}}{8}\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$= -\begin{bmatrix} 3 \\ 6 \end{bmatrix} - \frac{1}{8}\begin{bmatrix} 2 & 2 \\ 14 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$= -\begin{bmatrix} 3 \\ 6 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 9 \\ 33 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} 21 \\ 57 \end{bmatrix}$$

$$\frac{1}{8}\begin{bmatrix} 2 & 2 \\ 14 & 2 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} 21 \\ 57 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 7 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = -3\begin{bmatrix} 7 \\ 19 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 9 \\ 9 & 33 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = -3\begin{bmatrix} 7 \\ 19 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 3 & 11 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = -\begin{bmatrix} 7 \\ 19 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = -\frac{\begin{bmatrix} 11 & -3 \\ -3 & 1 \end{bmatrix}}{2}\begin{bmatrix} 7 \\ 19 \end{bmatrix} = -\begin{bmatrix} 10 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}^{-1}\left(-\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \end{bmatrix}\right)$$

$$= \frac{1}{8}\begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix}\left(\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 10 \\ -1 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \end{bmatrix}\right) = \underline{\underline{\begin{pmatrix} -1 \\ 2 \end{pmatrix}}}$$

(b) $4x_1 + x_2^2 + \lambda(x_1^2 + x_2^2 - 9)$

$$\partial_{x_1}, \partial_{x_2} \Rightarrow \begin{cases} 4 + 2x_1\lambda = 0 \\ 2x_2 + 2x_2\lambda = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 2 + x_1\lambda = 0 \\ x_2(1+\lambda) = 0 \end{cases} \qquad \underline{\lambda \neq 0}$$

$$\Rightarrow \quad x_1 = -\frac{2}{\lambda}$$

<u>If $\lambda \neq -1$</u>, then $x_2 = 0$

$$\Rightarrow \quad \frac{4}{\lambda^2} = 9 \Rightarrow \lambda = \pm\frac{2}{3}$$

$$\Rightarrow x_1 = \mp 3$$

<u>If $\lambda = -1$</u>, then $x_1 = 2$,

$$\Rightarrow x_2^2 = 9 - 4 = 5 \Rightarrow x_2 = \pm\sqrt{5}$$

Possible pts: $(\pm 3, 0)$, $(2, \pm\sqrt{5})$

Obj fct: $4x_1 + x_2^2 \longrightarrow$ max. at $(2, \pm\sqrt{5})$, $\underline{\underline{13}}$

(i) $x_1x_2+\lambda(x_1^2+x_2^2-1)$

$\partial x_1, \partial x_2 \Longrightarrow \quad x_2+2x_1\lambda=0$

$x_1+2x_2\lambda=0$

M1

$$\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=-2\lambda\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

or $$\underbrace{\begin{bmatrix}2\lambda&1\\1&2\lambda\end{bmatrix}}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$\Longrightarrow \quad 4\lambda^2-1=0 \Rightarrow \lambda=\pm\frac{1}{2}$

$\lambda=\frac{1}{2} \Longrightarrow \begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Longrightarrow x_1=-x_2$

$x_1^2+x_2^2=1 \Rightarrow x_1^2=\frac{1}{2},$

$\underline{x_1=\pm\frac{1}{\sqrt{2}},\ x_2=\mp\frac{1}{\sqrt{2}}}$

$\lambda=-\frac{1}{2} \Longrightarrow \begin{bmatrix}-1&1\\1&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Longrightarrow x_1=x_2$

$x_1^2+x_2^2=1 \Rightarrow \underline{x_1=\pm\frac{1}{\sqrt{2}},\ x_2=\pm\frac{1}{\sqrt{2}}}$

Possible pts: $\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\right)$, $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$

$x_1 x_2 \longrightarrow \quad -\frac{1}{2} \qquad \underbrace{\frac{1}{2}}_{\text{max.}}$

M2

$$\left.\begin{aligned} x_2 + 2x_1\lambda &= 0 \\ x_1 + 2x_2\lambda &= 0 \end{aligned}\right\} \Rightarrow x_1, x_2, \lambda \neq 0$$

$(x_1^2 + x_2^2 = 1)$

$$\Rightarrow \frac{x_2}{x_1} = \frac{x_1}{x_2}$$

$$\Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$$

$$\Rightarrow x_1 = \pm\frac{1}{\sqrt{2}}, \quad x_2 = \pm\frac{1}{\sqrt{2}}$$

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20.3 Find minimizers and maximizers of the function

$$f(\boldsymbol{x}) = (\boldsymbol{a}^\top \boldsymbol{x})(\boldsymbol{b}^\top \boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^3,$$

subject to

$$x_1 + x_2 = 0$$
$$x_2 + x_3 = 0,$$

where

$$\boldsymbol{a} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

$$f(x) = x_2(x_1 + x_3)$$

$$x_2(x_1 + x_3) + \lambda_1(x_1 + x_2) + \lambda_2(x_2 + x_3)$$

$$\partial x_1, \partial x_2, \partial x_3 \Rightarrow \begin{array}{l} x_2 + \lambda_1 = 0 \\ x_1 + x_3 + \lambda_1 + \lambda_2 = 0 \\ x_2 + \lambda_2 = 0 \end{array}$$

$$\left(\Leftrightarrow \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{\text{not invertible}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \right)$$

