Primal and Dual Problems

[V] p.59 $\max \zeta(x) = 4x_1 + x_2 + 3x_3$

s.t.

$$\begin{aligned} x_1 + 4x_2 \qquad\quad &\le 1 \\ 3x_1 - x_2 + x_3 &\le 3 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

Primal and Dual Problems

[V] p.59 $\max \ \zeta(x) = 4x_1 + x_2 + 3x_3$

$$\text{s.t.} \quad \begin{aligned} x_1 + 4x_2 \phantom{{}+x_3} &\leq 1 \\ 3x_1 - x_2 + x_3 &\leq 3 \\ x_1,\ x_2,\ x_3 &\geq 0 \end{aligned}$$

lin comb. of constraints

$$\left\{ \begin{aligned} y_1 (x_1 + 4x_2 \phantom{{}+x_3}) &\leq y_1 1 \\ y_2 (3x_1 - x_2 + x_3) &\leq y_2 3 \end{aligned} \right.$$

$$\underbrace{(y_1 + 3y_2)}_{\geq\, 4} x_3 + \underbrace{(4y_1 - y_2)}_{\geq\, 1} x_2 + \underbrace{(y_2)}_{\geq\, 3} x_3 \leq y_1 1 + y_2 3$$

Primal and Dual Problems

$$\underbrace{(y_1+3y_2)}_{\geq 4} x_3 + \underbrace{(4y_1-y_2)}_{\geq 1} x_2 + \underbrace{(y_2)}_{\geq 3} x_3 \leq y_1 1 + y_2 3$$

$$\begin{aligned} \zeta(x) &= 4x_1 + 1x_2 + 3x_3 \\ &\leq (y_1+3y_2)x_3 + (4y_1-y_2)x_2 + (y_2)x_3 \\ &\leq 1y_1 + 3y_2 \\ &= \xi(y) \end{aligned}$$

$$\boxed{\zeta(x) \leq \xi(y)}$$

for any feasible x, y

Primal and Dual Problems

$$(P) \quad \max \zeta(x) = 4x_1 + x_2 + 3x_3$$

$$\text{s.t.} \quad \begin{aligned} x_1 + 4x_2 \qquad & \leq 1 \\ 3x_1 - x_2 + x_3 & \leq 3 \\ x_1, x_2, x_3 & \geq 0 \end{aligned}$$

$$(D) \quad \min \xi(y) = 1y_1 + 3y_2$$

$$\text{s.t.} \quad \begin{aligned} y_1 + 3y_2 & \geq 4 \\ 4y_1 - y_2 & \geq 1 \\ y_2 & \geq 3 \\ y_1, y_2, y_3 & \geq 0 \end{aligned}$$

Primal and Dual Problems

$$(P) \quad \max \ \zeta(x) = 4x_1 + x_2 + 3x_3 \qquad \begin{pmatrix} 4 & 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\text{s.t.} \quad \begin{aligned} x_1 + 4x_2 \quad &\le 1 \\ 3x_1 - x_2 + x_3 &\le 3 \\ x_1, x_2, x_3 &\ge 0 \end{aligned} \qquad \begin{bmatrix} 1 & 4 & 0 \\ 3 & -2 & 1 \end{bmatrix}$$

$$(D) \quad \min \ \xi(y) = 1y_1 + 3y_2 \qquad \begin{pmatrix} 1 & 3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\text{s.t.} \quad \begin{aligned} y_1 + 3y_2 &\ge 4 \\ 4y_1 - y_2 &\ge 1 \\ y_2 &\ge 3 \\ y_1, y_2, y_3 &\ge 0 \end{aligned} \qquad \begin{bmatrix} 1 & 3 \\ 4 & -1 \\ 0 & 1 \end{bmatrix}$$

Primal and Dual Problems

(P)

$$\boxed{\begin{aligned} &\max \quad \zeta(x) = c^T x \\ &\text{subject to} \quad Ax \le b \\ &\qquad\qquad\qquad x \ge 0 \end{aligned}} \sim \sum_j c_j x_j$$

$$\sum_i y_i \left(\sum_j a_{ij} x_j \le b_i \right), \quad y_i \ge 0$$

$$\sum_j \underbrace{\left(\sum_i a_{ij} y_i \right)}_{c_j \le} x_j \le \sum_i b_i y_i$$

Thm 5.1 Weak Duality

$$\boxed{\zeta(x) = \sum_j c_j x_j \le \sum_i b_i y_i = \xi(y)}$$

Primal and Dual Problems

(D)

$$\boxed{\begin{aligned} &\min \quad \xi(x) = b^T y \\ &\text{subject to} \quad A^T y \geq c \\ &\qquad\qquad\qquad\; y \geq 0 \end{aligned}}$$

