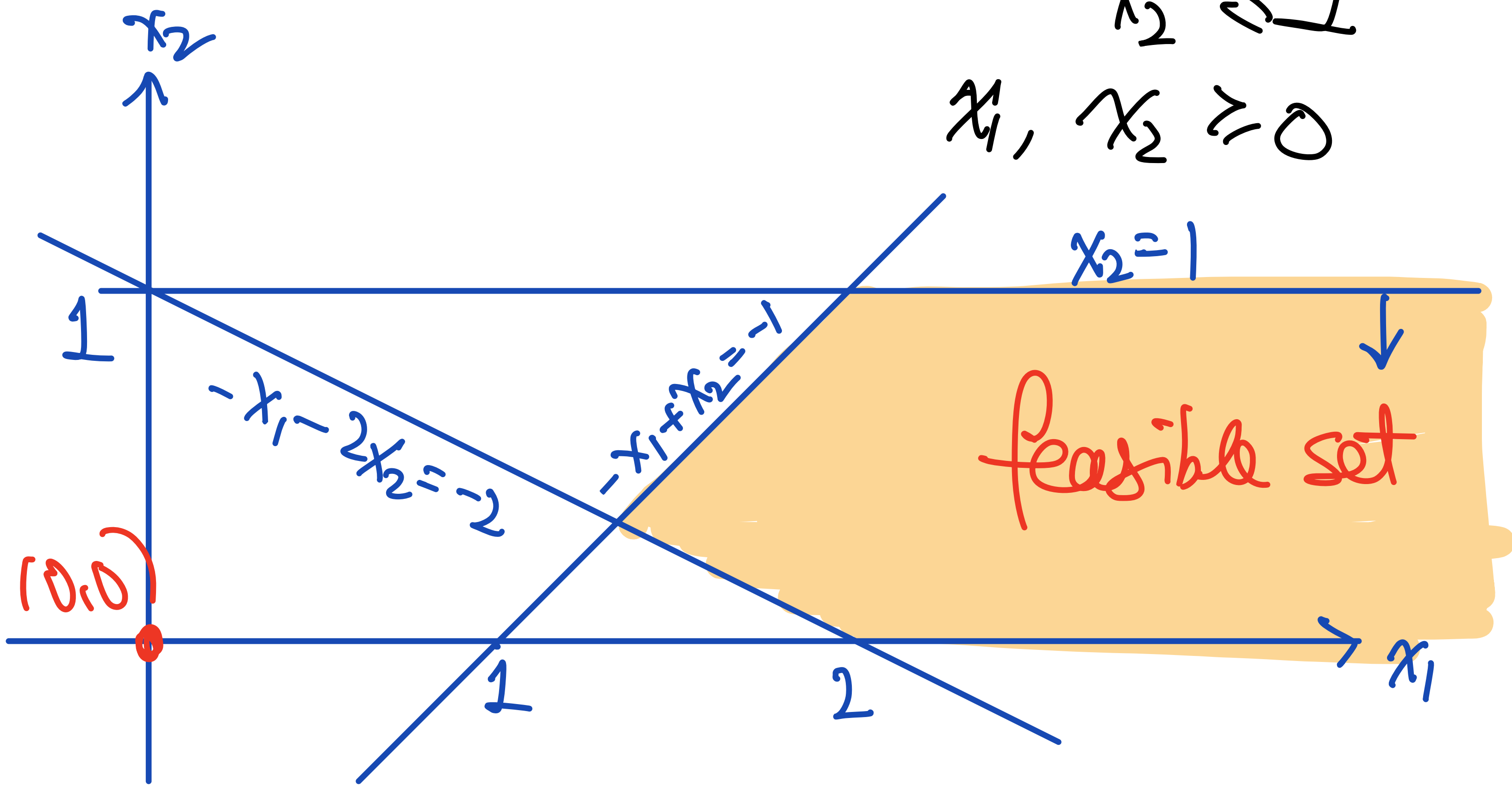
What if the origin is not feasible?