Solve directly: $x_1 = -x_2, \; x_3 = -x_2$

$$f(x) = x_2(x_1 + x_3) = x_2(-2x_2) = -2x_2^2$$

$$\max f(x) = 0 \quad \text{at } x_2 = 0$$

$$\min f(x) = -\infty$$

20.5 Consider the problem

$$\begin{aligned} \text{minimize} \quad & \|\boldsymbol{x} - \boldsymbol{x}_0\|^2 \\ \text{subject to} \quad & \|\boldsymbol{x}\|^2 = 9, \end{aligned}$$

where $\boldsymbol{x}_0 = [1, \sqrt{3}]^\top$.

a. Find all points satisfying the Lagrange condition for the problem.

$$(x_1-1)^2 + (x_2-\sqrt{3})^2 + \lambda(x_1^2+x_2^2-9)$$

$$\begin{cases} 2(x_1-1) + 2\lambda x_1 = 0 \\ 2(x_2-\sqrt{3}) + 2\lambda x_2 = 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} (1+\lambda)x_1 = 1 \\ (1+\lambda)x_2 = \sqrt{3} \end{cases} \quad \begin{cases} x_1 = \frac{1}{1+\lambda} \\ x_2 = \frac{\sqrt{3}}{1+\lambda} \end{cases} \qquad \lambda + 1 \neq 0$$

$$\Rightarrow \quad \frac{1}{(1+\lambda)^2} + \frac{3}{(1+\lambda)^2} = 9$$

$$\Rightarrow \quad (1+\lambda)^2 = \frac{4}{9} \Rightarrow 1+\lambda = \pm\frac{2}{3}$$

$$\left(x_1 = \frac{3}{2},\ x_2 = \frac{3\sqrt{3}}{2}\right)$$

or $\left(x_1 = -\frac{3}{2},\ x_2 = -\frac{3\sqrt{3}}{2}\right)$

20.7 Find local extremizers of

a. $f(x_1, x_2, x_3) = x_1^2 + 3x_2^2 + x_3$ subject to $x_1^2 + x_2^2 + x_3^2 = 16$.

b. $f(x_1, x_2) = x_1^2 + x_2^2$ subject to $3x_1^2 + 4x_1x_2 + 6x_2^2 = 140$.

(a) $$\begin{cases} 2x_1 + 2\lambda x_1 = 0 \\ 6x_2 + 2\lambda x_2 = 0 \\ 1 + 2\lambda x_3 = 0 \end{cases} \Longleftrightarrow \begin{array}{l} (1+\lambda) x_1 = 0 \\ (3+\lambda) x_2 = 0 \\ 1 + 2\lambda x_3 = 0 \end{array}$$

$$\lambda = -1 \Rightarrow x_2 = 0,\ x_3 = +\frac{1}{2},\ \Rightarrow x_1 = \pm\sqrt{16 - \frac{1}{4}}$$
$$= \pm\sqrt{\frac{63}{4}}$$

$$\lambda = -3 \Rightarrow x_1 = 0,\ x_3 = \frac{3}{2},\ \Rightarrow x_2 = \pm\sqrt{16 - \frac{9}{4}}$$
$$= \pm\sqrt{\frac{55}{4}}$$

$$\lambda \neq -1, -3 \Rightarrow x_1 = x_2 = 0 \Rightarrow x_3 = \pm 4$$
$$\lambda = \pm\frac{1}{8}$$

(b)

$$2x_1 + \lambda(6x_1 + 4x_2) = 0$$
$$2x_2 + \lambda(4x_1 + 12x_2) = 0$$

$$\lambda \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\frac{1}{\lambda} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 3+\frac{1}{\lambda} & 2 \\ 2 & 6+\frac{1}{\lambda} \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\mu = \frac{1}{\lambda}$

$$\mu^2 + 9\mu + 18 - 4 = 0$$
$$\mu^2 + 9\mu + 14 = 0$$
$$(\mu + 7)(\mu + 2) = 0$$
$$\mu = -7, -2$$

$$\mu = -7 \Longrightarrow \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\underline{x_2 = 2x_1}$$

