

Simplex Method [V] p. 20

$$\max \quad \underline{f = 3x_1 + 2x_2}$$

$$\text{Subject to} \quad -x_1 + 3x_2 \leq 12$$

$$x_1 + x_2 \leq 8$$

$$2x_1 - x_2 \leq 10$$

$$x_1, x_2 \geq 0$$

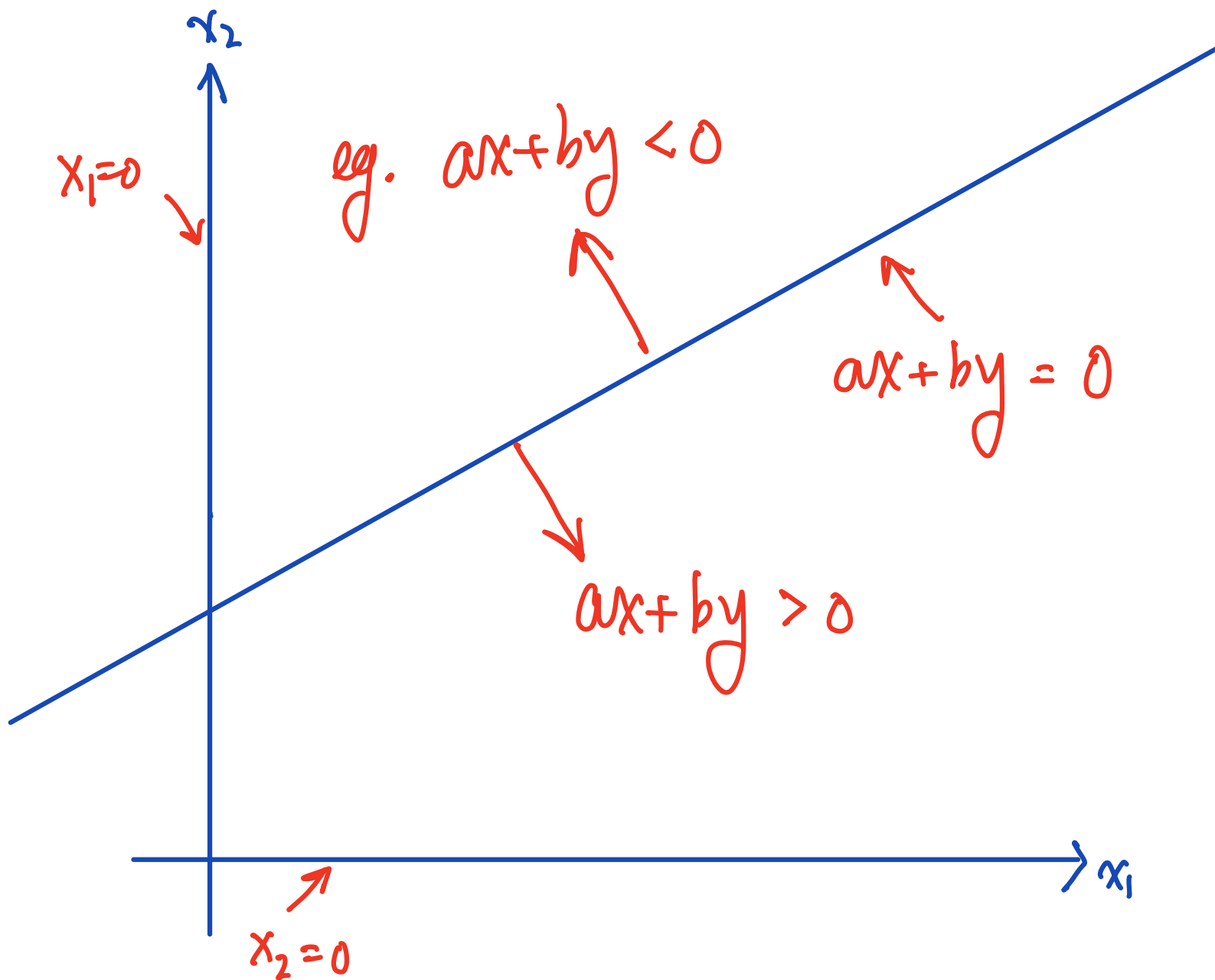
Slack variable form:

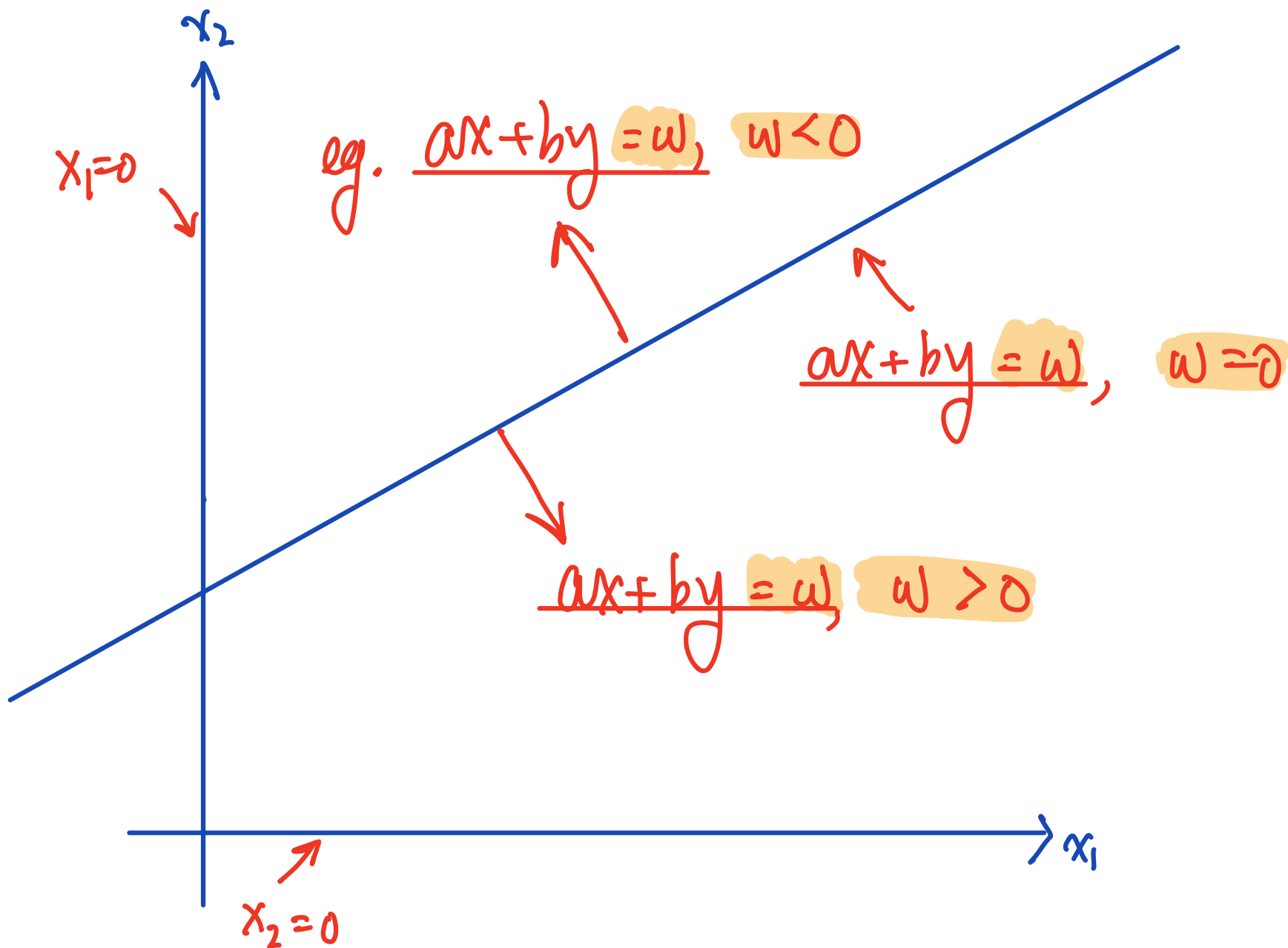
$$w_1 = 12 + x_1 - 3x_2$$

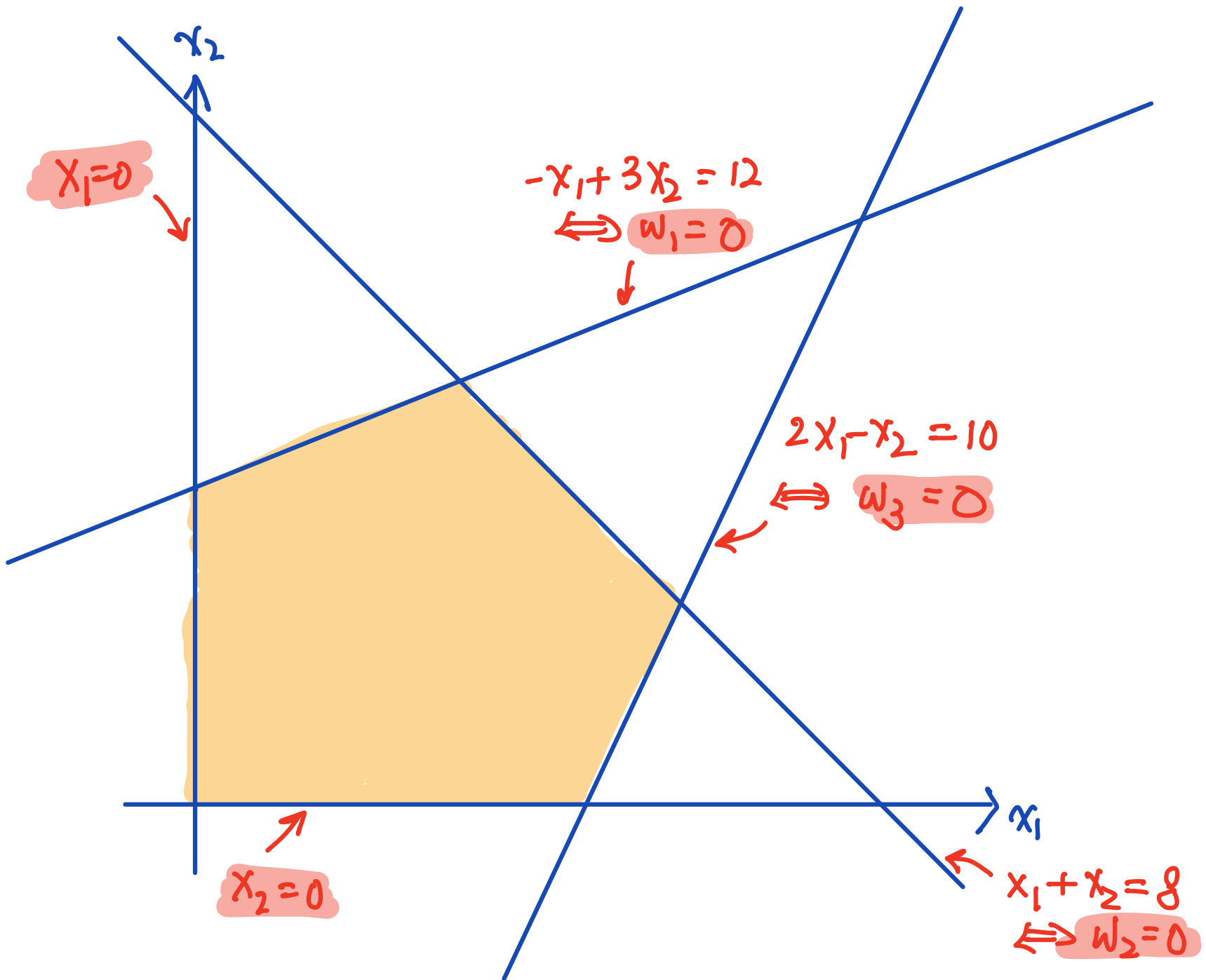
$$w_2 = 8 - x_1 - x_2$$

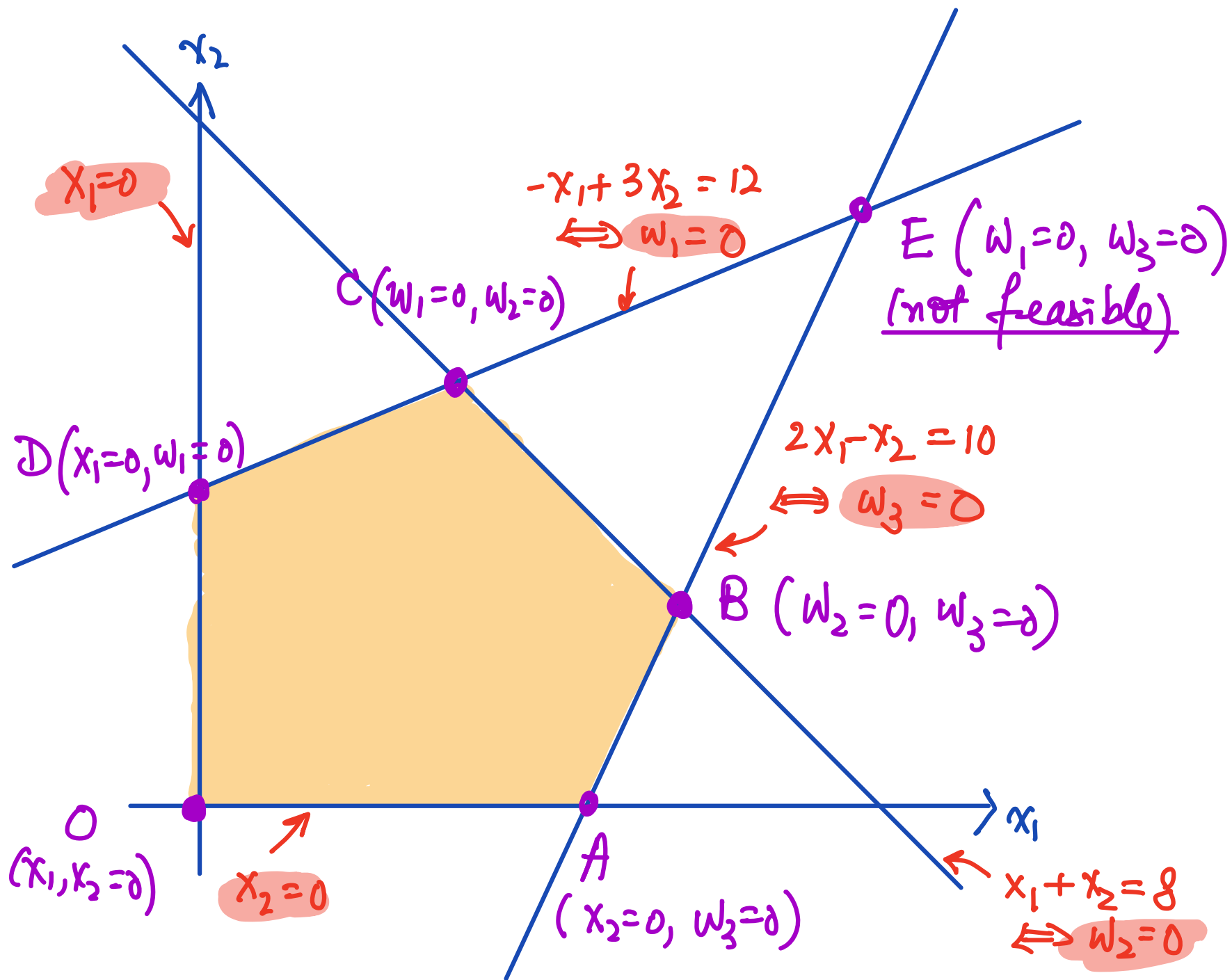
$$w_3 = 10 - 2x_1 + x_2$$

$$\underline{x_1, x_2, x_3, w_1, w_2, w_3 \geq 0}$$









$$(0) \quad \max \quad J = 3x_1 + 2x_2$$

$$\text{subject to} \quad \begin{aligned} w_1 &= 12 + x_1 - 3x_2 \\ w_2 &= 8 - x_1 - x_2 \\ w_3 &= 10 - 2x_1 + x_2 \end{aligned}$$

$$\underline{x_1, x_2, x_3, w_1, w_2, w_3 \geq 0}$$

x_1, x_2, x_3 on the RHS — non-basic variables

w_1, w_2, w_3 on the LHS — basic variable

Idea of Simplex Method:

(1) Set all non-basic var. to zero — correspond to a vertex

(2) choose a new set of (non)basic vars, to improve J .

(1) & (2) is the same as moving from vertex to vertex.

①

Initial Step/Guess

$$\max \quad J = 3x_1 + 2x_2$$