[V] p. 18

$$\max \ \zeta = -2x_1 - x_2$$

subject to
$$\begin{aligned} -x_1 + x_2 &\le -1 \\ -x_1 - 2x_2 &\le -2 \\ x_2 &\le 1 \\ x_1, x_2 &\ge 0 \end{aligned}$$



Auxiliary Problem

$$\max \ \xi = -x_0$$

subject to

$$-x_1 + x_2 - x_0 \le -1$$

$$-x_1 - 2x_2 - x_0 \le -2$$

$$x_2 - x_0 \le 1$$

$$x_1, x_2, x_0 \ge 0$$

(1) Must be feasible: choose x_0 large enough

(2) Original problem is feasible

$\Longleftrightarrow \quad \max \xi = 0 \qquad$ (Note: $\max \xi \le 0$)

Auxiliary Problem

$$\max \ \zeta = -x_0$$

subject to

$$w_1 = -1 + x_1 - x_2 + x_0$$
$$w_2 = -2 + x_1 + 2x_2 + x_0$$
$$w_3 = 1 - x_2 + x_0$$

Setting $x_1, x_2, x_0 = 0$

$w_1 = -1$

$w_2 = -2$ ← most infeasible

$w_3 = 1$ (interchange x_0 & w_2)

Auxiliary Problem

$$\max \ \xi = -x_0$$

subject to
$$w_1 = -1 + x_1 - x_2 + x_0$$
$$w_2 = -2 + x_1 + 2x_2 + x_0$$
$$w_3 = 1 - x_2 + x_0$$

$$\boxed{x_0 = 2 - x_1 - 2x_2 + w_2}$$

$$w_1 = -1 + x_1 - x_2 + 2 - x_1 - 2x_2 + w_2$$
$$= 1 - 3x_2 + w_2$$
$$w_3 = 1 - x_2 + 2 - x_1 - 2x_2 + w_2$$
$$= 3 - x_1 - 3x_2 + w_2$$

Auxiliary Problem

(I)

max $\xi = -2 + x_1 + 2x_2 - w_2$

subject to

$$x_0 = 2 - x_1 - 2x_2 + w_2$$

$$w_1 = 1 - 3x_2 + w_2$$

$$w_3 = 3 - x_1 - 3x_2 + w_2$$

Set $x_1, x_2, w_2 = 0$

feasible !!

Auxiliary Problem

$$\max \ \xi = -2 + x_1 + 2x_2 - w_2$$

subject to

$$x_0 = 2 - x_1 - 2x_2 + w_2$$
$$w_1 = 1 - 3x_2 + w_2$$
$$w_3 = \underbrace{3 - x_1 - 3x_2 + w_2}_{\text{Set } x_1, x_2, w_2 = 0}$$

feasible !!

