LP in Standard form

$$\max \ \zeta = c^T X$$

$$\text{s.t.} \quad AX \le b$$

$$X \ge 0$$

General LP Formulation

[V] ch. 9, p. 149
[C] ch. 8 p. 118

$$\max \ \zeta = c^T X$$

$$\text{s.t.} \quad \left.\begin{aligned} A_1 \vec{X} &\le \vec{b}_1 \\ A_2 \vec{X} &= \vec{b}_2 \\ \vec{l} \le X &\le \vec{u} \end{aligned}\right\} (*)$$

(1) Assume $\vec{l} = \vec{0}$, i.e. $0 \le \vec{X} \le \vec{u}$

Then $(*) \iff$

$$\begin{cases} A_1 X \le b_1 \\ A_2 X \le b_2 \\ -A_2 X \le -b_2 \\ \quad X \le \vec{u}, \quad (X \ge 0) \end{cases}$$

$$\Longleftrightarrow \underbrace{\begin{bmatrix} A_1 \\ A_2 \\ -A_2 \\ I \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} X \end{bmatrix} \le \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ -b_2 \\ \vec{u} \end{bmatrix}}_{\tilde{b}}$$

$$(X \geq 0)$$

$$\Longleftrightarrow \quad \tilde{A} X \le \tilde{b}$$

$$X \geq 0$$

(2) For $\vec{l} \neq \vec{0}$,

$$\vec{l} \le \vec{X} \le \vec{u} \Longleftrightarrow 0 \le \underbrace{\vec{X} - \vec{l}}_{\tilde{X}} \le \underbrace{\vec{u} - \vec{l}}_{\tilde{u}}$$

new var. $\longrightarrow$ $\tilde{X}$ $\quad$ $\tilde{u}$

$$A_1 X \le b_1 \Longleftrightarrow A_1 \underline{\tilde{X}} \le b_1 - A_1 \tilde{l} \; (= \tilde{b}_1)$$

$$A_2 X = b_2 \Longleftrightarrow A_2 \underline{\tilde{X}} = b_2 - A_2 \tilde{l} \; (= \tilde{b}_2)$$

(3) Sometimes X has no constraints, not even positivity constraint.

Set $X = X^{+} - X^{-}$, $X^{+}, X^{-} \geqslant 0$

[V]

CHAPTER 9

Problems in General Form

Up until now, we have always considered our problems to be given in standard form. However, for real-world problems it is often convenient to formulate problems in the following form:

$$
\begin{array}{lrcccl}
\text{maximize} & & & c^T x & & \\
\text{subject to} & a & \leq & Ax & \leq & b \\
 & l & \leq & x & \leq & u.
\end{array} \tag{9.1}
$$

Two-sided constraints such as those given here are called constraints with *ranges*. The vector l is called the vector of *lower bounds*, and u is the vector of *upper bounds*. We allow some of the data to take infinite values; that is, for each $i = 1, 2, \ldots, m$,

$$-\infty \leq a_i \leq b_i \leq \infty,$$

and, for each $j = 1, 2, \ldots, n$,

$$-\infty \leq l_j \leq u_j \leq \infty.$$

In this chapter, we shall show how to modify the simplex method to handle problems presented in this form.

1. The Primal Simplex Method

It is easiest to illustrate the ideas with an example:

$$
\begin{array}{lrcrcl}
\text{maximize} & & & 3x_1 - x_2 & & \\
\text{subject to} & 1 & \leq & -x_1 + x_2 & \leq & 5 \\
 & 2 & \leq & -3x_1 + 2x_2 & \leq & 10 \\
 & & & 2x_1 - x_2 & \leq & 0 \\
 & -2 & \leq & x_1 & & \\
 & 0 & \leq & x_2 & \leq & 6.
\end{array}
$$

R. J. Vanderbei, *Linear Programming*, International Series in Operations Research & Management Science 285,
https://doi.org/10.1007/978-3-030-39415-8_9

With this formulation, zero no longer plays the special role it once did. Instead, that role is replaced by the notion of a variable or a constraint being at its upper or lower bound. Therefore, instead of defining slack variables for each constraint, we use w_i simply to denote the value of the ith constraint:

$$\begin{aligned} w_1 &= \ -x_1 + \ x_2 \\ w_2 &= -3x_1 + 2x_2 \\ w_3 &= \ \ 2x_1 - \ x_2. \end{aligned}$$

← slacks

The constraints can then be interpreted as upper and lower bounds on these variables. Now when we record our problem in a dictionary, we will have to keep explicit track of the upper and lower bound on the original x_j variables and the new w_i variables. Also, the value of a nonbasic variable is no longer implicit; it could be at either its upper or its lower bound. Hence, we shall indicate which is the case by putting a box around the relevant bound. Finally, we need to keep track of the values of the basic variables. Hence, we shall write our dictionary as follows:

tight (initial)

$$\begin{array}{rr|l}
 & l & \qquad\quad \boxed{-2} \qquad \boxed{0} \\
 & u & \qquad\quad\ \ \infty \qquad\ \ 6 \\
\hline
 & & \zeta = \ \ 3x_1 - \ x_2 = -6 \\
\hline
1 & 5 & w_1 = \ -x_1 + \ x_2 = \ \ 2 \\
2 & 10 & w_2 = -3x_1 + 2x_2 = \ \ 6 \\
-\infty & 0 & w_3 = \ \ 2x_1 - \ x_2 = -4\,.
\end{array}$$

bounds for vars. (non-basic)

bounds for slacks (basic var.)