$$\text{subject to} \quad \begin{aligned} w_1 &= 12 + x_1 - 3x_2 \\ w_2 &= 8 - x_1 - x_2 \\ w_3 &= 10 - 2x_1 + x_2 \end{aligned}$$

$$\underline{x_1, x_2, x_3, w_1, w_2, w_3 \geq 0}$$

$$\underline{\text{Set } x_1, x_2 = 0 \text{ (0)} \Rightarrow J = 0}$$

①

Improve (0) ($x_1=0, x_2=0$)

$$\max \quad z = 3x_1 + 2x_2$$

increase x_1 , keep $x_2=0$
(max. coeff. rule)

subject to

$$\begin{aligned} w_1 &= 12 + x_1 - 3x_2 \\ w_2 &= 8 - x_1 - x_2 \\ w_3 &= 10 - 2x_1 + x_2 \end{aligned}$$

$$\underline{x_1, x_2, x_3, w_1, w_2, w_3 \geq 0}$$

w_1 : x_1 can be as large as possible

$$w_2: 8 - x_1 \geq 0 \Rightarrow x_1 \leq 8$$

$$w_3: 10 - 2x_1 \geq 0 \Rightarrow x_1 \leq 5$$

Set $x_1 = 5 \Rightarrow \underline{w_3 = 0 \Rightarrow A = (x_2=0, w_3=0)}$

⑦ A ($x_2=0, w_3=0$)

Interchange x_1 and w_3 with