$\sim \quad \sum b_i y_i$

$$\sum_i y_i \left(\sum_j a_{ij} x_j \leq b_i \right), \qquad y_i \geq 0$$

$$\sum_j \underbrace{\left(\sum_i a_{ij} y_i \right)}_{c_j \leq} x_j \leq \sum_i b_i y_i$$

Thm 5.1 Weak Duality

$$\boxed{\zeta(x) = \sum_j c_j x_j \leq \sum_i b_i y_i = \xi(y)}$$

Primal and Dual Problems

(P)

$$\begin{aligned} \max \quad & \zeta(x) = c^T x \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

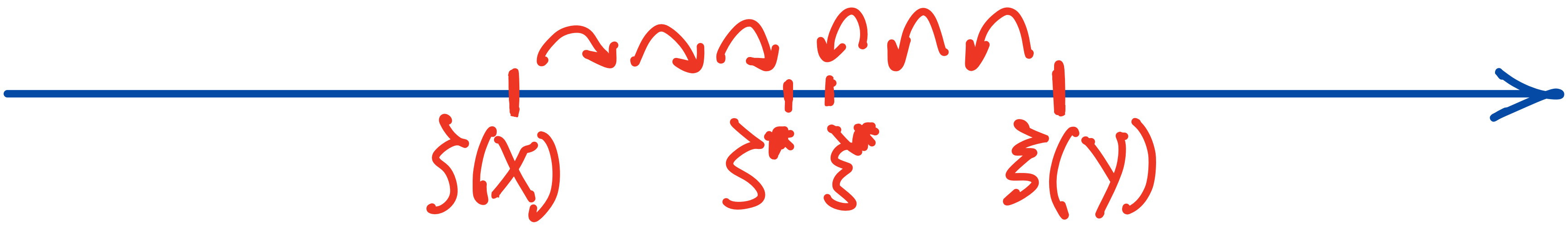
Thm 5.1 Weak Duality

$$\zeta(x) \leq \xi(y)$$

$$(\zeta^* \leq \xi^*)$$

(D)

$$\begin{aligned} \min \quad & \xi(y) = b^T y \\ \text{subject to} \quad & A^T y \geq c \\ & y \geq 0 \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \max \quad & -\xi(y) = -b^T y \\ \text{subject to} \quad & (-A^T) y \leq -c \\ & y \geq 0 \end{aligned}$$



Matrix Representations

(P) $\max \ \zeta(x) = \zeta_0 + c^T x$

$$W = b - Ax \geq 0$$

$$x \geq 0$$

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix}$$

ζ_0	c_1	c_2	$\cdots$	c_n
b_1	$-a_{11}$	$-a_{12}$	$\cdots$	$-a_{1n}$
b_2	$-a_{21}$	$-a_{22}$	$\cdots$	$-a_{2n}$
$\vdots$		$\vdots$		$\vdots$
b_m	$-a_{m1}$	$-a_{m2}$	$\cdots$	$-a_{mn}$

(D) $\max \ -\xi(y) = -\xi_0 - b^T y$

$$Z = -c + A^T y \geq 0$$

$$y \geq 0$$

$$\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

$-\xi_0$	$-b_1$	$-b_2$	$\cdots$	$-b_m$
$-c_1$	a_{11}	a_{21}	$\cdots$	a_{m1}
$-c_2$	a_{12}	a_{22}	$\cdots$	a_{m2}
$\vdots$				
$-c_n$	a_{1n}	a_{2n}	$\cdots$	a_{mn}

Matrix Representations

(P) $\max \ \zeta(x) = \zeta_0 + c^T x$

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix} \qquad w = b - Ax \geq 0$$

$$x \geq 0$$

$$\left[\begin{array}{c|c} \zeta_0 & c^T \\ \hline b & -A \end{array}\right]$$

negative transpose

(D) $\max \ -\xi(y) = -\zeta_0 - b^T y$

$$\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \qquad z = -c + A^T y \geq 0$$

$$y \geq 0$$

$$\left[\begin{array}{c|c} -\zeta_0 & -b^T \\ \hline -c & A^T \end{array}\right]$$

Complementary Variables

$$(x_1, x_2, \ldots, x_n, w_1, w_2, \ldots w_m)$$

$y_i \sim w_i$ $\qquad$ $x_j \sim z_j$

$$(y_1, y_2, \ldots y_m, z_1, z_2, \ldots z_n)$$

(P) $\qquad$ x_j enter (leaves), w_i enter (leaves)

$$\Updownarrow$$

(D) $\qquad$ z_j leaves (enters), y_i leaves (enters)

