Dual of a general LP

[V, p.81]

5.1 What is the dual of the following linear programming problem:

$$\begin{array}{lrcrcrcrcl} \text{maximize} & x_1 & - & 2x_2 & & & & & & \\ \text{subject to} & x_1 & + & 2x_2 & - & x_3 & + & x_4 & \geq & 0 \\ & 4x_1 & + & 3x_2 & + & 4x_3 & - & 2x_4 & \leq & 3 \\ & -x_1 & - & x_2 & + & 2x_3 & + & x_4 & = & 1 \\ & & & x_2, & x_3 & \geq & 0 & . & & \end{array}$$

(Note: x_1 is not constrained to be +ve.)

M1

$$y_1(-x_1 - 2x_2 + x_3 - x_4 \le 0) \qquad y_1 \ge 0$$

$$y_2(4x_1 + 3x_2 + 4x_3 - 2x_4 \le 3) \qquad y_2 \ge 0$$

$$y_3(-x_1 - x_2 + 2x_3 + x_4 = 1)$$

$$\underbrace{(-y_1 + 4y_2 - y_3)}_{=1} x_1 + \underbrace{(-2y_1 + 3y_2 - y_3)}_{-2 \le} x_2 + \underbrace{(y_1 + 4y_2 + 2y_3)}_{0 \le} x_3 + \underbrace{(-y_1 - 2y_2 + y_3)}_{0 =} x_4 \le 3y_2 + y_3$$

① $$x_1 \le (-y_1 + 4y_2 - y_3)x_1$$

$$x_1 > 0 \Longrightarrow 1 \le -y_1 + 4y_2 - y_3$$

$$x_1 < 0 \Longrightarrow 1 \ge -y_1 + 4y_2 - y_3$$

$$\Longrightarrow \underline{-y_1 + 4y_2 - y_3 = 1}$$

② $$-2x_2 \le (-2y_1 + 3y_2 - y_3)x_2$$

$$x_2 \ge 0 \Longrightarrow \underline{-2 \le -2y + 3y_2 - y_3}$$

$$0 \le (y_1 + 4y_2 + 2y_3)x_3$$

③ $$x_3 \ge 0 \Longrightarrow \underline{0 \le y_1 + 4y_2 + 2y_3}$$

④ $$x_4 \begin{matrix} \ge 0 \\ \le 0 \end{matrix} \Longrightarrow \underline{0 = -y_1 - 2y_2 + y_3}$$

Dual problem:

$$\min \quad 3y_2 + y_3$$

$$\text{s.t.} \quad -y_1 + 4y_2 - y_3 = 1$$

$$-2y + 3y_2 - y_3 \geq -2 \qquad (x_2 \geq 0)$$

$$y_1 + 4y_2 + 2y_3 \geq 0 \qquad (x_3 \geq 0)$$

$$-y_1 - 2y_2 + y_3 = 0$$

$$y_1, y_2 \geq 0$$

5.1 What is the dual of the following linear programming problem:

M2

$$
\begin{array}{lrcl}
\text{maximize} & x_1 - 2x_2 & & \\
\text{subject to} & x_1 + 2x_2 - x_3 + x_4 & \geq & 0 \\
 & 4x_1 + 3x_2 + 4x_3 - 2x_4 & \leq & 3 \\
 & -x_1 - x_2 + 2x_3 + x_4 & = & 1 \\
 & x_2,\ x_3 & \geq & 0\ .
\end{array}
$$

≤ 1

≥ 1

$x_1 = x_1^+ - x_1^+$, $x_1^+, x_1^- \geq 0$

$x_4 = x_4^+ - x_4^-$, $x_4^+, x_4^- \geq 0$

$$\max \quad x_1^+ - x_1^- - 2x_2$$

$$\text{s.t.} \quad
\begin{aligned}
-x_1^+ + x_1^- - 2x_2 + x_3 - x_4^+ + x_4^- &\le 0 \\
4x_1^+ - 4x_1^- + 3x_2 + 4x_3 - 2x_4^+ + 2x_4^- &\le 3 \\
-x_1^+ + x_1^- - x_2 + 2x_3 + x_4^+ - x_4^- &\le 1 \\
x_1^+ - x_1^- + x_2 - 2x_3 - x_4^+ + x_4^- &\le -1
\end{aligned}$$

y_1

y_2

y_3^+

y_3^-

$$y_1, y_2, y_3^+, y_3^- \ge 0$$

$$(-y_1 + 4y_2 - y_3^+ + y_3^-)\,x_1^+$$
$$+\,(+y_1 - 4y_2 + y_3^+ - y_3^-)\,x_1^-$$
$$+\,(-2y_1 + 3y_2 - y_3^+ + y_3^-)\,x_2$$
$$+\,(y_1 + 4y_2 + 2y_3^+ - 2y_3^-)\,x_3$$
$$+\,(-y_1 - 2y_2 + y_3^+ - y_3^-)\,x_4^+$$
$$+\,(y_1 + 2y_2 - y_3^+ + y_3^-)\,x_4^-$$
$$\leq 3y_2 + y_3^+ - y_3^-$$
$$= 3y_2 + y_3$$