 w_3 leaving basic and x_1 entering basic

$$J = 3x_1 + 2x_2$$

$$w_1 = 12 + x_1 - 3x_2$$

$$w_2 = 8 - x_1 - x_2$$

$$w_3 = 10 - 2x_1 + x_2$$

$$x_1 = 5 - \frac{w_3}{2} + \frac{1}{2}x_2$$

$$w_1 = 12 + 5 - \frac{w_3}{2} + \frac{1}{2}x_2 - 3x_2 = 17 - \frac{w_3}{2} - \frac{5x_2}{2}$$

$$w_2 = 8 - 5 + \frac{w_3}{2} - \frac{1}{2}x_2 - x_2 = 3 + \frac{w_3}{2} - \frac{3x_2}{2}$$

$$J = 3\left(5 - \frac{w_3}{2} + \frac{1}{2}x_2\right) + 2x_2 = 15 - \frac{3w_3}{2} + \frac{7}{2}x_2$$

⑦ A ($x_2=0, w_3=0$)

Interchange x_1 and w_3 with
 w_3 leaving basic and x_1 entering basic

$$\max z = 15 - \frac{3}{2}w_3 + \frac{7}{2}x_2$$

$$\text{subject to } x_1 = 5 - \frac{1}{2}w_3 + \frac{x_2}{2}$$

$$w_1 = 17 - \frac{1}{2}w_3 - \frac{5}{2}x_2$$

$$w_3 = 3 + \frac{1}{2}w_3 - \frac{3}{2}x_2$$

basic var. non-basic var.