Neg. Transpose Property is Preserved in each Simplex Step

[V] p.64

X_3 enters, w_2 leaves (greatest increase rule)

$$
\text{(P)}_1 \quad
\begin{array}{rcrcrcrcr}
\zeta & = & & + & 4x_1 & + & 1x_2 & + & 3x_3 \\
\hline
w_1 & = & 1 & - & x_1 & - & 4x_2 & & \\
w_2 & = & 3 & - & 3x_1 & + & x_2 & - & x_3
\end{array}
$$

neg. transpose

$$
\text{(D)}_1 \quad
\begin{array}{rcrcrcr}
-\xi & = & & - & 1y_1 & - & 3y_2 \\
\hline
z_1 & = & -4 & + & y_1 & + & 3y_2 \\
z_2 & = & -1 & + & 4y_1 & - & y_2 \\
z_3 & = & -3 & & & + & y_2
\end{array}
$$

Z_3 leaves, y_2 enters

Neg. Transpose Property is Preserved in each Simplex Step

[V] p.64

$$(P)_2 \qquad \begin{array}{rcrrrrrrr}
\zeta & = & 9 & - & 5\,x_1 & + & 4\,x_2 & - & 3\,w_2 \\
\hline
w_1 & = & 1 & - & x_1 & - & 4\,x_2 & + & \\
x_3 & = & 3 & - & 3\,x_1 & + & x_2 & - & w_2
\end{array}$$

x_2 enters, w_1 leaves

neg. transpose

$$(D)_2 \qquad \begin{array}{rcrrrrr}
-\xi & = & -9 & - & 1\,y_1 & - & 3\,z_3 \\
\hline
z_1 & = & 5 & + & y_1 & + & 3\,z_3 \\
z_2 & = & -4 & + & 4\,y_1 & - & z_3 \\
y_2 & = & 3 & + & & + & z_3
\end{array}$$

z_2 leaves, y_1 enters

Neg. Transpose Property is Preserved in each Simplex Step

[V] p.64

$$
(P)_3 \qquad
\begin{array}{rcrcrcrcr}
\zeta & = & 10 & - & 6 & x_1 & - & 1 & w_1 & - & 3 & w_2 \\
\hline
x_2 & = & 0.25 & - & 0.25 & x_1 & - & 0.25 & w_1 & & & \\
x_3 & = & 3.25 & - & 3.25 & x_1 & - & 0.25 & w_1 & - & & w_2
\end{array}
$$

Optimal for (P)

$\left(x_1 = 0,\ X_2 = 0.25,\ X_3 = 3.25,\ W_1 = W_2 = 0 \right)$

neg. transpose

$$
(D)_3 \qquad
\begin{array}{rcrcrcrcr}
-\xi & = & -10 & - & 0.25 & z_2 & - & 3.25 & z_3 \\
\hline
z_1 & = & 6 & + & 0.25 & z_2 & + & 3.25 & z_3 \\
y_1 & = & 1 & + & 0.25 & z_2 & + & 0.25 & z_3 \\
y_2 & = & 3 & & & & + & 1 & z_3
\end{array}
$$

Optimal for (D)

$\left(z_2,\ z_3 = 0,\ y_1 = 1,\ y_2 = 3,\ z_1 = 6 \right)$

Optimal values for y_1, y_2

Neg. Transpose Property is Preserved in each Simplex Step

[V] p.64

$$\zeta(x^*) = 4x_1^* + x_2^* + 3x_3^*$$

$$= 4(0) + (0.25) + 3(3.25) = 10$$

$$\xi(y^*) = 1y_1^* + 3y_2^*$$

$$= 1(1) + 3(3) = 10$$

Strong Duality

Neg. Transpose Property is Preserved in each Simplex step

at Optimal:

$c_j^* \le 0$

(P)

$$
\begin{array}{c|cccc}
\zeta^* & c_1^* & c_2^* & \cdots & c_n^* \\
\hline
b_1^* & -a_{11}^* & -a_{12}^* & \cdots & -a_{1n}^* \\
b_2^* & & & & \\
 & & -a_{ij}^* & & \\
\vdots & & & & \\
b_m^* & -a_{m1}^* & -a_{m2}^* & \cdots & -a_{mn}^*
\end{array}
$$

$b_i^* \ge 0$

Neg. Transpose Property is Preserved in each Simplex Step

at Optimal:

$-b_i^* \le 0$

(D)

$$\begin{array}{|c|cccc|}
\hline
-\zeta^* & -b_1^* & -b_2^* & \cdots & -b_m^* \\
\hline
-c_1^* & a_{11}^* & a_{21}^* & \cdots & a_{m1}^* \\
-c_2^* & & & & \\
\vdots & & a_{ji}^* & & \\
-c_n^* & a_{1n}^* & a_{2n}^* & \cdots & a_{mn}^* \\
\hline
\end{array}$$