Since all the w_i's are between their upper and lower bounds, this dictionary is feasible. But it is not optimal, since x_1 could be increased from its present value at the lower bound, thereby increasing the objective function's value. Hence, x_1 shall be the entering variable for the first iteration. Looking at w_1, we see that x_1 can be raised by only 1 unit before w_1 hits its lower bound. Similarly, x_1 can be raised by $4/3$ units, at which point w_2 hits its lower bound. Finally, if x_1 were raised by 2 units, then w_3 would hit its upper bound. The tightest of these constraints is the one on w_1, and so w_1 becomes the leaving variable—which, in the next iteration, will then be at its lower bound. Performing the usual row operations, we get

$x_1 \leftrightarrow w_1$

update

$$\begin{array}{rr|l}
 & l & \qquad\qquad \boxed{1} \qquad \boxed{0} \\
 & u & \qquad\qquad\ 5 \qquad\ \ 6 \\
\hline
 & & \zeta = -3w_1 + 2x_2 = -3 \\
\hline
-2 & \infty & x_1 = \ -w_1 + \ x_2 = -1 \\
2 & 10 & w_2 = \ \ 3w_1 - \ x_2 = \ \ 3 \\
-\infty & 0 & w_3 = -2w_1 + \ x_2 = -2\,.
\end{array}$$

Note, of course, that the objective function value has increased (from -6 to -3). For the second iteration, raising x_2 from its lower bound will produce an increase in ζ. Hence, x_2 is the entering variable. Looking at the basic variables (x_1, w_2, and w_3), we see that w_2 will be the first variable to hit a bound, namely, its lower bound. Hence, w_2 is the leaving variable, which will become nonbasic at its lower bound:

$$
\begin{array}{rr|lrcrl}
l & & & \boxed{1} & & \boxed{2} & \\
 & u & & 5 & & 10 & \\
\hline
 & & \zeta = & 3w_1 & - & 2w_2 & = -1 \\
\hline
-2 & \infty & x_1 = & 2w_1 & - & w_2 & = 0 \\
0 & 6 & x_2 = & 3w_1 & - & w_2 & = 1 \\
-\infty & 0 & w_3 = & w_1 & - & w_2 & = -1\,.
\end{array}
$$

For the third iteration, w_1 is the entering variable, and w_3 is the leaving variable, since it hits its upper bound before any other basic variables hit a bound. The result is

$$
\begin{array}{rr|lrcrl}
l & & & -\infty & & \boxed{2} & \\
 & u & & \boxed{0} & & 10 & \\
\hline
 & & \zeta = & 3w_3 & + & w_2 & = 2 \\
\hline
-2 & \infty & x_1 = & 2w_3 & + & w_2 & = 2 \\
0 & 6 & x_2 = & 3w_3 & + & 2w_2 & = 4 \\
1 & 5 & w_1 = & w_3 & + & w_2 & = 2\,.
\end{array}
$$

Now for the next iteration, note that the coefficients on both w_3 and w_2 are positive. But w_3 is at its upper bound, and so if it were to change, it would have to decrease. However, this would mean a decrease in the objective function. Hence, only w_2 can enter the basis, in which case x_2 is the leaving variable getting set to its upper bound:

$$
\begin{array}{rr|lrcrl}
l & & & -\infty & & 0 & \\
 & u & & \boxed{0} & & \boxed{6} & \\
\hline
 & & \zeta = & 1.5w_3 & + & 0.5x_2 & = 3 \\
\hline
-2 & \infty & x_1 = & 0.5w_3 & + & 0.5x_2 & = 3 \\
2 & 10 & w_2 = & -1.5w_3 & + & 0.5x_2 & = 3 \\
1 & 5 & w_1 = & -0.5w_3 & + & 0.5x_2 & = 3\,.
\end{array}
$$

← upper bound

For this dictionary, both w_3 and x_2 are at their upper bounds and have positive coefficients in the formula for ζ. Hence, neither can be moved off from its bound to increase the objective function. Therefore, the current solution is optimal.