III Improve: A ($x_2=0, w_3=0$)

$$\max J = 15 - \frac{3}{2}w_3 + \frac{7}{2}x_2$$

increase x_2 ,
keeping $w_3=0$

$$\text{subject to } x_1 = 5 - \frac{1}{2}w_3 + \frac{x_2}{2}$$

$$w_1 = 17 - \frac{1}{2}w_3 - \frac{5}{2}x_2$$

$$w_2 = 3 + \frac{1}{2}w_3 - \frac{3}{2}x_2$$

basic var. non-basic var.

x_1 : x_2 can be as large as possible

$$w_1: 17 - \frac{5}{2}x_2 \geq 0 \Rightarrow x_2 \leq \frac{34}{5}$$

$$w_2: 3 - \frac{3}{2}x_2 \geq 0 \Rightarrow x_2 \leq 2$$

Set $x_2=2 \Rightarrow w_3=0$ and $w_2=0 \Rightarrow B$

III

B ($w_2=0, w_3=0$)

Interchange x_2 and w_2 with
 w_2 leaving basic and x_2 entering basic.

$$\max z = 15 - \frac{3}{2}w_3 + \frac{7}{2}x_2$$

$$\text{subject to } x_1 = 5 - \frac{1}{2}w_3 + \frac{x_2}{2}$$

$$w_1 = 17 - \frac{1}{2}w_3 - \frac{5}{2}x_2$$

$$w_2 = 3 + \frac{1}{2}w_3 - \frac{3}{2}x_2$$

$$x_2 = 2 + \frac{1}{3}w_3 - \frac{2}{3}w_2$$

$$x_1 = 5 - \frac{1}{2}w_3 + \frac{1}{2}\left(2 + \frac{1}{3}w_3 - \frac{2}{3}w_2\right) = 6 - \frac{1}{3}w_3 - \frac{1}{3}w_2$$

$$w_1 = 17 - \frac{1}{2}w_3 - \frac{5}{2}\left(2 + \frac{1}{3}w_3 - \frac{2}{3}w_2\right) = 12 - \frac{4}{3}w_3 + \frac{5}{3}w_2$$

III

B ($w_2=0, w_3=0$)

Interchange x_2 and w_2 with
 w_2 leaving basic and x_2 entering basic.

$$\max Z = 15 - \frac{3}{2}w_3 + \frac{7}{2}x_2$$

$$\text{subject to } x_1 = 5 - \frac{1}{2}w_3 + \frac{x_2}{2}$$

$$w_1 = 17 - \frac{1}{2}w_3 - \frac{5}{2}x_2$$

$$w_2 = 3 + \frac{1}{2}w_3 - \frac{3}{2}x_2$$

$$x_2 = 2 + \frac{1}{3}w_3 - \frac{2}{3}w_2$$

$$Z = 15 - \frac{3}{2}w_3 + \frac{7}{2}\left(2 + \frac{1}{3}w_3 - \frac{2}{3}w_2\right)$$

$$= 22 - \frac{1}{3}w_3 - \frac{7}{3}w_2$$

③ B ($w_2=0, w_3=0$)

$$\max \quad J = 22 - \frac{1}{3}w_3 - \frac{7}{3}w_2$$

Subject to $x_1 = 6 - \frac{1}{3}w_3 - \frac{1}{3}w_2$

$$w_1 = 12 - \frac{4}{3}w_3 + \frac{5}{3}w_2$$

$$\underbrace{x_2}_{\text{basic}} = \underbrace{2 + \frac{1}{3}w_3 - \frac{2}{3}w_2}_{\text{non-basic}}$$

$$x_1, x_2, w_1, w_2, w_3 \geq 0$$

→ B is optimal!! $\max J = 22$

$$J = 22 - \frac{1}{3}w_3 - \frac{7}{3}w_2$$

③ B ($w_2=0, w_3=0$)