increase x_2

$x_0 : \quad x_2 \leq 1$

$w_1 : \quad x_2 \leq 1/3$ ⟵ interchange x_2, w_1

$w_3 : \quad x_2 \leq 1$

Auxiliary Problem

$$\max \quad \xi = -2 + x_1 + 2x_2 - w_2$$

subject to

$$x_0 = 2 - x_1 - 2x_2 + w_2$$

$$w_1 = 1 - 3x_2 + w_2$$

$$w_3 = 3 - x_1 - 3x_2 + w_2$$

$$\boxed{x_2 = \frac{1}{3} - \frac{1}{3}w_1 + \frac{1}{3}w_2}$$

$$w_3 = 2 - x_1 + w_1$$

$$x_0 = \frac{4}{3} - x_1 + \frac{2}{3}w_1 + \frac{1}{3}w_2$$

Auxiliary Problem

$$\max \quad \xi = -2 + x_1 + 2x_2 - w_2$$

subject to

$$x_0 = 2 - x_1 - 2x_2 + w_2$$

$$w_1 = 1 - 3x_2 + w_2$$

$$w_3 = 3 - x_1 - 3x_2 + w_2$$

$$\boxed{x_2 = \frac{1}{3} - \frac{1}{3}w_1 + \frac{1}{3}w_2}$$

$$\xi = -\frac{4}{3} + x_1 - \frac{2}{3}w_1 - \frac{w_2}{3}$$

Auxiliary Problem

II

$$\max \quad \xi = -\frac{4}{3} + x_1 - \frac{2}{3} w_1 - \frac{w_2}{3}$$

subject to

$$x_2 = \frac{1}{3} - \frac{1}{3} w_1 + \frac{1}{3} w_2$$

$$w_3 = 2 - x_1 + w_1$$

$$x_0 = \frac{4}{3} - x_1 + \frac{2}{3} w_1 + \frac{1}{3} w_2$$

increase x_1

w_3: $\quad x_1 \leq 2$

x_0: $\quad x_1 \leq \frac{4}{3}$ $\leftarrow$ interchange x_0, x_1

Auxiliary Problem

II

$$\max \quad \zeta = -\frac{4}{3} + x_1 - \frac{2}{3}w_1 - \frac{w_2}{3}$$

subject to

$$x_2 = \frac{1}{3} - \frac{1}{3}w_1 + \frac{1}{3}w_2$$

$$w_3 = 2 - x_1 + w_1$$

$$x_0 = \frac{4}{3} - x_1 + \frac{2}{3}w_1 + \frac{1}{3}w_2$$

$$\boxed{x_1 = \frac{4}{3} + \frac{2}{3}w_1 + \frac{1}{3}w_2 - x_0}$$

$$w_3 = \frac{2}{3} + \frac{1}{3}w_1 - \frac{1}{3}w_2 + x_0$$

$$x_2 = \frac{1}{3} - \frac{1}{3}w_1 + \frac{1}{3}w_2$$

Auxiliary Problem

II

max $\xi = -\frac{4}{3} + x_1 - \frac{2}{3}w_1 - \frac{w_2}{3}$

subject to

$$x_2 = \frac{1}{3} - \frac{1}{3}w_1 + \frac{1}{3}w_2$$

$$w_3 = 2 - x_1 + w_1$$

$$x_0 = \frac{4}{3} - x_1 + \frac{2}{3}w_1 + \frac{1}{3}w_2$$

$$\boxed{x_1 = \frac{4}{3} + \frac{2}{3}w_1 + \frac{1}{3}w_2 - x_0}$$

$$w_3 = \frac{2}{3} + \frac{1}{3}w_1 - \frac{1}{3}w_2 + x_0$$

$$x_2 = \frac{1}{3} - \frac{1}{3}w_1 + \frac{1}{3}w_2$$

$$\xi = -\frac{4}{3} + x_1 - \frac{2}{3}w_1 - \frac{w_2}{3}$$

$$= -x_0$$

Auxiliary Problem

III

$$\max \quad \xi = -x_0$$

subject to

$$x_1 = \frac{4}{3} + \frac{2}{3}w_1 + \frac{1}{3}w_2 - x_0$$

$$w_3 = \frac{2}{3} + \frac{1}{3}w_1 - \frac{1}{3}w_2 + x_0$$

$$x_2 = \frac{1}{3} - \frac{1}{3}w_1 + \frac{1}{3}w_2$$

Setting $w_1 = w_2 = x_0 = 0$

$$\max \quad \xi = -x_0 = 0 \qquad \text{Optimal!}$$

Back to Original Problem

$$\max \zeta = -2x_1 - x_2$$

subject to

$$\begin{cases} -x_1 + x_2 \leq -1 \\ -x_1 - 2x_2 \leq -2 \\ x_2 \leq 1 \\ x_1, x_2 \geq 0 \end{cases}$$

We already have

$$\begin{cases} x_1 = \frac{4}{3} + \frac{2}{3} w_1 + \frac{1}{3} w_2 - \cancel{x_0} \quad 0 \\ w_3 = \frac{2}{3} + \frac{1}{3} w_1 - \frac{1}{3} w_2 + \cancel{x_0} \quad 0 \\ x_2 = \frac{1}{3} - \frac{1}{3} w_1 + \frac{1}{3} w_2 \end{cases}$$

Back to Original Problem

$$\max \zeta = -2x_1 - x_2$$

$$\begin{cases} x_1 = \frac{4}{3} + \frac{2}{3}w_1 + \frac{1}{3}w_2 \\ w_3 = \frac{2}{3} + \frac{1}{3}w_1 - \frac{1}{3}w_2 \\ x_2 = \frac{1}{3} - \frac{1}{3}w_1 + \frac{1}{3}w_2 \end{cases}$$

$$x_1, x_2, w_1, w_2, w_3 \geq 0$$

$$\zeta = -2x_1 - x_2$$

$$= -3 - w_1 - w_2$$

Already at Optimal

$(w_1 = w_2 = 0)$