MA 421 Lec 1 Intro to Opt

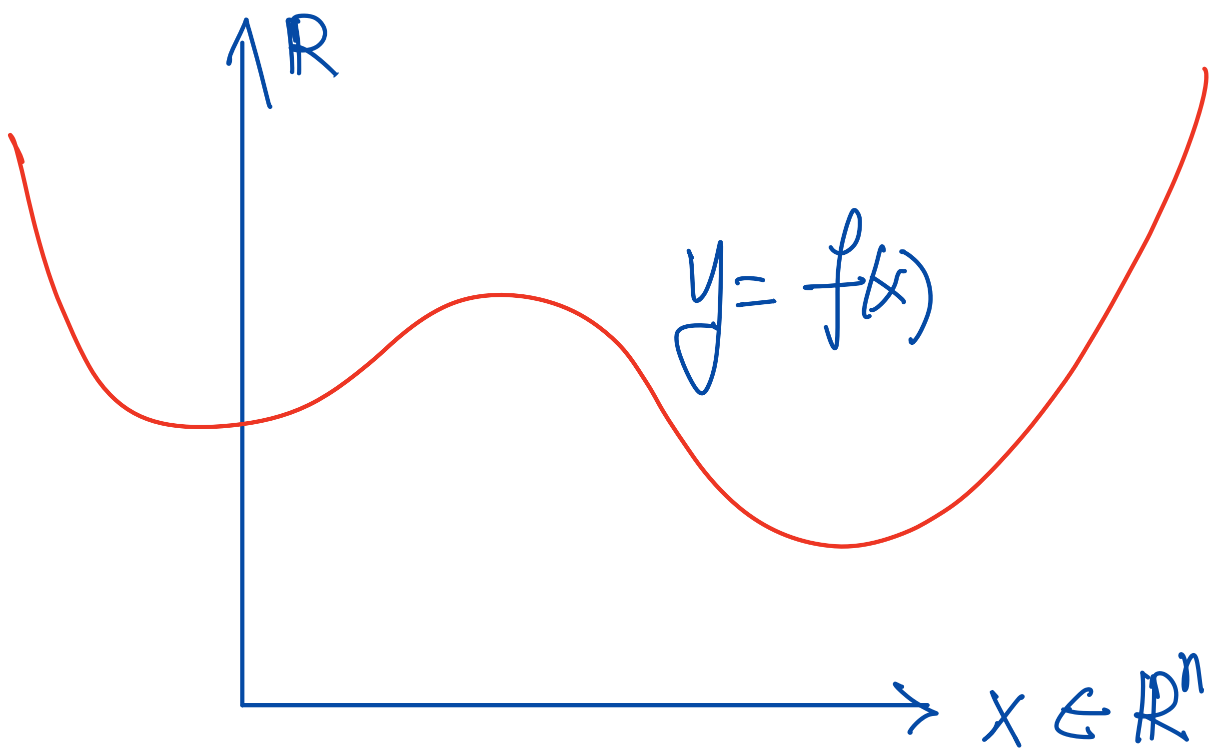
- max/min
- lin/nonlin functions
- with/without constraints
- equalities/inequalities

max $f(x)$ $\Longleftrightarrow$ min $(-f(x))$

(1) max (or min) $f(x)$

$\underline{x \in \mathbb{R}^n}$, $x = (x_1, x_2, \dots, x_n) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

no-constraints



In one dimension:

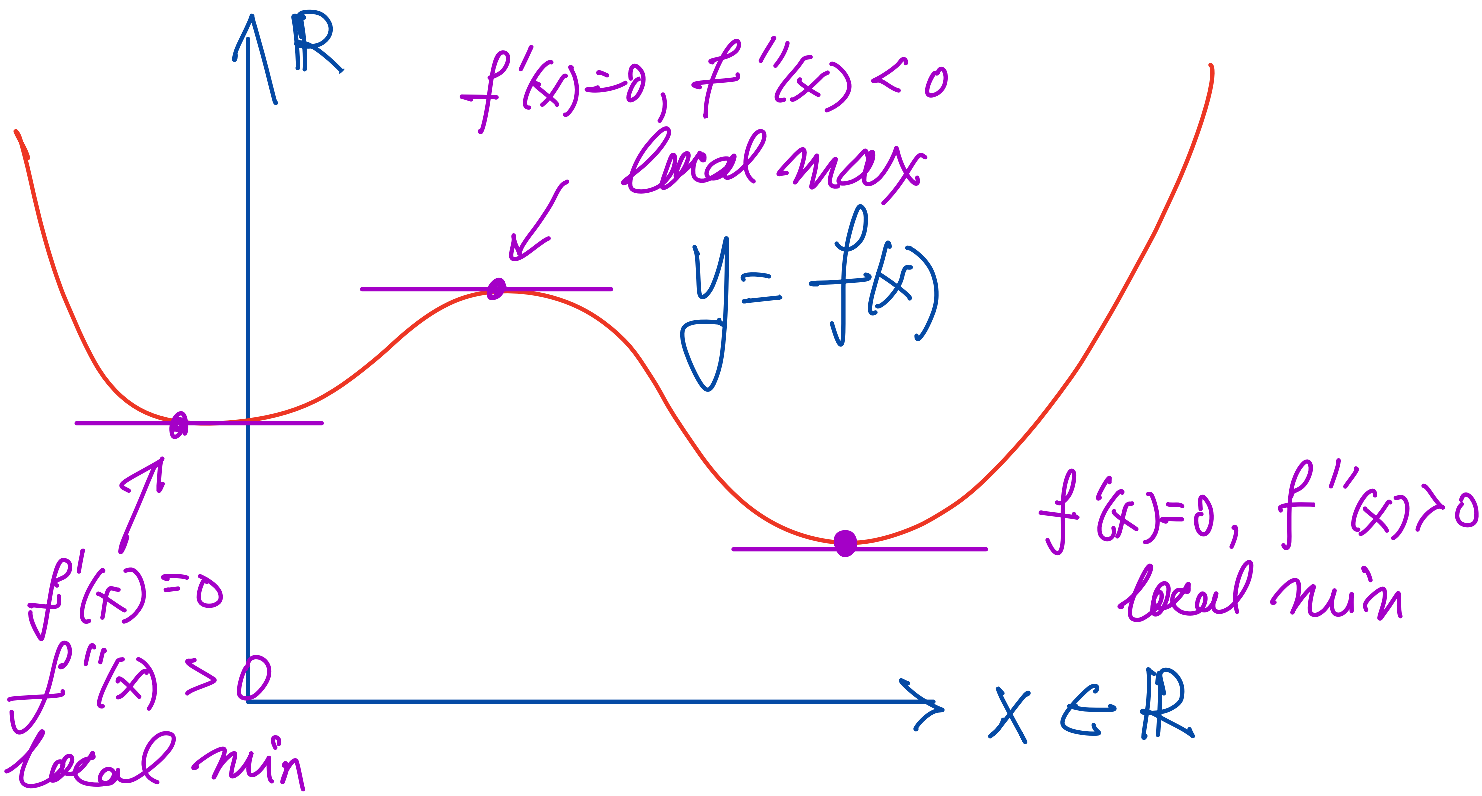
$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \qquad\qquad f(x)$$

First derivative test: (critical point)