$$\max \quad J = 22 - \frac{1}{3}w_3 - \frac{7}{3}w_2$$

$$\text{Subject to } x_1 = 6 - \frac{1}{3}w_3 - \frac{1}{3}w_2$$

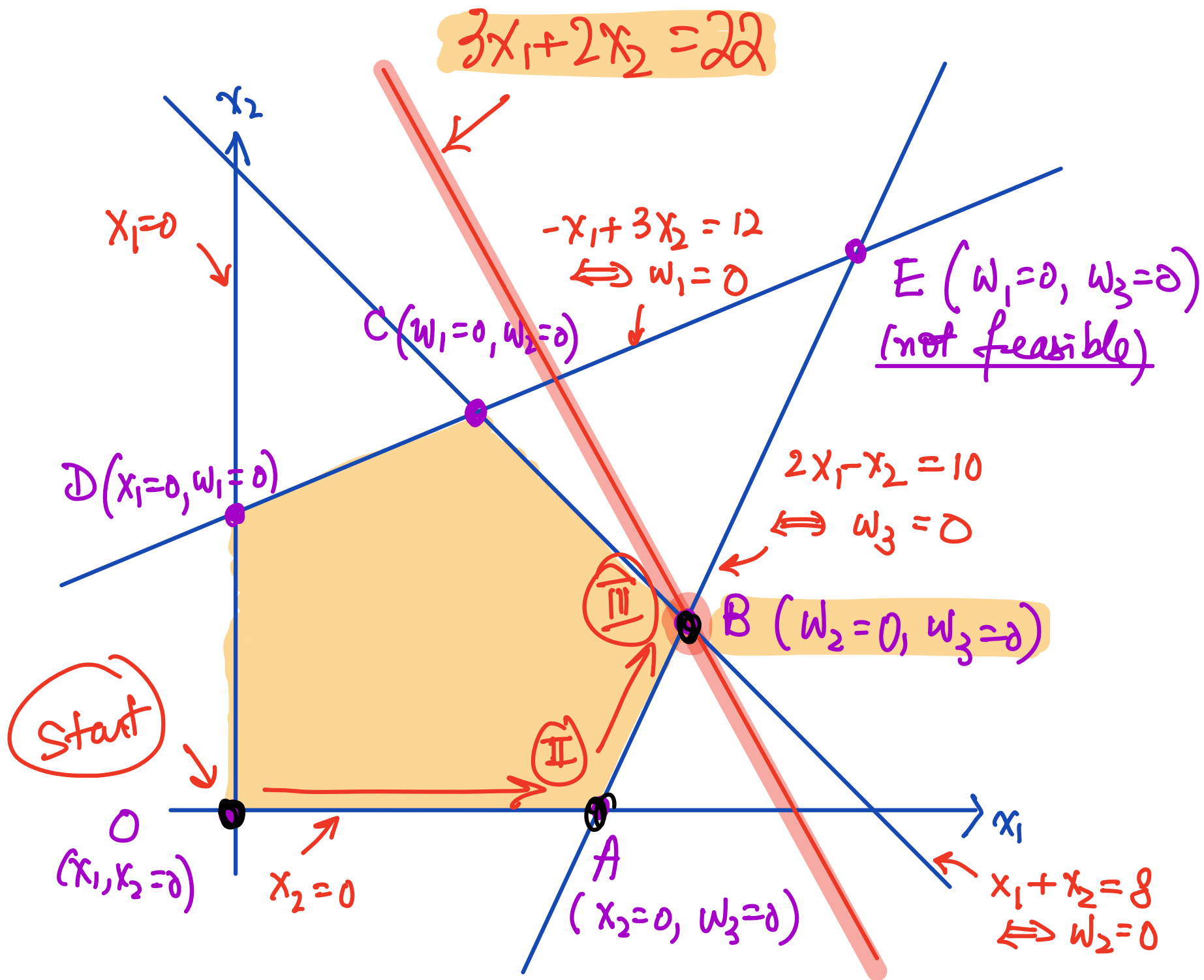
$$w_1 = 12 - \frac{4}{3}w_3 + \frac{5}{3}w_2$$

$$x_2 = 2 + \frac{1}{3}w_3 - \frac{2}{3}w_2$$

$$x_1, x_2, w_1, w_2, w_3 \geq 0$$

→ B is optimal!! $\max J = 22$

$$J = 22 - \frac{1}{3}w_3 - \frac{7}{3}w_2 \leq 22, = 22 \text{ at } w_2=0, w_3=0$$

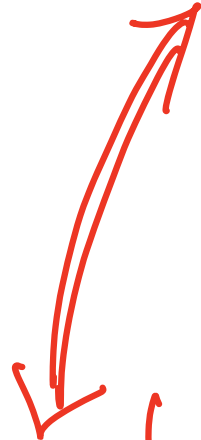


[V] p. 11

$$\max \quad z = 5x_1 + 4x_2 + 3x_3$$

subject to

$$\begin{cases} 2x_1 + 3x_2 + x_3 \leq 5 \\ 4x_1 + x_2 + 2x_3 \leq 11 \\ 3x_1 + 4x_2 + 2x_3 \leq 8 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$



$$w_1 = 5 - 2x_1 - 3x_2 - x_3$$

$$w_2 = 11 - 4x_1 - x_2 - 2x_3$$

$$w_3 = 8 - 3x_1 - 4x_2 - 2x_3$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