$$\Longrightarrow \quad 3x_1^2 + 8x_1^2 + 24x_1^2 = 140$$

$$35x_1^2 = 140$$

$$\Longrightarrow \quad x_1 = \pm 2, \quad x_2 = \pm 4$$

$$\mu = -2 \quad \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\underline{x_1 = -2x_2}$$

$$\Longrightarrow \quad 12x_2^2 - 8x_2^2 + 6x_2^2 = 140$$

$$10x_2^2 = 140$$

$$\Longrightarrow \quad x_2 = \pm\sqrt{14}, \quad x_1 = \mp 2\sqrt{14}$$

20.8 Consider the problem

$$\begin{aligned} \text{minimize} \quad & 2x_1 + 3x_2 - 4, \qquad x_1, x_2 \in \mathbb{R} \\ \text{subject to} \quad & x_1 x_2 = 6. \end{aligned}$$

a. Use Lagrange's theorem to find all possible local minimizers and maximizers.

$2x_1 + 3x_2 - 4 + \lambda(x_1x_2 - 6)$

$\checkmark$ $2 + \lambda x_2 = 0$ & $x_1 x_2 = 6$

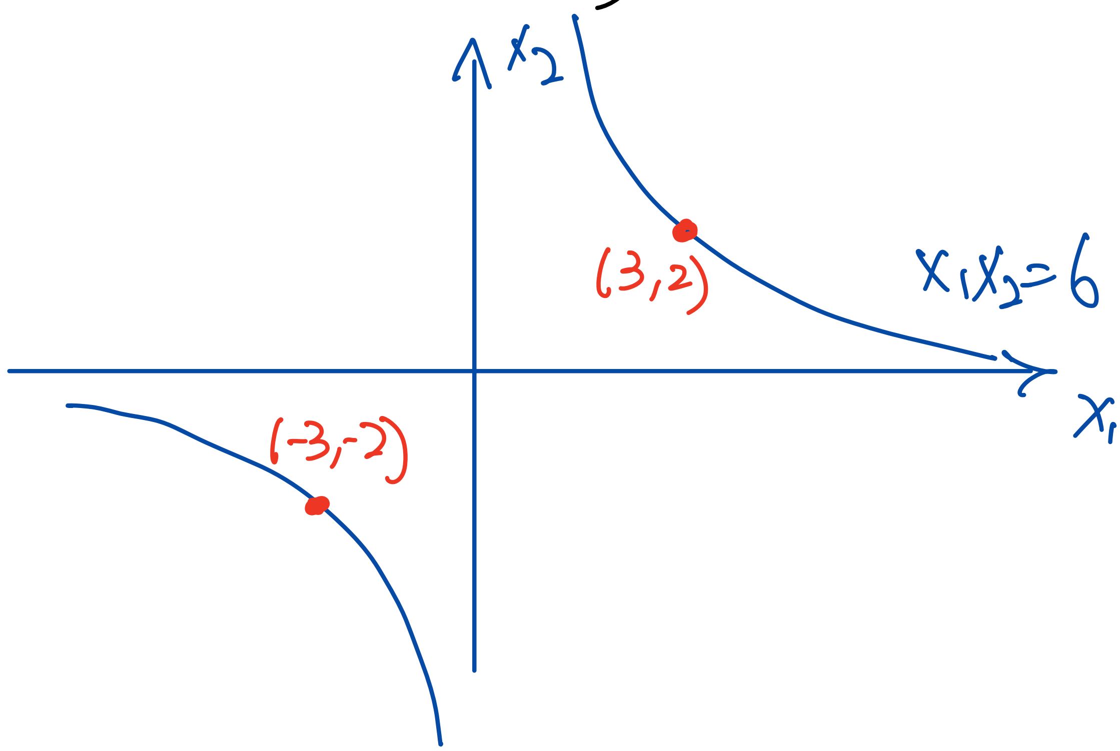
$3 + \lambda x_1 = 0$

$x_1 = -\frac{3}{\lambda}, \quad x_2 = -\frac{2}{\lambda}, \Rightarrow \frac{6}{\lambda^2} = 6$

$\Rightarrow \lambda = \pm 1$

$\lambda = 1 \Rightarrow (x_1 = -3, x_2 = -2) \leftarrow$ local max

$\lambda = -1 \Rightarrow (x_1 = 3, x_2 = 2) \leftarrow$ local min



20.9 Find all maximizers of the function

$$f(x_1, x_2) = \frac{18x_1^2 - 8x_1x_2 + 12x_2^2}{2x_1^2 + 2x_2^2}.$$

M1 max/min $f(x_1,x_2) = 9x_1^2 - 4x_1x_2 + 6x_2^2$

s.t. $x_1^2 + x_2^2 = 1$

$\partial x_1, \partial x_2 \Longrightarrow$

$$18x_1 - 4x_2 + \lambda x_1 = 0$$
$$-4x_1 + 12x_2 + \lambda x_2 = 0$$

$$\underbrace{\begin{bmatrix} 18+\lambda & -4 \\ -4 & 12+\lambda \end{bmatrix}}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(\lambda+18)(\lambda+12) - 16 = 0$$
$$\lambda^2 + 30\lambda + 200 = 0$$
$$(\lambda+20)(\lambda+10) = 0$$
$$\lambda = -20, -10$$