2. The Dual Simplex Method

The problem considered in the previous section had an initial dictionary that was feasible. But as always, we must address the case where the initial dictionary is not feasible. That is, we must define a Phase I algorithm. Following the ideas presented in Chapter 5, we base our Phase I algorithm on a dual simplex method. To this end, we need to introduce the dual of (9.1). So first we rewrite (9.1) as

Ex 1 [B&B, p. 61]

$$\max \quad 4x_1 + 2x_2 - x_3 + 2x_4$$

s.t. (*)
$$\begin{cases} x_1 + x_2 + 2x_3 + 4x_4 = 8 \\ x_1 + x_2 + x_3 + x_4 = 4 \end{cases}$$
$$x_1, x_2, x_3, x_4 \geq 0$$

M1 (*) $\Longleftrightarrow$

$$\begin{aligned} x_1 + x_2 + 2x_3 + 4x_4 &\leq 8 \\ -x_1 - x_2 - 2x_3 - 4x_4 &\leq -8 \\ x_1 + x_2 + x_3 + x_4 &\leq 4 \\ -x_1 - x_2 - x_3 - x_4 &\leq -4 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

need to use Phase I

Phase I: Auxiliary Problem [V, p. 18]

$$\max \quad \xi = -x_0 \quad (=0\,?)$$

s.t.
$$\begin{aligned} x_1 + x_2 + 2x_3 + 4x_4 - x_0 &\leq 8 \\ -x_1 - x_2 - 2x_3 - 4x_4 - x_0 &\leq -8 \\ x_1 + x_2 + x_3 + x_4 - x_0 &\leq 4 \\ -x_1 - x_2 - x_3 - x_4 - x_0 &\leq -4 \\ x_1, x_2, x_3, x_4, x_0 &\geq 0 \end{aligned}$$

MQ Use auxiliary variables

$$\begin{aligned} x_1 + x_2 + 2x_3 + 4x_4 + w_1 \quad &= 8 \\ x_1 + x_2 + x_3 + x_4 \quad + w_2 &= 4 \end{aligned}$$

is w_1, w_2 feasible?

Phase I: $\max \ \xi = -w_1 - w_2 \quad (= 0?)$

$$\text{s.t.} \quad \begin{aligned} x_1 + x_2 + 2x_3 + 4x_4 + w_1 \quad &= 8 \\ x_1 + x_2 + x_3 + x_4 \quad + w_2 &= 4 \end{aligned}$$

$$(x_1, x_2, x_3, x_4, w_1, w_2 \geq 0)$$

$$\Longleftrightarrow \quad \max \ \underline{\xi = -w_1 - w_2} \quad (= 0?)$$

$$\text{s.t.} \quad \begin{aligned} w_1 &= 8 - x_1 - x_2 - 2x_3 - 4x_4 \\ w_2 &= 4 - x_1 - x_2 - x_3 - x_4 \end{aligned}$$

$$(x_1, x_2, x_3, x_4, w_1, w_2 \geq 0)$$

feasible dictionary

$$\boxed{\begin{aligned} \xi = & -8 + x_1 + x_2 + 2x_3 + 4x_4 \\ & -4 + x_1 + x_2 + x_3 + x_4 \\ = & -12 + 2x_1 + 2x_2 + 3x_3 + 5x_4 \end{aligned}}$$

Ex 2
[BGB. p. 61]

$$\max \quad 2x_1 + 3x_2 + 2x_3$$

$$\text{s.t.} \quad 2x_1 + x_2 - 3x_3 \geq -2$$

$$x_2 - x_3 \leq 3$$

$$2x_1 + x_2 + x_3 = 4$$

$$x_1, x_3 \geq 0, \quad x_2 = \text{free}$$

$-2x_1 - x_2 + 3x_3 \leq 2$

$$-2x_1 - x_2 + 3x_3 + w_1 = 2$$

$$x_2 - x_3 + w_2 = 3$$

$$2x_1 + x_2 + x_3 + w_3 = 4$$

$w_3 = 0$?

$x_2 = x_2^+ - x_2^-$

Phase I (Auxiliary problem)

$$\max \quad \xi = -w_3 \quad (= 0?)$$

substitute.

$$\text{s.t.} \quad w_1 = 2 + 2x_1 + x_2^+ - x_2^- + 3x_3$$

$$w_2 = 3 - x_2^+ + x_2^- + x_3$$

$$w_3 = 4 - 2x_1 - x_2^+ + x_2^- - x_3$$

$$x_1, x_2^+, x_2^-, x_3, w_1, w_2, w_3 \geq 0$$

feasible dictionary

Ex 3

[BGB, p.95]

$$\max \; -2x_1 - x_2$$

s.t. $3x_1 + 4x_2 = 12$

$$\begin{cases} -x_1 + 2x_2 \leq 2 \\ x_1 + 4x_2 \geq 6 \\ x_1, x_2 \geq 0 \end{cases}$$

$$3x_1 + 4x_2 + w_1 = 12$$
$$-x_1 + 2x_2 + w_2 = 2$$

$w_1 = 0$?

$$-x_1 - 4x_2 \leq -6$$
$$-x_1 - 4x_2 + w_3 - w_4 = -6$$

slack

auxiliary var.

(Phase I) $\max \; -w_1 - w_4$ $(= 0?)$

substitute

s.t.
$$\begin{cases} w_1 = 12 - 3x_1 - 4x_2 \\ w_2 = 2 + x_1 - 2x_2 \\ w_4 = 6 - x_1 - 4x_2 + w_3 \end{cases}$$

$$x_1, x_2, w_1, w_2, w_3, w_4 \geq 0$$

feasible dictionary