[V] p. 11 Set $x_1 = x_2 = x_3 = 0$

(I) $\max \quad z = 5x_1 + 4x_2 + 3x_3$
subject to $\left\{ \begin{array}{l} 2x_1 + 3x_2 + x_3 \leq 5 \\ 4x_1 + x_2 + 2x_3 \leq 11 \\ 3x_1 + 4x_2 + 2x_3 \leq 8 \\ x_1, x_2, x_3 \geq 0 \end{array} \right.$

$\left\{ \begin{array}{l} w_1 = 5 - 2x_1 - 3x_2 - x_3 = 5 \geq 0 \\ w_2 = 11 - 4x_1 - x_2 - 2x_3 = 11 \geq 0 \\ w_3 = 8 - 3x_1 - 4x_2 - 2x_3 = 8 \geq 0 \\ x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{array} \right.$

[V] p. 11 Set $x_1 = x_2 = x_3 = 0$

$$\max J = 5x_1 + 4x_2 + 3x_3$$

increase x_1 ,
keep $x_2 = x_3 = 0$

subject to $w_1 = 5 - 2x_1 - 3x_2 - x_3$

$$w_2 = 11 - 4x_1 - x_2 - 2x_3$$

$$w_3 = 8 - 3x_1 - 4x_2 - 2x_3$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$w_3 \geq 0 \Rightarrow x_1 \leq 8/3$$

$$w_2 \geq 0 \Rightarrow x_1 \leq 11/4$$

$$w_1 \geq 0 \Rightarrow x_1 \leq 3/2 \leftarrow \text{most strict}$$

[V] p. 11 Set $x_1 = x_2 = x_3 = 0$

$$\max \quad J = 5x_1 + 4x_2 + 3x_3$$

increase x_1 ,
keep $x_2 = x_3 = 0$

subject to $w_1 = 5 - 2x_1 - 3x_2 - x_3$

$$w_2 = 11 - 4x_1 - x_2 - 2x_3$$

$$w_3 = 8 - 3x_1 - 4x_2 - 2x_3$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

Set $x_1 = \frac{5}{2} \Rightarrow w_1 = 0$, interchange x_1, w_1

$$x_1 = \frac{5}{2} - \frac{w_1}{2} - \frac{3}{2}x_2 - \frac{x_3}{2}$$

$$w_2 = 1 + 2w_1 + 5x_2$$

$$w_3 = \frac{1}{2} + \frac{3}{2}w_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3$$

[V] p. 11 Set $x_1 = x_2 = x_3 = 0$

$$\max \quad J = 5x_1 + 4x_2 + 3x_3$$

increase x_1 ,
keep $x_2 = x_3 = 0$

subject to $w_1 = 5 - 2x_1 - 3x_2 - x_3$

$$w_2 = 11 - 4x_1 - x_2 - 2x_3$$

$$w_3 = 8 - 3x_1 - 4x_2 - 2x_3$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

Set $x_1 = \frac{5}{2} \Rightarrow w_1 = 0$, interchange x_1, w_1

$$\begin{aligned} J &= 5x_1 + 4x_2 + 3x_3 \\ &= \frac{25}{2} - \frac{5}{2}w_1 - \frac{7}{2}x_2 + \frac{1}{2}x_3 \end{aligned}$$

[V] p. 11 Set $w_1=0, x_2=0, x_3=0$

(II)

$$\max. \quad J = \frac{25}{2} - \frac{5}{2}w_1 - \frac{7}{2}x_2 + \frac{1}{2}x_3$$

$$\text{subj.} \quad x_1 = \frac{5}{2} - \frac{w_1}{2} - \frac{3}{2}x_2 - \frac{x_3}{2}$$

$$w_2 = 1 + 2w_1 + 5x_2$$

$$w_3 = \frac{1}{2} + \frac{3}{2}w_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3$$

basic

non-basic

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

[V] p. 11 Set $w_1=0, x_2=0, x_3=0$

$$\text{max. } J = \frac{25}{2} - \frac{5}{2}w_1 - \frac{7}{2}x_2 + \frac{1}{2}x_3$$

increase

x_3 ,

keep

$w_1, x_2=0$

$$\text{subj. } x_1 = \frac{5}{2} - \frac{w_1}{2} - \frac{3}{2}x_2 - \frac{x_3}{2}$$

$$w_2 = 1 + 2w_1 + 5x_2$$

$$w_3 = \frac{1}{2} + \frac{3}{2}w_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3$$

basic

non-basic

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$w_3 \geq 0 \Rightarrow x_3 \leq 1 \leftarrow \text{most strict.}$$

$$w_2 \geq 0 \Rightarrow x_2 \text{ can be any number}$$

$$x_1 \geq 0 \Rightarrow x_3 \leq 5$$

[V] p. 11 Set $w_1=0, x_2=0, w_3=0$

III

max. $J = \frac{25}{2} - \frac{5}{2}w_1 - \frac{7}{2}x_2 + \frac{1}{2}x_3$

increase

x_3 ,

keep

$w_1, x_2=0$

subj. $\rightarrow x_1 = \frac{5}{2} - \frac{w_1}{2} - \frac{3}{2}x_2 - \frac{x_3}{2}$