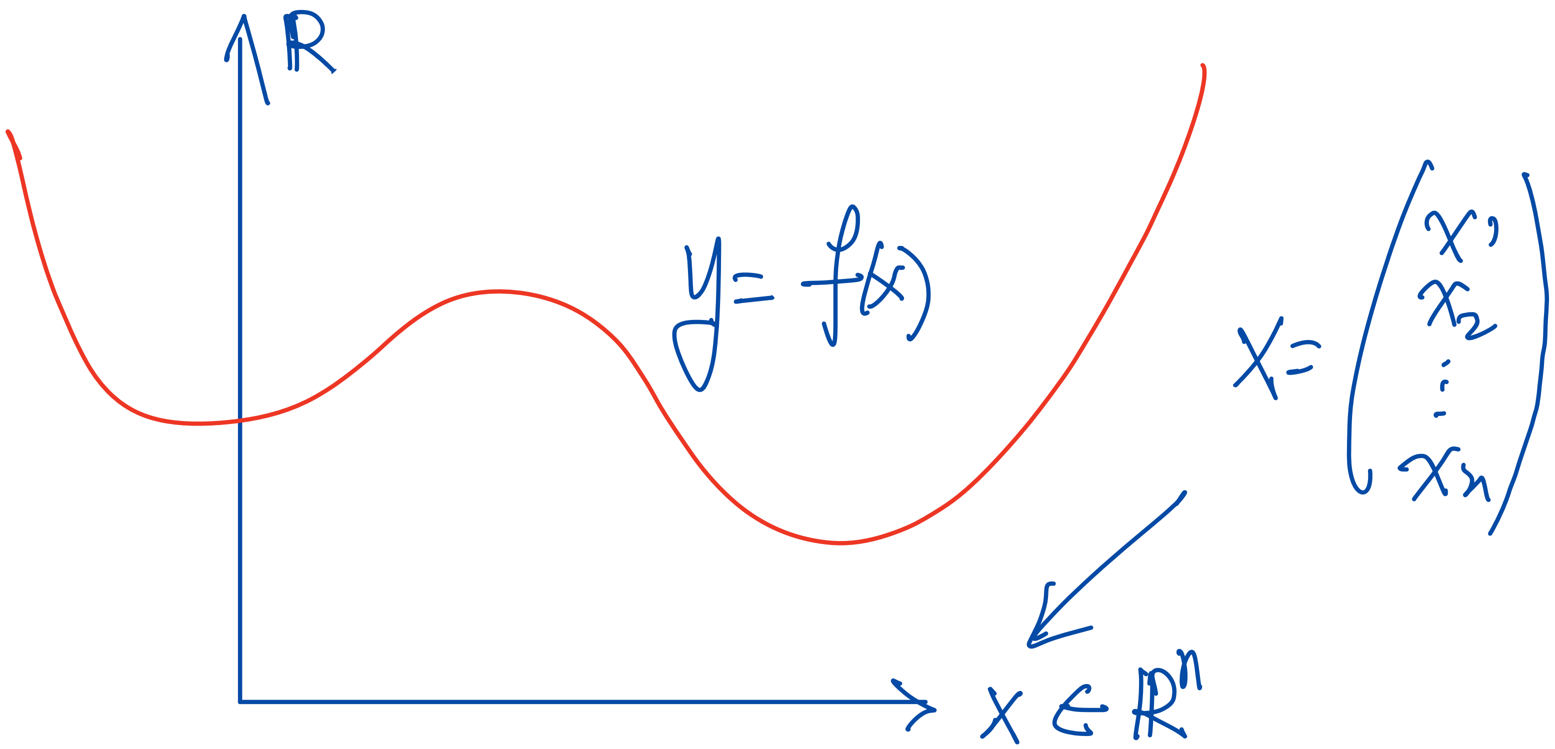
$$f'(x) = 0$$

Second derivative test:

$$f''(x) \begin{cases} > 0 & \Rightarrow \text{local min} \\ < 0 & \Rightarrow \text{local max.} \end{cases}$$



In n-dimension



1st derivative test $\Rightarrow$ $\nabla f(x) = 0$

critical pt(s)

gradient

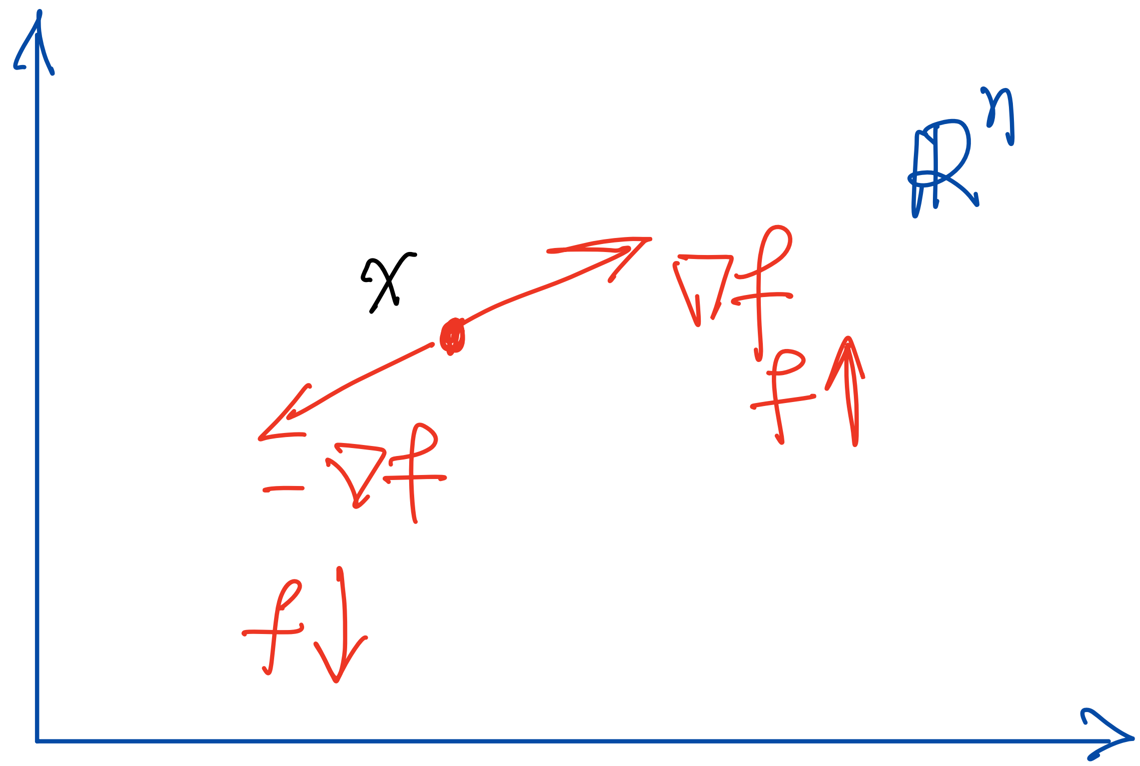
$$\nabla f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix},$$

column vector

$$Df(x) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \cdots \frac{\partial f}{\partial x_n} \right)$$

row vector

$$\nabla f = (Df)^T$$



$\mathbb{R}^n$
x
∇f
$f\uparrow$
$-\nabla f$
$f\downarrow$

2^{nd} derivative test

$\Longrightarrow$ $D^2 f(x) > 0$ (local min)

$[D^2 f] > 0$ (positive definite) $\longrightarrow$ (local min)

$D^2 f(x) < 0$ (local max)

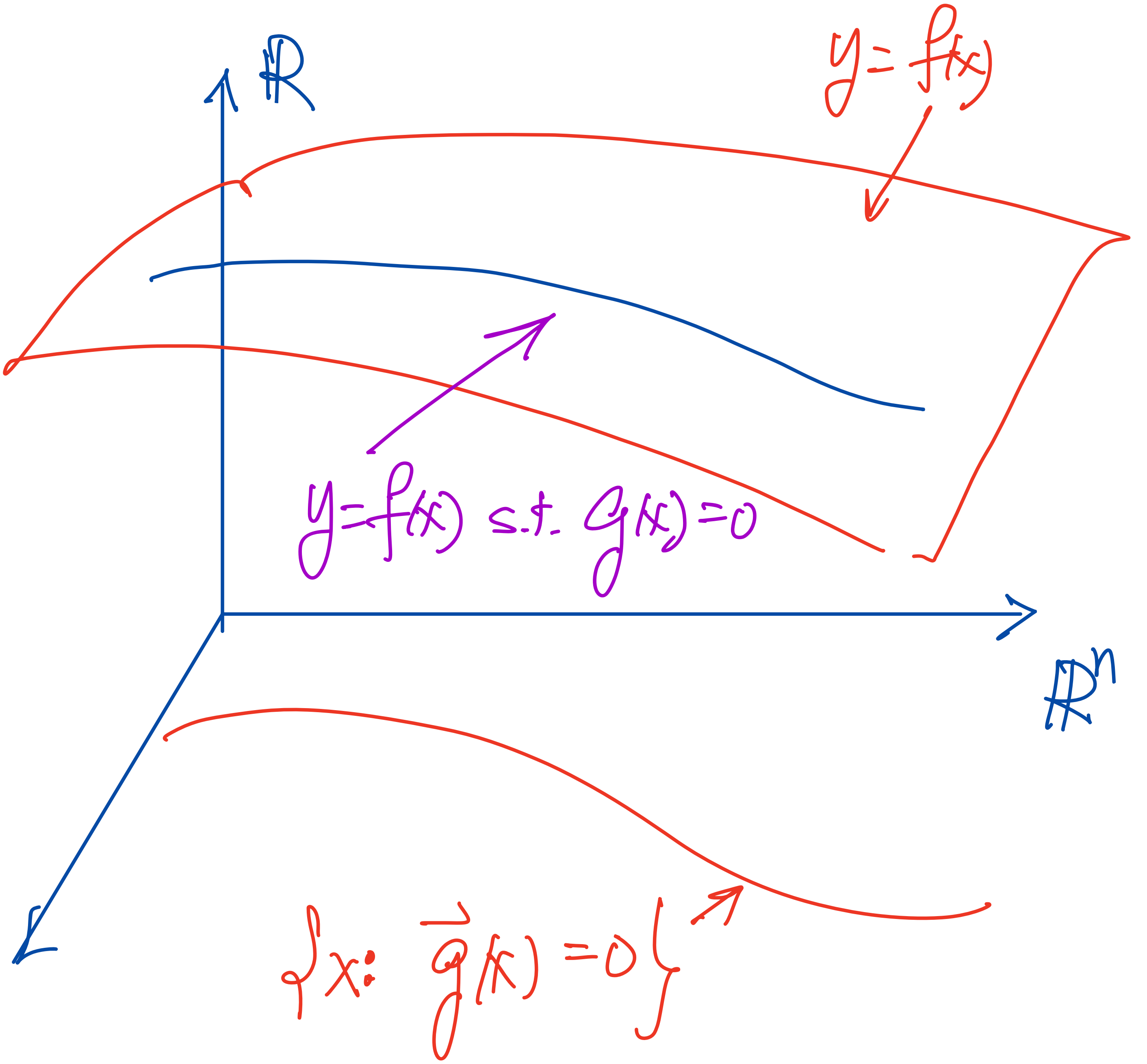
$[D^2 f] < 0$ (negative definite) $\longrightarrow$ (local max)