$$\left.\begin{array}{l} -y_1 + 4y_2 - y_3^+ + y_3^- \geqslant 1 \\ +y_1 - 4y_2 + y_3^+ - y_3^- \geqslant -1 \end{array}\right\} = 1$$

$$-2y_1 + 3y_2 - y_3^+ + y_3^- \geqslant -2$$

$$y_1 + 4y_2 + 2y_3^+ - 2y_3^- \geqslant 0$$

$$\left.\begin{array}{l} -y_1 - 2y_2 + y_3^+ - y_3^- \geqslant 0 \\ y_1 + 2y_2 - y_3^+ + y_3^- \geqslant 0 \end{array}\right\} = 0$$

$$\begin{cases} -y_1 + 4y_2 - y_3^+ + y_3^- = 1 \\ -2y_1 + 3y_2 - y_3^+ + y_3^- \geqslant -2 \\ y_1 + 4y_2 + 2y_3^+ - 2y_3^- \geqslant 0 \\ -y_1 - 2y_2 + y_3^+ - y_3^- = 0 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} -y_1 + 4y_2 - y_3 = 1 \\ -2y_1 + 3y_2 - y_3 \geqslant -2 \\ y_1 + 4y_2 + 2y_3 \geqslant 0 \\ -y_1 - 2y_2 + y_3 = 0 \end{cases}$$

[V]

CHAPTER 9

Problems in General Form

Up until now, we have always considered our problems to be given in standard form. However, for real-world problems it is often convenient to formulate problems in the following form:

Dual = ?

$$
\begin{array}{lrcccc}
\text{maximize} & & & c^T x & & \\
\text{subject to} & a & \leq & Ax & \leq & b \\
& l & \leq & x & \leq & u.
\end{array}
\tag{9.1}
$$

Two-sided constraints such as those given here are called constraints with *ranges*. The vector l is called the vector of *lower bounds*, and u is the vector of *upper bounds*. We allow some of the data to take infinite values; that is, for each $i = 1, 2, \ldots, m$,

$$-\infty \leq a_i \leq b_i \leq \infty,$$

and, for each $j = 1, 2, \ldots, n$,

$$-\infty \leq l_j \leq u_j \leq \infty.$$

"Standard Form"

$$
\begin{aligned}
\max \quad & c^T x \\
\text{s.t.} \quad & Ax \le b \\
& -Ax \le -a \\
& x \le u \\
& -x \le -l
\end{aligned}
$$

$$
\begin{aligned}
& x = x^+ - x^- \\
& x^+ \ge 0 \\
& x^- \ge 0
\end{aligned}
$$

$$
\begin{aligned}
\max \quad & c^T (x^+ - x^-) \\
\text{s.t.} \quad & A(x^+ - x^-) \le b \\
& -A(x^+ - x^-) \le -a \\
& x^+ - x^- \le u \\
& -x^+ + x^- \le -l
\end{aligned}
$$

$$p^T A (x^+ - x^-) \le p^T b$$
$$-q^T A (x^+ - x^-) \le -q^T a$$
$$r^T (x^+ - x^-) \le r^T u$$
$$s^T (-x^+ + x^-) \le -s^T l$$

$$(p^T A - q^T A + r^T - s^T) x^+ + (-p^T A + q^T A - r^T + s^T) x^- \le p^T b - q^T a + r^T u - s^T l$$

$c^T \le$

$-c^T \le$

Dual:
$$\begin{aligned} \min\quad & p^T b - q^T a + r^T u - s^T l \\ \text{s.t.}\quad & p^T A - q^T A + r^T - s^T \geq c^T \\ & -p^T A + q^T A - r^T + s^T \geq -c^T \\ & p, q, r, s \geq 0 \end{aligned}$$

$$\Longleftrightarrow \quad \begin{aligned} \min\quad & p^T b - q^T a + r^T u - s^T l \\ \text{s.t.}\quad & p^T A - q^T A + r^T - s^T = c^T \end{aligned}$$

$$\left(\Longleftrightarrow \quad \begin{aligned} \min\quad & b^T p - a^T q + u^T r - l^T s \\ \text{s.t.}\quad & A^T p - A^T q + r - s = c \end{aligned} \right)$$

Duality - Interpretations

Dual

$$\boxed{\begin{aligned} \max \quad & \zeta(x) = c^T x \\ \text{s.t.} \quad & Ax \le b \\ & x \ge 0 \end{aligned}}$$

$$\boxed{\begin{aligned} \min \quad & \xi(y) = b^T y \\ \text{s.t.} \quad & A^T y \ge c \\ & y \ge 0 \end{aligned}}$$