Strong Duality

$-c_j^* \ge 0$

Strong Duality Theorem

If (P) has an optimal solution (x_j^*) then (D) also has an optimal solution (y_i^*) and

$$(\zeta^* =) \sum_j c_j x_j^* = \sum_i b_i y_i^* (= \xi^*)$$

i.e. duality gap = 0

(The proof gives a formula for y_i^*)

Strong Duality Theorem

Pf

$\left(\begin{array}{l} N : \text{non-basic vars} \\ B : \text{basic vars} \end{array} \right)$

① $$\zeta = \sum_{j} c_j \underbrace{x_j}_{N} + \sum_{i} 0 \underbrace{w_i}_{B}$$ (original formulation)

Strong Duality Theorem

Pf

2 at (P)'s optimal

non-basic

basic

$$\zeta = \zeta^* + \sum_j \bar{c}_j \bar{x}_j + \sum_i 0\, \bar{\omega}_i$$ ← Opt. dict.

$(x_1, x_2, \ldots, x_n, w_1, \ldots w_m)$

$$= \zeta^* + \sum_j c_j^* x_j + \sum_i d_i^* \omega_i$$

$(c_j^* \le 0)$ $(d_i^* \le 0)$

$$= \zeta^* + \sum_j c_j^* x_j + \sum_i d_i^* \left(b_i - \sum_j a_{ij} x_j\right)$$

Strong Duality Theorem

Pf

2 at (P)'s optimal

$$\zeta = \zeta^* + \sum_j c_j^* x_j + \sum_i d_i^* \left(b_i - \sum_j a_{ij} x_j \right)$$

$$= \underline{\left(\zeta^* + \sum_i d_i^* b_i \right)}$$

$$+ \sum_j \underline{\left(c_j^* - \sum_i a_{ij} d_i^* \right)} x_j$$

Strong Duality Theorem

Pf Compare 1 and 2

Const term: $0 = \zeta^* + \sum_i d_i^* b_i$

x_j : $c_j = c_j^* - \sum_i a_{ij} d_i^*$

Let $y_i^* = -d_i^* \ (\geq 0)$. Then

$$\boxed{\zeta^* = \sum_i b_i y_i^*}$$

Strong Duality Theorem

$\{ y_i^* = -d_i^* \}$ feasible?

Pf Compare ① and ②

Const term: $$0 = \zeta^* + \sum_i d_i^* b_i$$

x_j : $$c_j = c_j^* - \sum_i a_{ij} d_i^*$$

$$c_j = c_j^* + \sum_i a_{ij} y_i^* \leq \sum_i a_{ij} y_i^*$$

$(c_j^* \leq 0)$

Strong Duality Theorem

<u>Pf</u> Hence there exists an y^* s.t.

$$\boxed{\zeta^* = \zeta(x^*) = \xi(y^*)}$$

This must be opt. for (D)

(By Weak Duality, $\zeta(x^*) \leq \xi(y^*)$)

Four and Only Four Possibilities

(1) (P) has an optimal solution

$\Longrightarrow$ (D) also has an optimal solution

$\zeta^* = \xi^*$, duality gap $= 0$

(2) (P) is feasible but unbounded

$\Longrightarrow$ (D) must be infeasible

$$\underbrace{\zeta(x)}_{\to +\infty} \leq \xi(y)$$

$\zeta^* = \xi^* = +\infty$

(min empty set $= +\infty$)

Four and Only Four Possibilities

(3) (D) is feasible but unbounded

$\Longrightarrow$ (P) must be infeasible

$\zeta(x) \leq \xi(y) \to -\infty$

$\zeta^* = \xi^* = -\infty$

(max empty set $= -\infty$)

(4) Both (P) and (D) are infeasible

$\Longrightarrow \quad \zeta^* = -\infty, \quad \xi^* = +\infty$

(duality gap $= +\infty$)

Four and Only Four Possibilities

[V, p.69]

(P)

$$\begin{aligned} \text{maximize} \quad & 2x_1 - x_2 \\ \text{subject to} \quad & x_1 - x_2 \le 1 \\ & -x_1 + x_2 \le -2 \\ & x_1,\ x_2 \ge 0. \end{aligned}$$

inconsistent

(D)

$$\begin{aligned} \min \quad & y_1 - 2y_2 \\ \text{subject to} \quad & y_1 - y_2 \geq 2 \\ & -y_1 + y_2 \geq -1 \\ & y_1,\ y_2 \geq 0 \end{aligned}$$

inconsistent