$$[D^2 f] = \begin{array}{cc} & \\ & \left[\begin{array}{cccc} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & & \\ & & & \\ i^{th}\text{ row} & \frac{\partial^2 f}{\partial x_i \partial x_j} & & \\ & & & \frac{\partial^2 f}{\partial x_n^2} \end{array}\right] \\ & \qquad\qquad j^{th}\text{ col} \end{array}$$

Hessian

② Equality constraints

$$\min \ f(x)$$
$$\text{s.t.} \quad \vec{g}(x) = \vec{0} \rightarrow \begin{pmatrix} g_1(x) = 0 \\ g_2(x) = 0 \\ \vdots \\ g_m(x) = 0 \end{pmatrix}$$



Lagrange Multiplier

$$\min_{x,\lambda} f(x) + \lambda g(x)$$

only 1 constraint

$g(x) = 0$

$$\nabla_x \Rightarrow \nabla f(x) + \lambda \nabla g(x) = 0$$

$$\partial_\lambda \Rightarrow g(x) = 0$$

$$\nabla f = -\lambda \nabla g$$

i.e. $\nabla f \parallel \nabla g$

Multiple constraints

$$\min f(x) + \lambda_1 g_1(x) + \cdots + \lambda_m g(x)$$

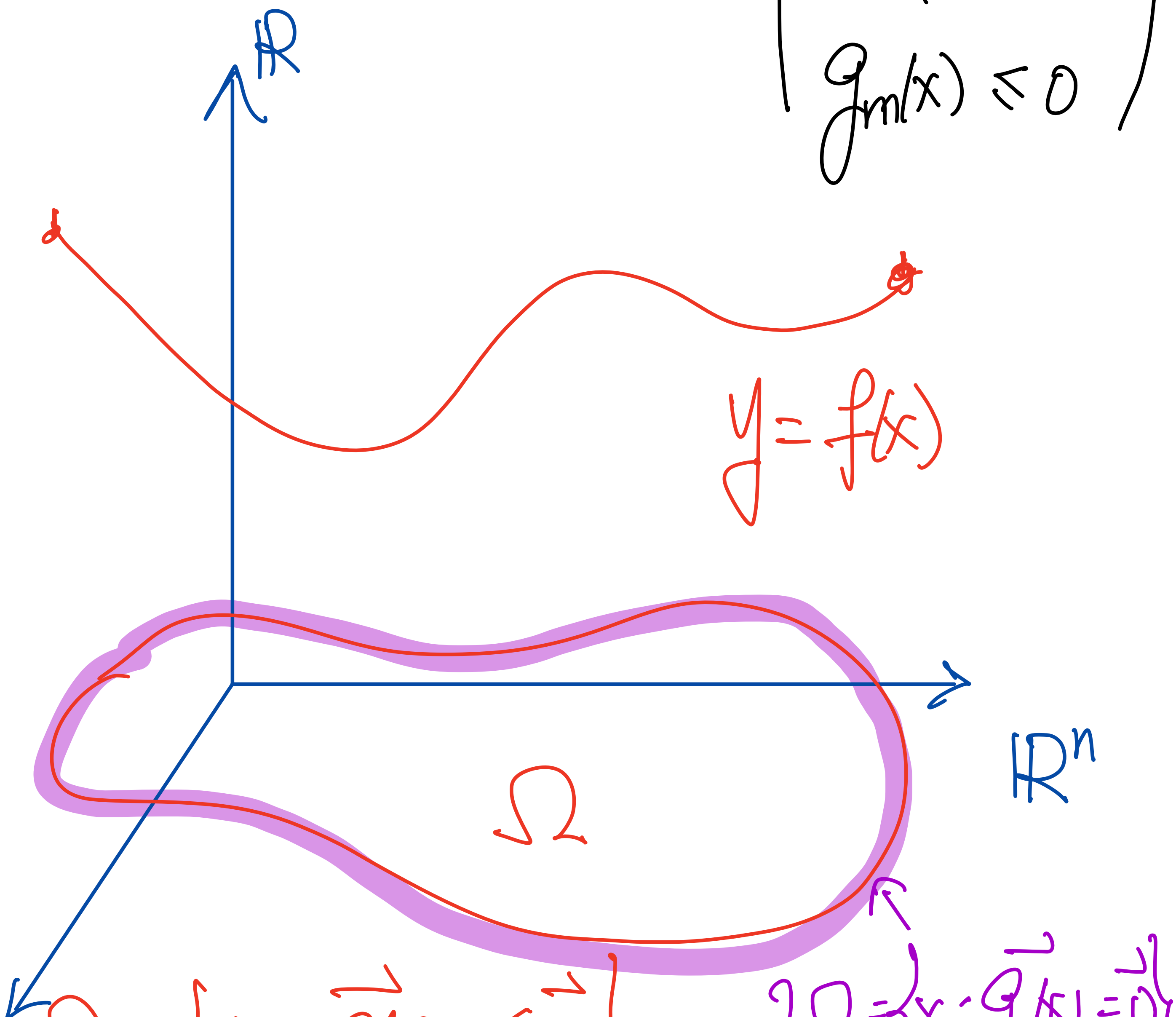
$x, \lambda_1, \cdots, \lambda_m$ — no constraints

$$\nabla_x \Rightarrow \nabla f(x) + \lambda_1 \nabla g(x) + \cdots + \lambda_m \nabla g(x) = 0$$

$$\partial_{\lambda_i} \Rightarrow g_i(x) = 0$$

3 With inequality constraints

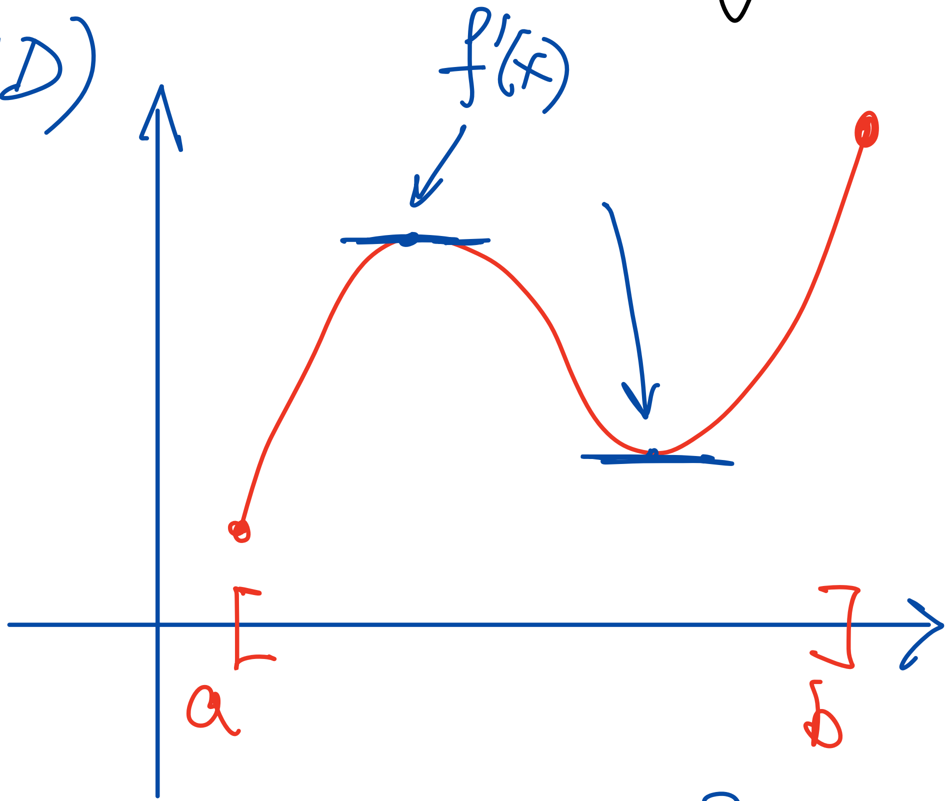
$$\min \quad f(x)$$
$$\text{s.t.} \quad \vec{g}(x) \leq 0 \rightarrow \begin{pmatrix} g_1(x) \leq 0 \\ g_2(x) \leq 0 \\ \vdots \\ g_m(x) \leq 0 \end{pmatrix}$$