Diet Problem

m Nutrients (eg. Vitamins) N_i : $i = 1, 2, \cdots, m$

n Foods F_j : $j = 1, 2, \cdots, n$

a_{ij} = amount of N_i in 1 unit of F_j

b_i = minimum (monthly) intake of N_i

c_j = price (cost) of F_j

$$\boxed{x_j = \text{amount of consumption of } F_j}$$

Diet Problem

		c_1	c_2	c_3	c_j	$- - - -$	c_n
	$N_i \diagdown F_j$	1	2	3	j	$- - - \cdot \cdot$	n
b_1	1						
b_2	2						
b_i	i				a_{ij}		
$\vdots$	$\vdots$						
b_m	m						

Diet Problem

		c_1	c_2	c_3	c_j	$\cdots$	c_n
	$N_i \backslash F_j$	1	2	3	j	$\cdots$	n
b_1	1						
b_2	2						
b_i	i				a_{ij}		
$\vdots$	$\vdots$						
b_m	m						

Diet Problem

cost of C_j

consumption of F_j

minimize cost: $f(x) = \sum_j c_j x_j$

s.t.

$$\sum_j a_{ij} x_j \geq b_i$$

min. intake of N_i

$$x_j \geq 0$$

Diet Problem

		c_1	c_2	c_3	c_j	- - - -	c_n
	N_i \ F_j	1	2	3	j	- - - -	n
b_1	1						
b_2	2						
b_i	i				a_{ij}		
$\vdots$	$\vdots$						
b_m	m						

Diet Problem

Replacement of food by pills for each N_i's

N_i = pills $\quad i = 1, 2, \ldots, m$

$\boxed{y_i = \text{unit price of pill for } N_i}$

1 unit of F_j contains a_{ij} unit of N_i

Price/Value of (1 unit) $F_j = \sum_i y_i a_{ij}$

Maximize profit by selling pills

Diet Problem

Maximize profit by selling pills

$$\max(\text{profit}) \quad \xi(Y) = \sum_i b_i y_i$$

← price of N_i pill

min. sale of N_i pill

s.t. $\quad \sum_i y_i a_{ij} \leq c_j$ ← price of "pill equivalent" of F_j $\leq$ cost of F_j

$$y_i \geq 0$$

Transportation Problem (Monge-Kantorovich)

m Plants | **n Markets**

P_1 $\qquad\qquad$ M_1

P_2 $\quad \sigma_i$ (production) $\qquad\qquad$ M_2 $\quad \delta_j$ (demand)

$$P_i \xrightarrow[x_{ij} \text{ amount transported}]{c_{ij} \text{ cost (1 unit)}} M_j$$

$\vdots$

P_m $\qquad\qquad$ M_n

$$\min \quad \sum_i \sum_j c_{ij} x_{ij} \qquad (\text{Total cost})$$

$$\text{s.t.} \quad \sum_j x_{ij} \leq \sigma_i \qquad i = 1, \dots, m$$

$$\sum_i x_{ij} \geq \delta_j \qquad j = 1, \dots, n$$

$$x_{ij} \geq 0$$

$$\min \sum_i \sum_j c_{ij} x_{ij} \quad (\text{Total cost})$$

$$\text{s.t.} \quad \tilde{y}_i \left(\sum_j -x_{ij} \geq -\sigma_i \right) \quad i = 1, \cdots, m$$

$$+ \quad y_j \left(\sum_i x_{ij} \geq \delta_j \right) \quad j = 1, \cdots, n$$

$$\sum_i \sum_j \left(y_j x_{ij} - \tilde{y}_i x_{ij} \right) \geq \sum_j y_j \delta_j - \sum_i \tilde{y}_i \tilde{\sigma}_i$$

$$\sum_i \sum_j \left(y_j - \tilde{y}_i \right) x_{ij} \geq \sum_j y_j \delta_j - \sum_i \tilde{y}_i \tilde{\sigma}_i$$

Dual

$$
\begin{aligned}
\max \quad & \sum_j y_j \delta_j - \sum_i \tilde{y}_i \sigma_i \\
\text{s.t.} \quad & \left.\begin{aligned} y_j - \tilde{y}_i &\leq C_{ij} \\ y_j,\ \tilde{y}_i &\geq 0 \end{aligned}\right\} \quad \begin{aligned} i &= 1, \dots, m \\ j &= 1, \dots, n \end{aligned}
\end{aligned}
$$

New Company (with new resources)

P_i

M_j

buy

sell

unit price $= \tilde{y}_i$

unit price $= y_j$

$$\max \sum_j y_j \delta_j - \sum_i \tilde{y}_i \sigma_i$$

s.t.

$$\left.\begin{array}{l} y_j - \tilde{y}_i \leq c_{ij} \\ y_j, \tilde{y}_i \geq 0 \end{array}\right\} \begin{array}{l} i = 1, \dots, m \\ j = 1, \dots, n \end{array}$$

(to be competitive)