$\rightarrow w_2 = 1 + 2w_1 + 5x_2$

$\rightarrow w_3 = \frac{1}{2} + \frac{3}{2}w_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3$

basic

non-basic

Set $x_3=1 \Rightarrow w_3=0$. Interchange x_3, w_3

$\rightarrow x_3 = 1 + 3w_1 + x_2 - 2w_3$

$\rightarrow x_1 = 2 - 2w_1 - 2x_2 + w_3$

$\rightarrow w_2 = 1 + 2w_1 + 5x_2$

[V] p. 11 Set $w_1=0, x_2=0, w_3=0$

max. $\int = \frac{25}{2} - \frac{5}{2}w_1 - \frac{7}{2}x_2 + \frac{1}{2}x_3$

increase

x_3 ,

keep

$w_1, x_2=0$

subj. $x_1 = \frac{5}{2} - \frac{w_1}{2} - \frac{3}{2}x_2 - \frac{x_3}{2}$

$$w_2 = 1 + 2w_1 + 5x_2$$

$$w_3 = \frac{1}{2} + \frac{3}{2}w_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3$$

basic

non-basic

Set $x_3=1 \Rightarrow w_3=0$. Interchange x_3, w_3

$\int = \frac{25}{2} - \frac{5}{2}w_1 - \frac{7}{2}x_2 + \frac{1}{2}x_3$

$x_3 = 1 + 3w_1 + x_2 - 2w_3$

$$= 13 - w_1 - 3x_2 - w_3$$

[V] p. 11 Set $w_1=0, x_2=0, w_3=0$

max. $\rightarrow J = \frac{25}{2} - \frac{5}{2}w_1 - \frac{7}{2}x_2 + \frac{1}{2}x_3$

increase

x_3 ,

keep

$w_1, x_2=0$

subj. $x_1 = \frac{5}{2} - \frac{w_1}{2} - \frac{3}{2}x_2 - \frac{x_3}{2}$

$$w_2 = 1 + 2w_1 + 5x_2$$

$$w_3 = \frac{1}{2} + \frac{3}{2}w_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3$$

basic

non-basic

Set $x_3=1 \Rightarrow w_3=0$. Interchange x_3, w_3

$\rightarrow J = 13 - w_1 - 3x_2 - w_3 \leq 13$

$\rightarrow$ max $J = 13$, achieved at $w_1 = x_2 = w_3 = 0$

Concept of a Dictionary of Variables

Initially $(\underbrace{x_1, x_2, \dots, x_n}_{n \text{ non-basic vars}}, \underbrace{w_1, w_2, \dots, w_m}_{m \text{ basic variables}})$

$(\underbrace{x_1, x_2, \dots, x_n}_{n \text{ non-basic vars}}, \underbrace{x_{n+1}, x_{n+2}, \dots, x_{n+m}}_{m \text{ basic variables}})$

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \leq \begin{pmatrix} x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+m} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & & a_{ij} & \\ & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$z = c_0 + c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

Concept of a Dictionary of Variables

Initially $(\underbrace{x_1, x_2, \dots, x_n}_{n \text{ non-basic vars}}, \underbrace{w_1, w_2, \dots, w_m}_{m \text{ basic variables}})$

$(\underbrace{x_1, x_2, \dots, x_n}_{n \text{ non-basic vars}}, \underbrace{x_{n+1}, x_{n+2}, \dots, x_{n+m}}_{m \text{ basic variables}})$

interchange one basic and one non-basic variable.

Concept of a Dictionary of Variables

$$\left(\underbrace{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n}_{n \text{ non-basic}}, \underbrace{\bar{x}_{n+1}, \bar{x}_{n+2}, \dots, \bar{x}_{n+m}}_{m \text{ basic}} \right)$$

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \leq \begin{pmatrix} \bar{x}_{n+1} \\ \bar{x}_{n+2} \\ \vdots \\ \bar{x}_{n+m} \end{pmatrix} = \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \vdots \\ \bar{b}_m \end{pmatrix} - \begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} & \dots & \bar{a}_{1n} \\ & & \bar{a}_{ij} & \\ & & & \\ \bar{a}_{m1} & \bar{a}_{m2} & \dots & \bar{a}_{mn} \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{pmatrix}$$

$$z = \bar{c}_0 + \bar{c}_1 \bar{x}_1 + \bar{c}_2 \bar{x}_2 + \dots + \bar{c}_n \bar{x}_n$$

Concept of a Dictionary of Variables

(1) In each dictionary, setting n non-basic variables to zero, corresponds to a vertex in a polyhedron.

(2) Total number of dictionaries

$$= \binom{n+m}{n} = \frac{(n+m)!}{n! m!}$$

= "# of vertices"

(Note: not all vertices are feasible)