$$\Omega = \{x : \vec{g}(x) \leq \vec{0}\} \qquad \partial\Omega = \{x : \vec{g}(x) = \vec{0}\}$$

Interior vs Boundary pts

(1D)



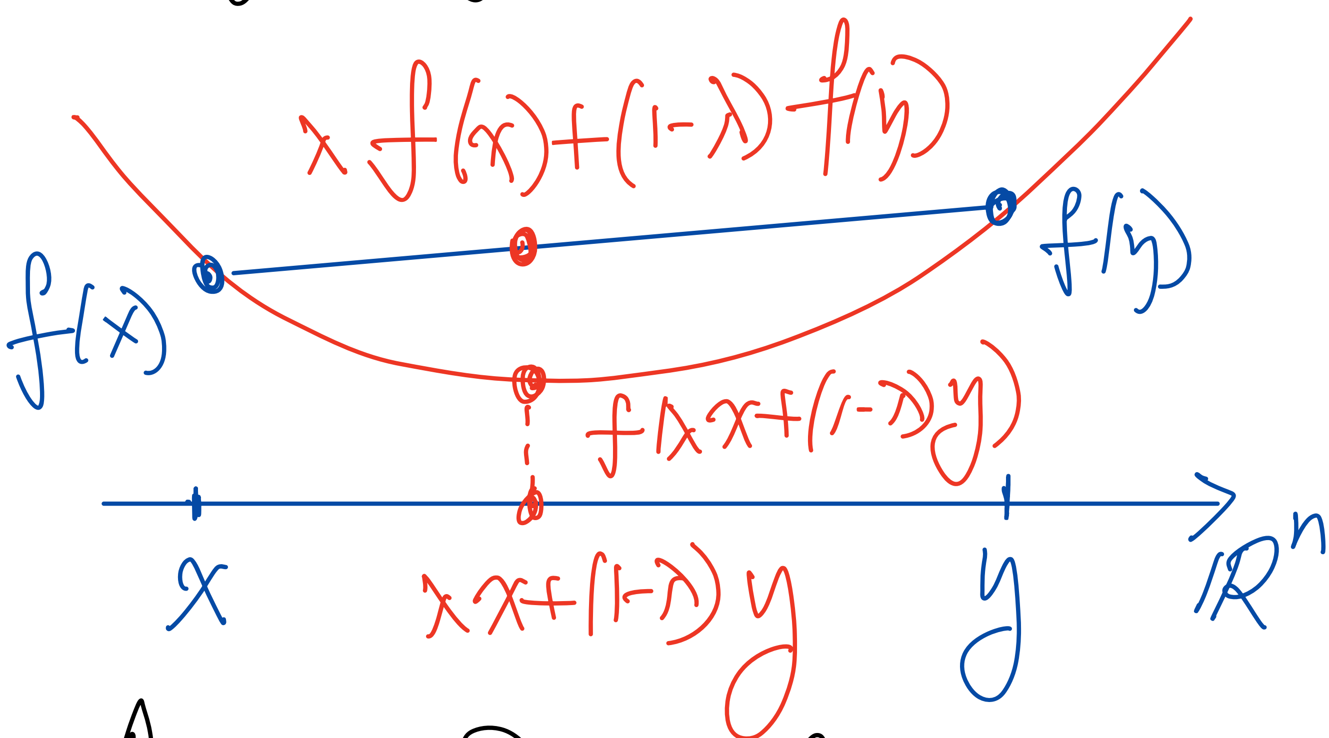
$f(a) = ?$ $\quad f(b) = ?$

Convexity

① $f:\mathbb{R}^n \longrightarrow \mathbb{R}$ is convex if

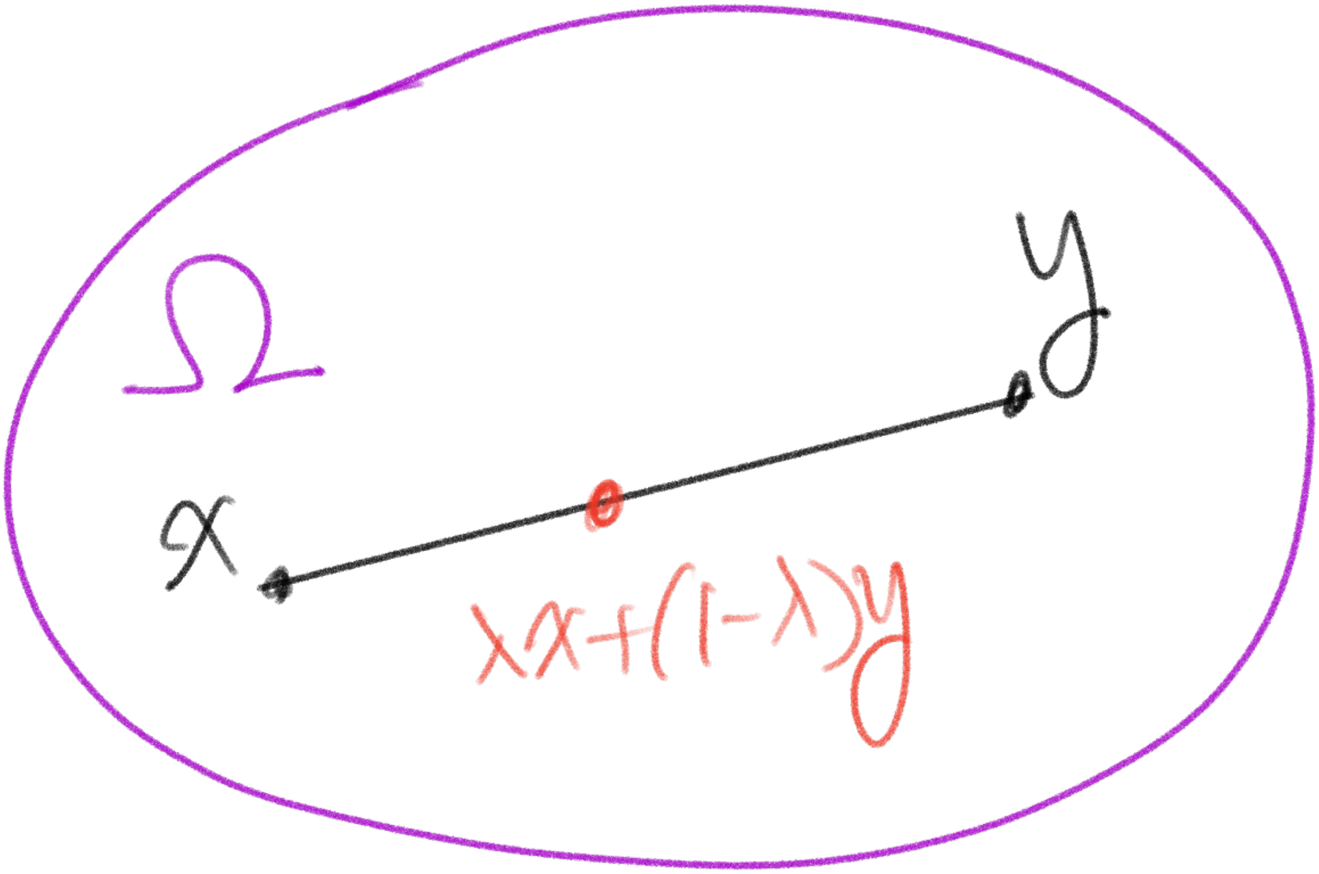
$$f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y)$$