$\lambda = -20$, $\begin{bmatrix} -2 & -4 \\ -4 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\Longrightarrow x_1 = -2x_2$

$x_1^2 + x_2^2 = 1$

$\Longrightarrow x_2^2 = \frac{1}{5}$

$9x_1^2 - 4x_1x_2 + 6x_2^2 = 36x_2^2 + 8x_2^2 + 6x_2^2 = 10$ max

$\lambda = -10$ $\begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_2 = 2x_1$

$\Rightarrow x_1^2 = \frac{1}{5}$

$9x_1^2 - 4x_1x_2 + 6x_2^2 = 9x_1^2 - 8x_1^2 + 24x_1^2 = 5$ min

20.9 Find all maximizers of the function

$$f(x_1, x_2) = \frac{18x_1^2 - 8x_1x_2 + 12x_2^2}{2x_1^2 + 2x_2^2}.$$

M2

$$\begin{pmatrix} x_1 = 0 \Rightarrow f(0, x_2) = 6 \\ x_2 = 0 \Rightarrow f(x_1, 0) = 9 \end{pmatrix}$$

Assume $x_1, x_2 \neq 0$. Let $t = \frac{x_2}{x_1}$

$$f(x_1, x_2) = \frac{18 - 8t + 12t^2}{2 + 2t^2} = \frac{9 - 4t + 6t^2}{1 + t^2} = g(t)$$

$$g'(t) = \frac{(1+t^2)(-4+12t) - (9-4t+6t^2)2t}{(1+t^2)^2}$$

$$= \frac{-4 - 4t^2 + 12t - 18t + 8t^2}{(1+t^2)^2}$$

$$= \frac{-4 - 6t + 4t^2}{(1+t^2)^2} = 0 \qquad \left(t = 2, -\frac{1}{2}\right)$$

$$2t^2 - 3t - 2 = 0 \Rightarrow (t-2)(2t+1) = 0$$

$$t = 2 \Rightarrow f = \frac{9 - 8 + 24}{1 + 4} = 5 \quad \text{(min)}$$

$$t = -\frac{1}{2} \Rightarrow f = \frac{9 + 2 + 3/2}{1 + \frac{1}{4}} = \frac{36 + 8 + 6}{5} = 10 \quad \text{(max)}$$

20.10 Find all solutions to the problem

$$\begin{aligned} &\text{maximize} && \boldsymbol{x}^\top \begin{bmatrix} 3 & 4 \\ 0 & 3 \end{bmatrix} \boldsymbol{x} \\ &\text{subject to} && \|\boldsymbol{x}\|^2 = 1. \end{aligned}$$

$$(x_1 \ \ x_2)\begin{pmatrix} 3 & 4 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1 \ \ x_2)\begin{pmatrix} 3x_1+4x_2 \\ 3x_2 \end{pmatrix}$$

$$= 3x_1^2 + 4x_1x_2 + 3x_2^2$$

$$6x_1 + 4x_2 + \lambda x_1 = 0$$
$$4x_1 + 6x_2 + \lambda x_2 = 0$$

$$\begin{bmatrix} 6 & 4 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{pmatrix} 6+\lambda & 4 \\ 4 & 6+\lambda \end{pmatrix}}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\lambda^2 + 12\lambda + 36 - 16 = 0$$
$$\lambda^2 + 12\lambda + 20 = 0$$
$$(\lambda + 10)(\lambda + 2) = 0$$

$$\lambda = -10 \Rightarrow \begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 = x_2$$

$$x_1^2 + x_2^2 = 1 \Rightarrow x_1 \pm \frac{1}{\sqrt{2}},\ x_2 = \pm\frac{1}{\sqrt{2}}$$

$$3x_1^2 + 4x_1x_2 + 3x_2^2 = \frac{3+4+3}{2} = 5$$ max

$$\lambda = -2 \Rightarrow \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 = -x_2$$

$$x_1^2 + x_2^2 = 1 \Rightarrow x_1 = \pm\frac{1}{\sqrt{2}},\ x_2 = \mp\frac{1}{\sqrt{2}}$$

$$3x_1^2 + 4x_1x_2 + 3x_2^2 = \frac{3-4+3}{2} = 1$$ min