for any $x, y \in \mathbb{R}^n$, $0 \leq \lambda \leq 1$

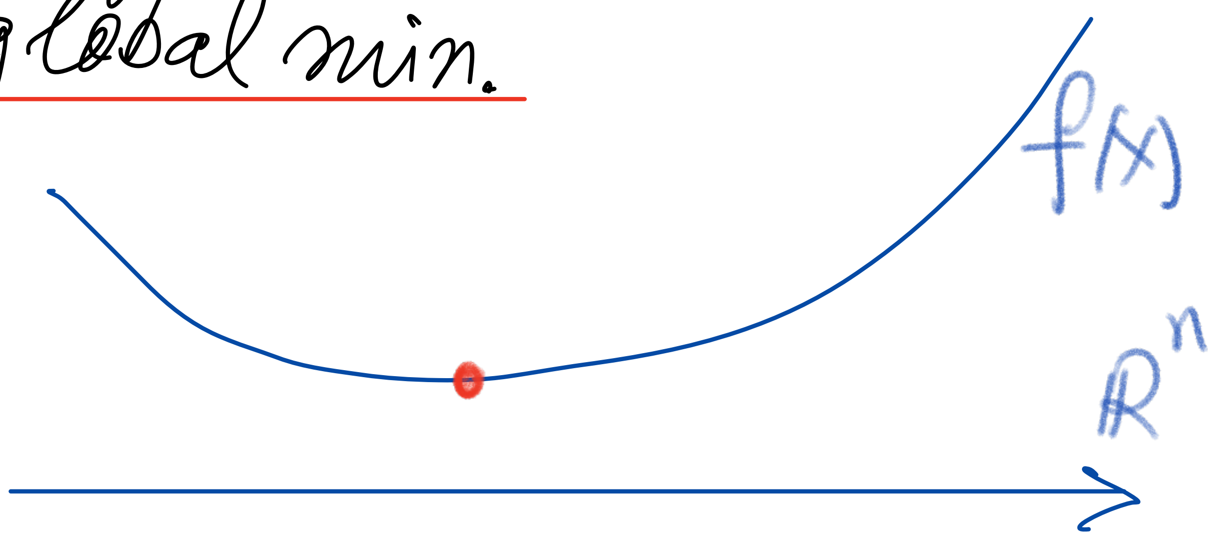


② A set $\Omega \subseteq \mathbb{R}^n$ is convex if $x, y \in \Omega$, then

$$\lambda x+(1-y)y \in \Omega$$



③ If a convex function has a critical point ($\nabla f(x)=0$), then the critical point is unique and it is global min.



Linear programming

Objective functions $f(x)$ and constraints $g(x)$ are all linear

Dietary Problem

	Vitamin A	Vitamin C	Price
Carrot (g)	2 (mg)	1 (mg)	2 ¢
Cabbage (g)	1 (mg)	3 (mg)	3 ¢
Min. Daily consumption	6 (mg)	8 (mg)	

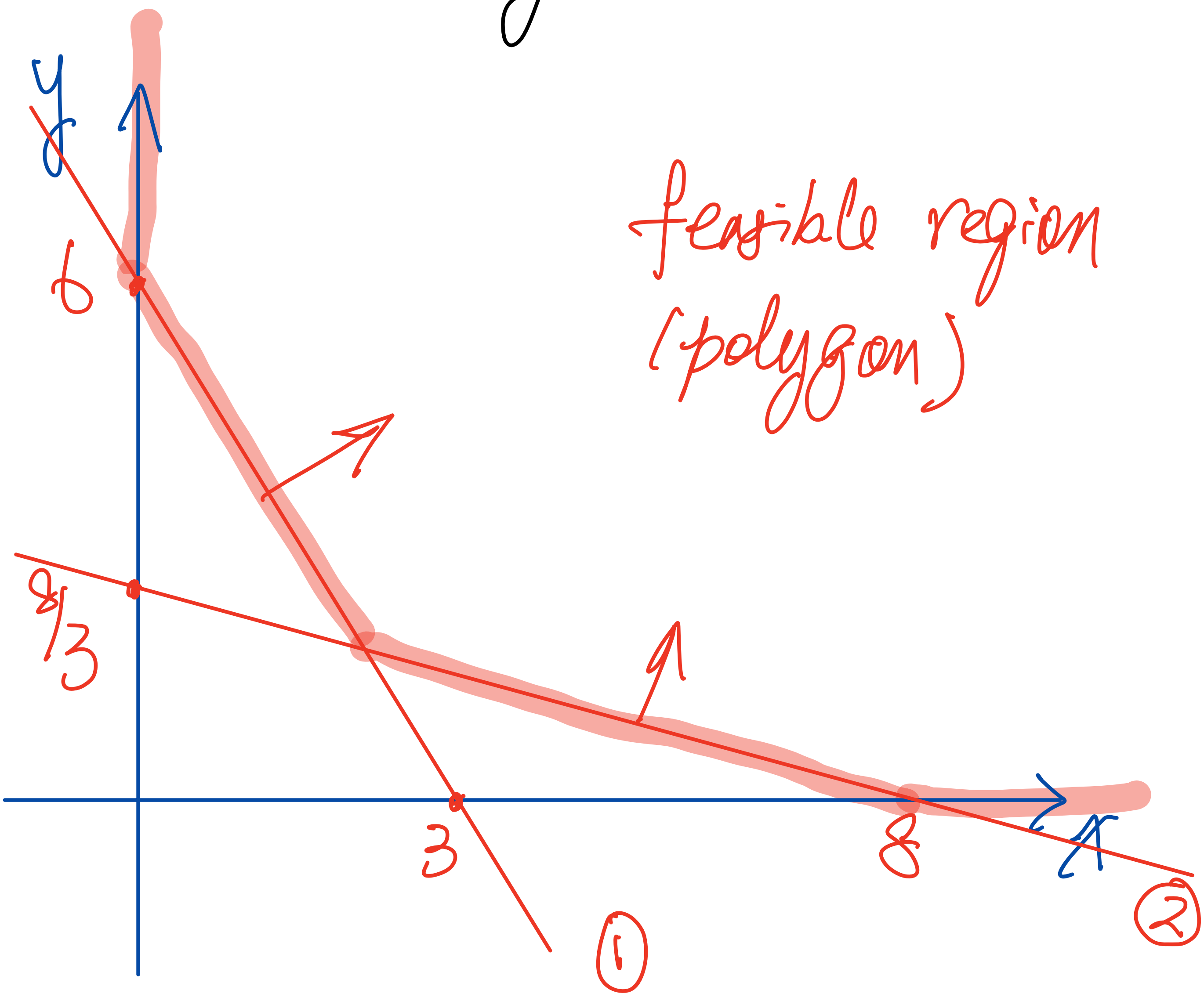
I 1 g of Carrot contains 2mg A & 1mg C

How much carrot & cabbage to consume so as to min cost?

Let x = carrot (g), y = cabbage (g)

obj: min $Z = 2x + 3y$ ← price/cost

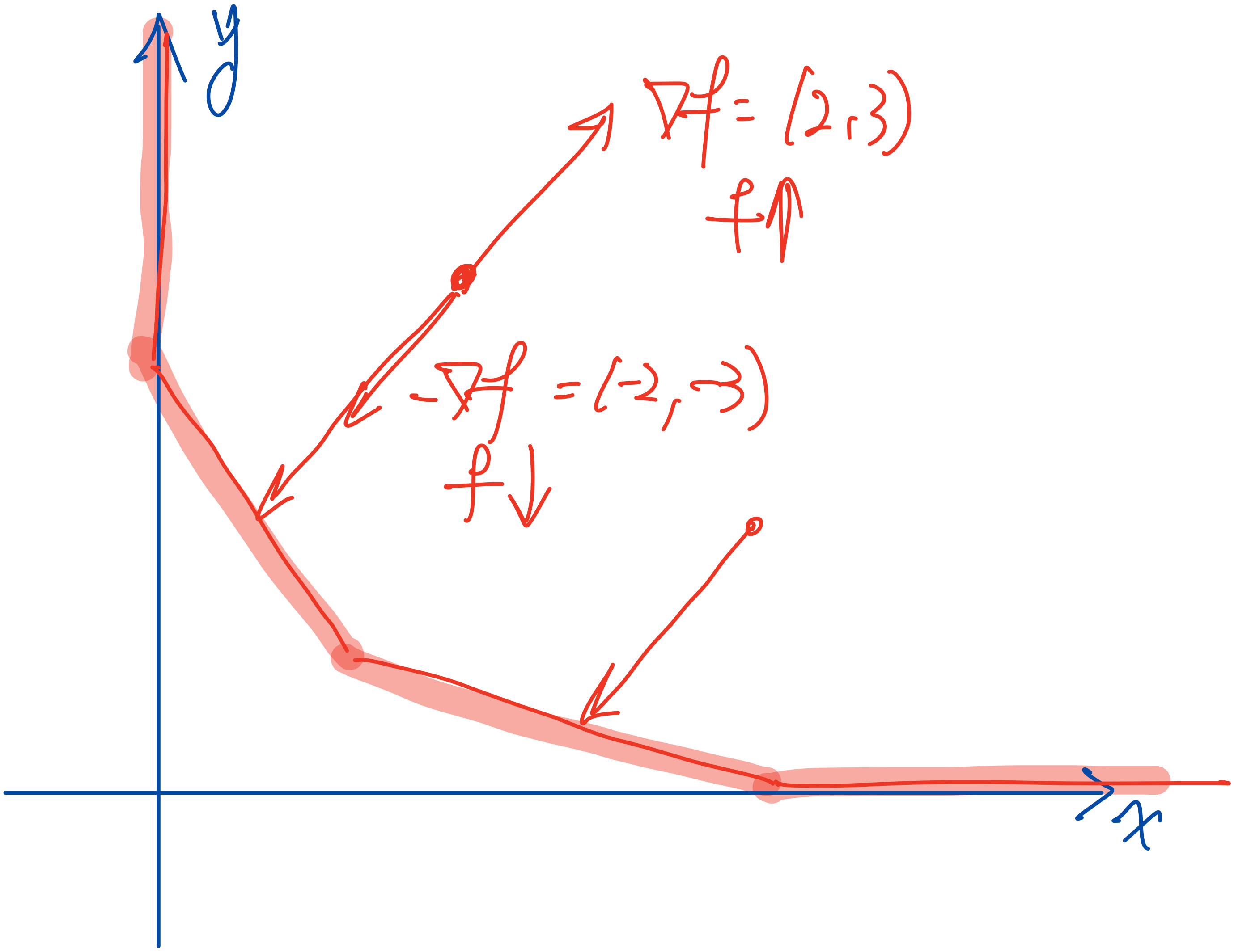
constraints
$$\begin{cases} 2x + y \geq 6 & (1) \\ x + 3y \geq 8 & (2) \\ x \geq 0,\ y \geq 0 \end{cases}$$



Objective function

$$Z = f(x,y) = 2x + 3y$$

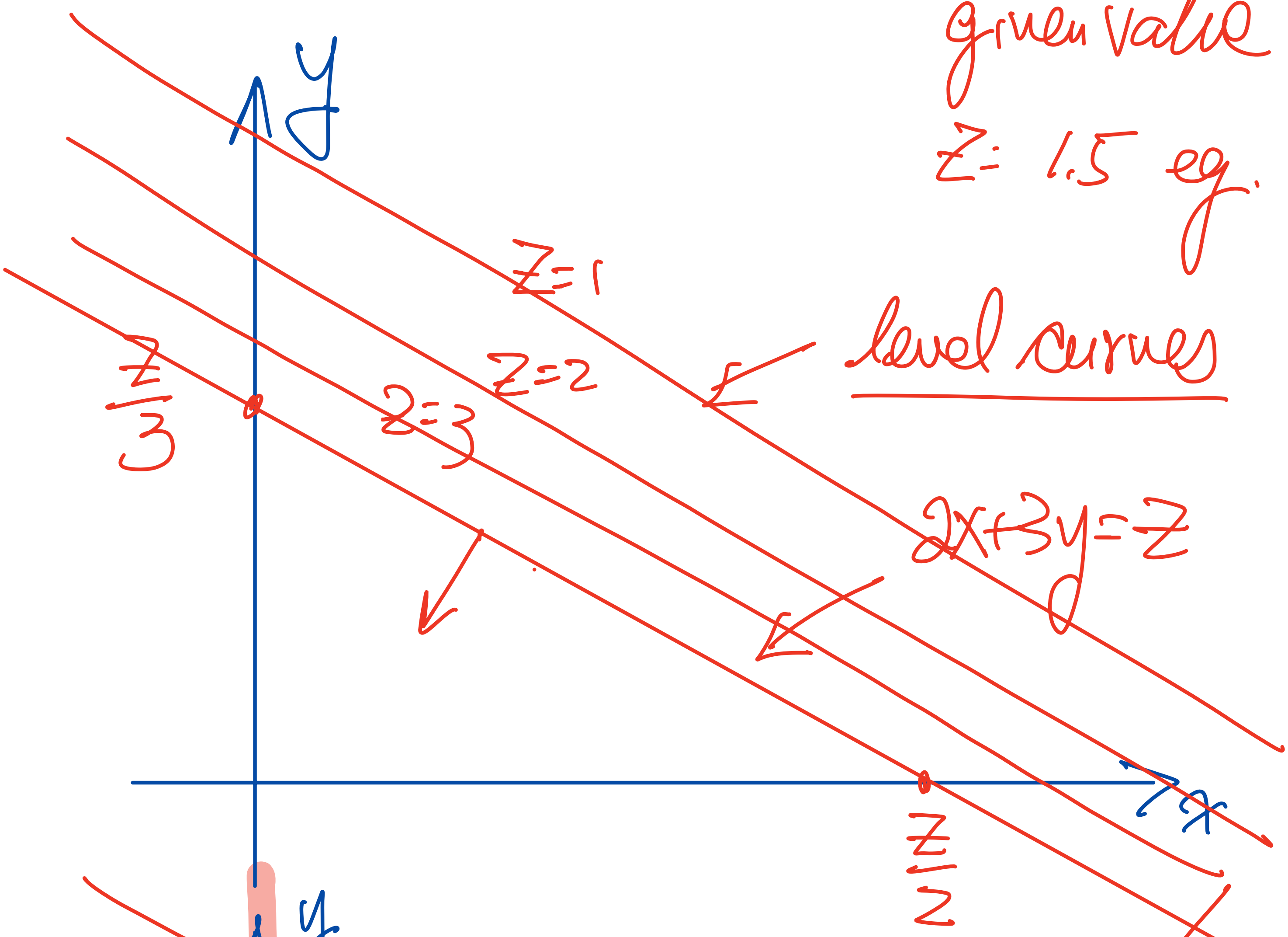
(i) $\nabla f = (\partial_x f, \partial_y f) = (2, 3)$

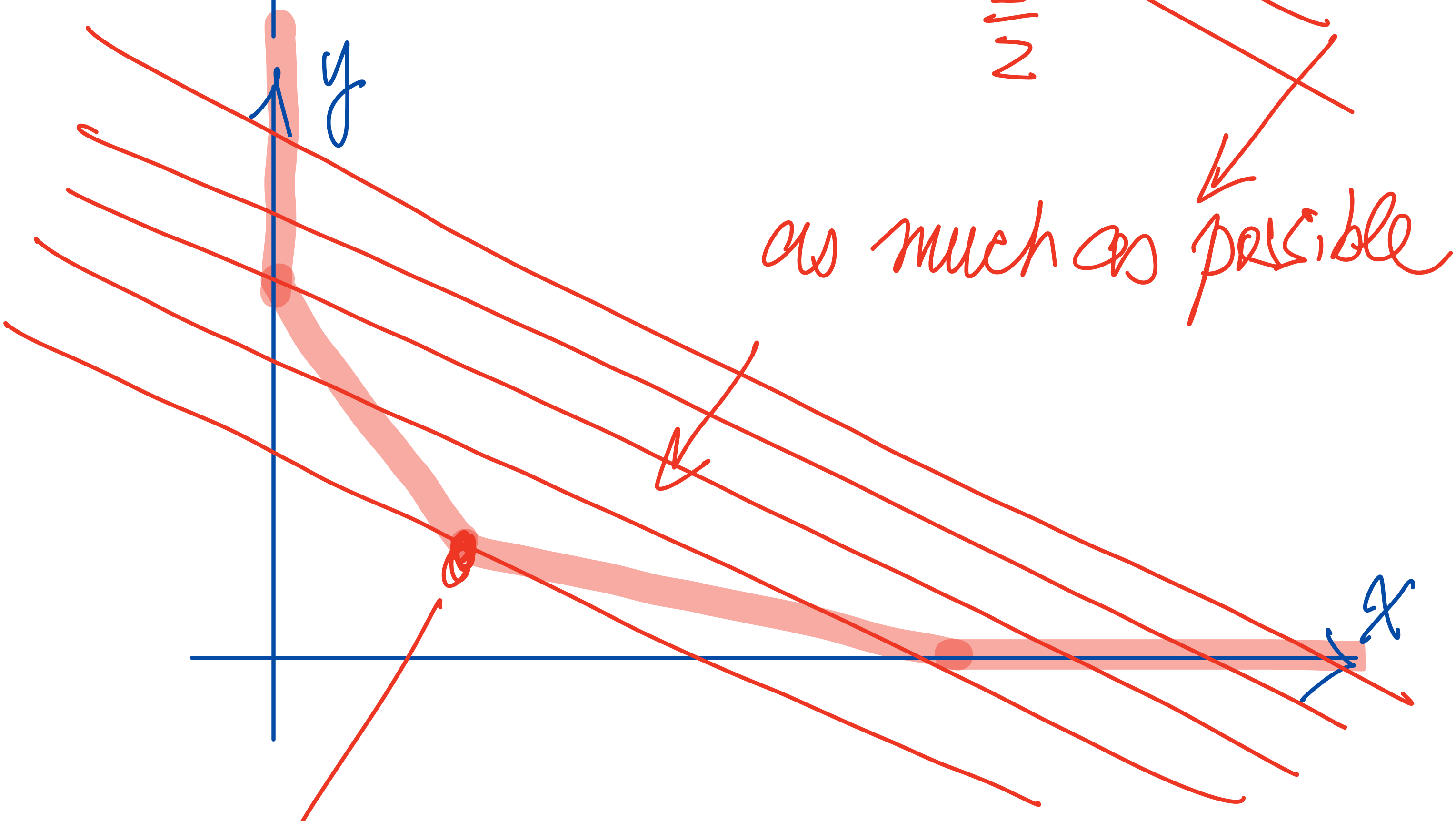


ii) $f(x,y) = 2x+3y = z$

given value

$z = 1.5$ eg.





intersection between ① & ②:

$$2x+y=6$$
$$x+3y=8$$
$$5x=10 \Rightarrow x=2$$
$$\Rightarrow y=2$$

But at $(2,2)$, $\nabla f(2,2)=(2,3)\neq 0$

Hence $(2,2)$ is not a critical pt.

Note: $(2,2)$ is at the boundary of the feasible set, not an interior pt.

$(2,2)$ is at corner pt.

$$\boxed{f(\underbrace{2,2}_{\text{min pt}}) = 2(2)+3(2) = \underbrace{10\,\$}_{\text{min value.}}}$$

II Dual Problem

Pharmaceutical company

- develop Vitamin A, C pills (to replace carrot & cabbage)
- determine the prices of A, C pills.

Let p = price of A
q = price of C

objective function: max $Z = 6p + 8q$

constraint:
$$\begin{cases} 2p + q \le 2 & \leftarrow \text{carrot price} \quad (1) \\ p + 3q \le 3 & \leftarrow \text{cabbage price} \quad (2) \\ p, q \ge 0 \end{cases}$$

The price of carrot from the company's pt. of view

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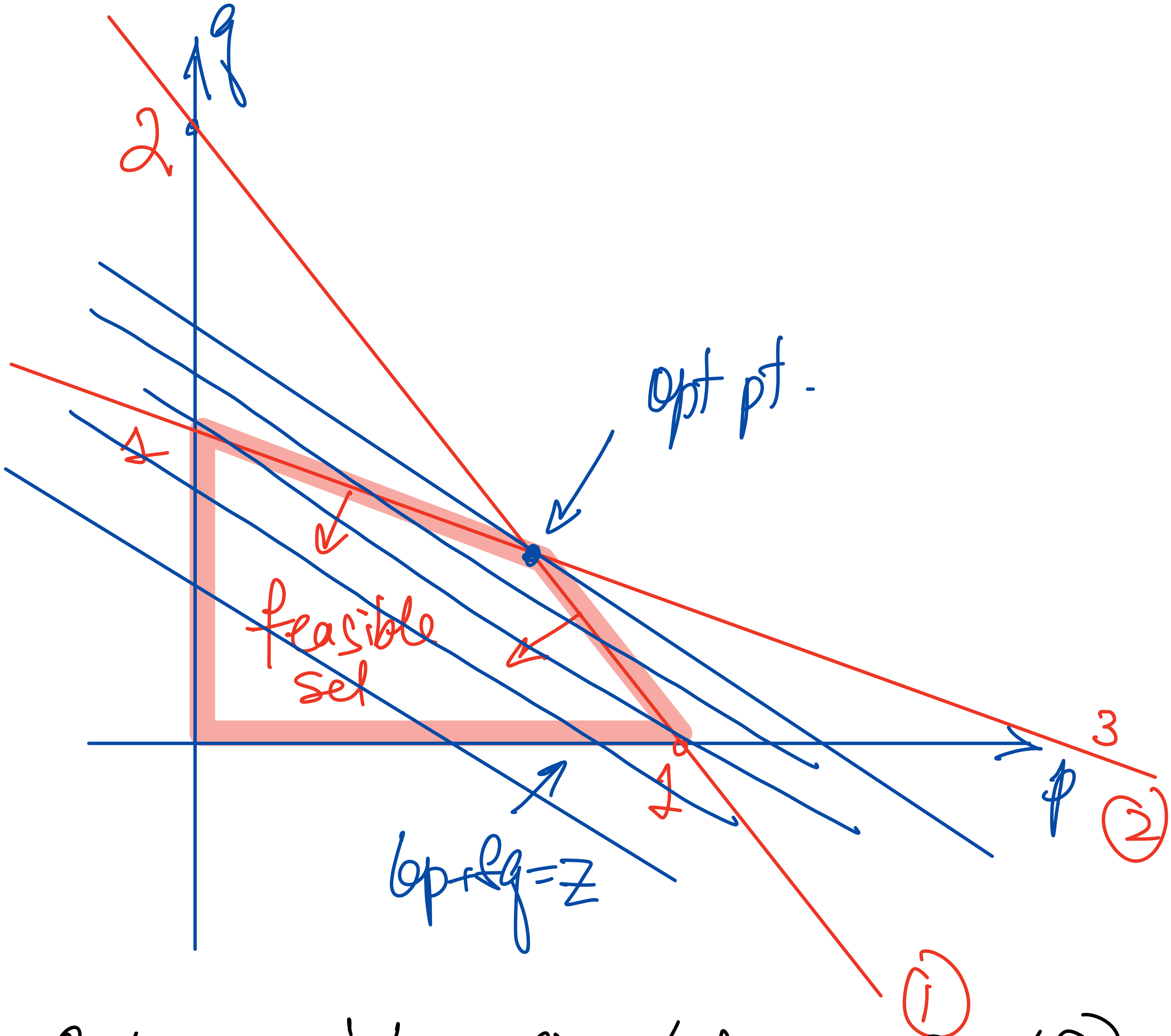


Opt pt: intersection between (1) & (2)

$$2p+q=2$$
$$p+3q=3$$
$$5p=3 \Rightarrow \quad p=\frac{3}{5}, \quad q=\frac{4}{5}$$

Opt value $\quad Z = 6p + 8q$

$$= 6\left(\frac{3}{5}\right) + 8\left(\frac{4}{5}\right)$$

$$= \underline{10}$$

the same as the $\underbrace{\text{origin problem}}_{\text{primal}}$

$$\boxed{\text{opt primal value} = \text{opt. dual value}}$$

III Change of Parameters

	Vitamin A	Vitamin C	Price
Carrot (g)	2 (mg)	1 (mg)	2 ¢
Cabbage (g)	1 (mg)	3 (mg)	3 ¢
Min. Daily Consumption	6 (mg)	8 (mg)	

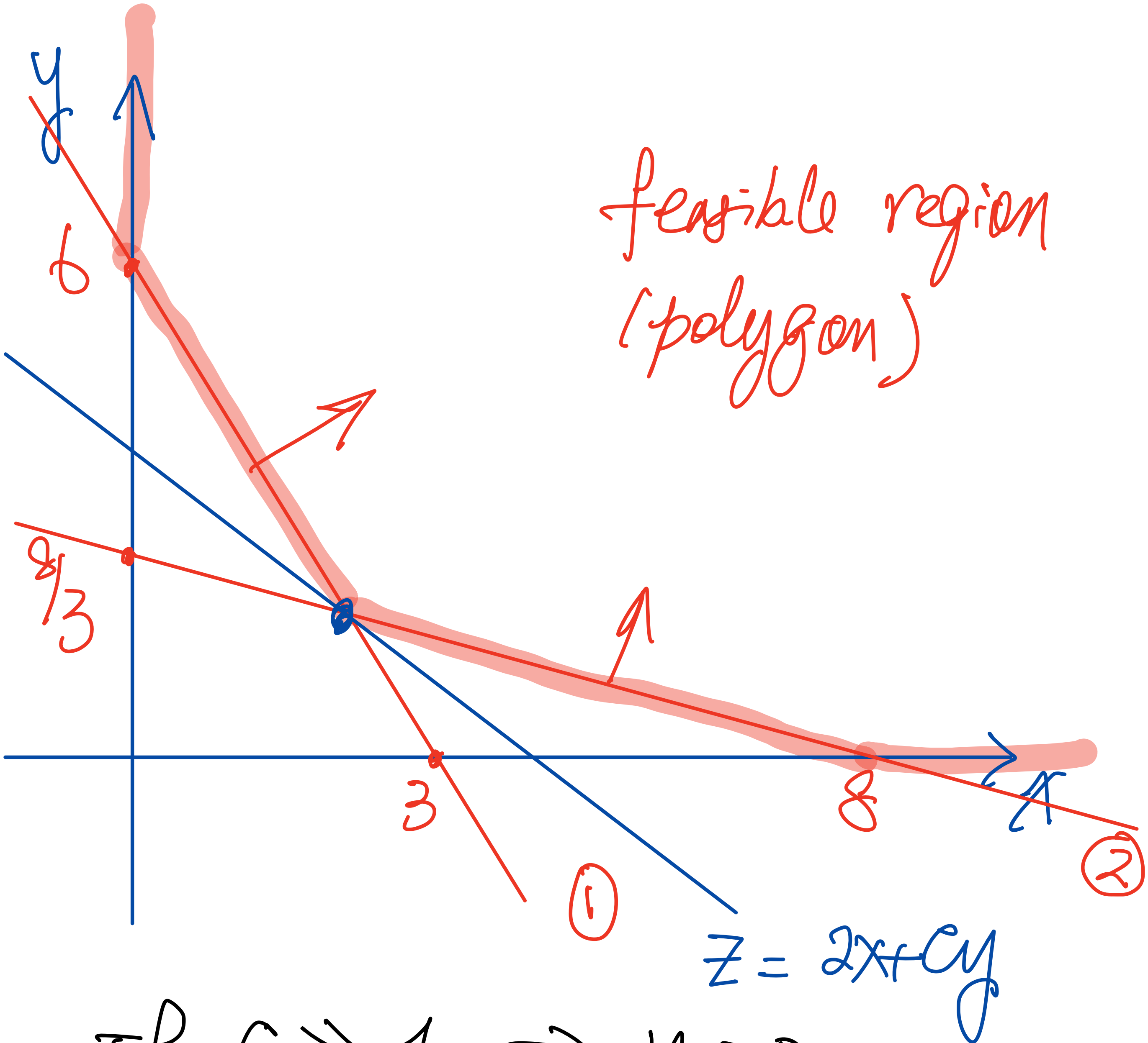
What if the price of cabbage has changed?

$$\min Z = 2x + cy$$

$$2x + y \geq 6$$

$$x + 3y \geq 8$$

For what range of c is $(2,2)$ still the opt. pt?



If $c \gg 1, \Rightarrow y=0$

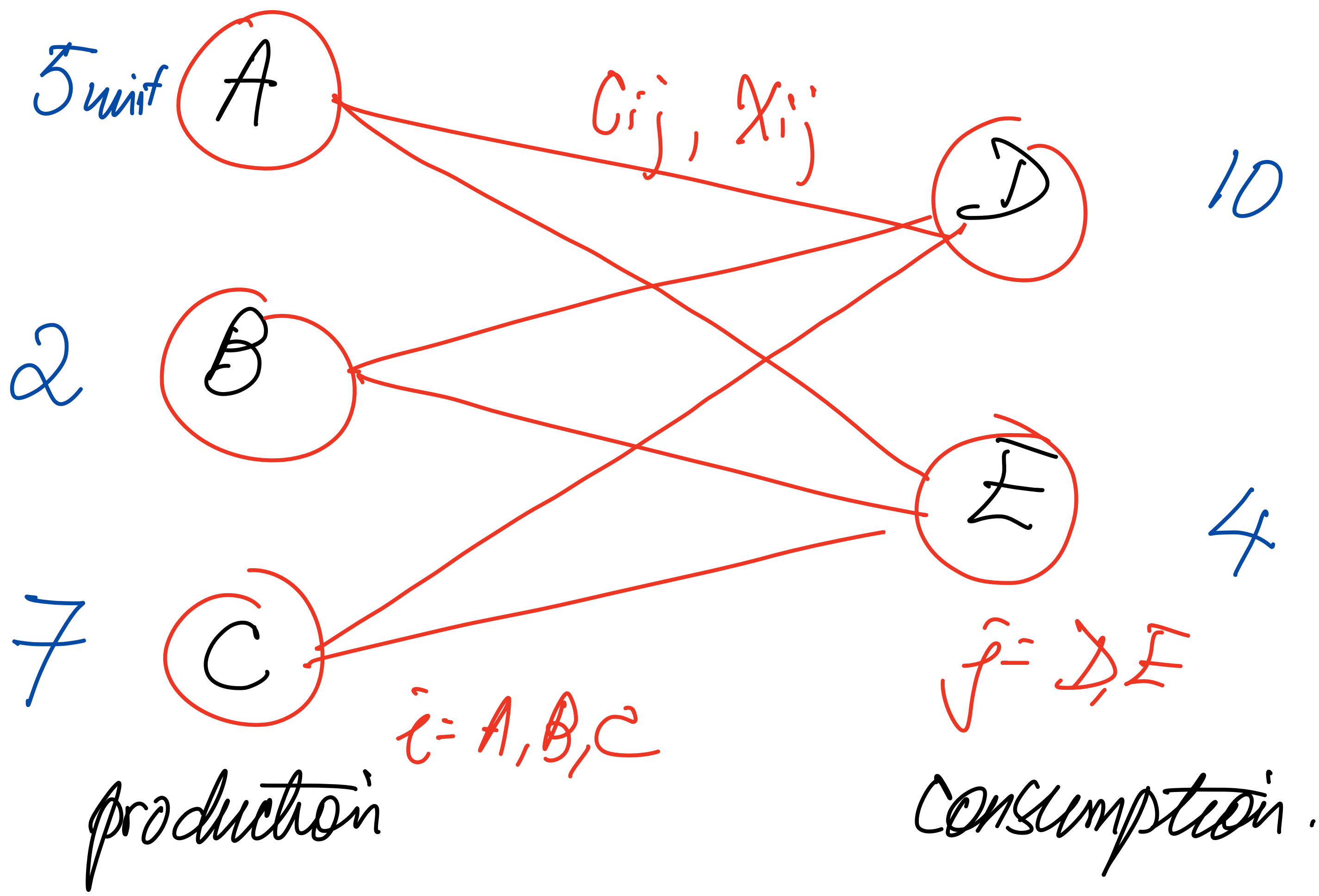
$$\frac{2}{c} = \frac{1}{3} \Rightarrow \underline{\underline{c=6}}$$

If $c \ll 1 \Rightarrow x=0$

$$\frac{2}{c} = 2 \Rightarrow c=1$$

$$\boxed{1 < c < 6}$$

Transportation Problem



($5+2+7 = 10+4$, production balances consumption.)

X_{ij} = amount of transportation from i to j

C_{ij} = unit cost of transportation from i to j

Total cost $z = \sum_{ij} c_{ij} x_{ij}$

Constraints:
$$\begin{pmatrix} \forall i, & \sum_j x_{ij} = p_i \\ \forall j & \sum_i x_{ij} = q_j \end{pmatrix}$$

$$x_{AD} + x_{AE} = 5$$

$$x_{BD} + x_{BE} = 2$$

$$x_{CD} + x_{CE} = 7$$

$$x_{AD} + x_{BD} + x_{CD} = 10$$

$$x_{AE} + x_{BE} + x_{CE} = 4$$

$$x_{ij} \geq 0$$