

Simplex in Matrix Notation


$$\begin{array}{ll}\max & f(x) = c^T x \\ \text{s.t.} & Ax \leq b \\ & x \geq 0\end{array}$$

$$x \in \mathbb{R}^n, \quad c \in \mathbb{R}^n \\ A^{m \times n}, \quad b \in \mathbb{R}^m$$

$$\begin{array}{ll}\max & f(x) = c^T x \\ \text{s.t.} & W = b - Ax \\ & x, W \geq 0\end{array}$$

$$W \in \mathbb{R}^m$$

Simplex in Matrix Notation

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$$w \in \mathbb{R}^m$$

Simplex
iterates

$$\begin{array}{ll} \max & f(x) = \bar{f} + \bar{c}^T \bar{x} \\ \text{s.t.} & \bar{w} = \bar{b} - \bar{A} \bar{x} \\ & \bar{x}, \bar{w} \geq 0 \end{array}$$

Simplex in Matrix Notation

$$\max \quad J(x) = c^T x$$

$$\text{s.t.} \quad Ax + W = b$$

$$x, W \geq 0$$

$m \times (n+m)$

$$\begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} x \\ W \end{bmatrix} = b$$

$\mathbb{R}^{n+m}$

Simplex
iterates

$$\max \quad J(x) = \bar{J} + \bar{c}^T \bar{x}$$

$$\text{s.t.} \quad \bar{A} \bar{x} + \bar{W} = \bar{b}$$

$$\bar{x}, \bar{W} \geq 0$$

$m \times (n+m)$

$$\begin{bmatrix} \bar{A} & I \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{W} \end{bmatrix} = \bar{b}$$

$\mathbb{R}^{n+m}$

Basic and Non-Basic Variables

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ w_1 \\ \vdots \\ w_m \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{pmatrix} = \begin{pmatrix} x_N \\ x_B \end{pmatrix}$$

N - non-basic
 B - basic
 Simplex iterations

$$\begin{aligned}
 \underline{C^T X} &= \sum_{j=1}^n C_j x_j = \underbrace{\sum_{j=1}^n C_j x_j}_N + \underbrace{\sum_{j=n+1}^{n+m} 0 \cdot x_j}_B \\
 &= \sum_{j \in B} C_j x_j + \sum_{j \in N} C_j x_j \\
 &= \underline{C_B^T X_B + C_N^T X_N}
 \end{aligned}$$

Basic and Non-Basic Variables

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ w_1 \\ \vdots \\ w_m \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{pmatrix} = \begin{pmatrix} x_N \\ x_B \end{pmatrix}$$

$\swarrow$ simplex iterations N - non-basic
 $\searrow$ B - basic

$$AX + W = b \iff \sum_{j=1}^n a_{ij} x_j + x_{n+i} = b_i$$

$$\boxed{[A \quad I]} \begin{bmatrix} X \\ W \end{bmatrix} = b \iff \sum_{j=1}^{n+m} a_{ij} x_j = b_i$$

$\swarrow$ new matrix $A^{m \times (n+m)}$

$$\sum_{j \in B} a_{ij} x_j + \sum_{j \in N} a_{ij} x_j = b_i$$

Basic and Non-Basic Variables

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ w_1 \\ \vdots \\ w_m \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{pmatrix} = \begin{pmatrix} x_N \\ x_B \end{pmatrix}$$

$\swarrow$ simplex iterations N - non-basic
 $\searrow$ B - basic

$$AX + W = b \iff \sum_{j=1}^n a_{ij} x_j + x_{n+i} = b_i$$

$$\boxed{[A \quad I]} \begin{bmatrix} X \\ W \end{bmatrix} = b \iff \underbrace{\sum_{j \in B} a_{ij} x_j} + \underbrace{\sum_{j \in N} a_{ij} x_j} = b_i$$

$\swarrow$
 new matrix $A^{m \times (n+m)}$

$$\underline{B} x_B + \underline{N} x_N = b$$

Basic and Non-Basic Variables

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ w_1 \\ \vdots \\ w_m \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{pmatrix} = \begin{pmatrix} x_N \\ x_B \end{pmatrix}$$

simplex iterations N - non-basic
 B - basic

$$[A \quad I] \begin{bmatrix} x \\ w \end{bmatrix} = b \iff [B \quad N] \begin{bmatrix} x_B \\ x_N \end{bmatrix} = b$$

B = basic var. col's
of $[A \quad I]$

N = non-basic var. cols
of $[A \quad I]$

Simplex Iterations in Matrix Form

$$\textcircled{1} \quad B X_B + N X_N = b \implies X_B = B^{-1} b - B^{-1} N X_N$$

$$\begin{aligned} \textcircled{2} \quad J &= C_B^T X_B + C_N^T X_N \\ &= C_B^T (B^{-1} b - B^{-1} N X_N) + C_N^T X_N \\ &= C_B^T B^{-1} b - (C_B^T B^{-1} N - C_N^T) X_N \\ &= C_B^T B^{-1} b - ((B^{-1} N)^T C_B - C_N)^T X_N \\ &= \bar{J} + \bar{C}^T X_N \end{aligned}$$

Simplex Iterations in Matrix Form

$$\begin{aligned} \max \quad & f(x) = C^T x \\ \text{s.t.} \quad & W = b - AX \\ & x, W \geq 0 \end{aligned}$$

$$x \in \mathbb{R}^n,$$

$$W \in \mathbb{R}^m$$

Simplex iterations

$$\begin{aligned} \max \quad & f(x) = \underline{C_B^T \bar{B}^{-1} b} - \underline{((\bar{B}^{-1} N)^T C_B - C_N)^T} x_N \\ \text{s.t.} \quad & x_B = \underline{\bar{B}^{-1} b} - \underline{\bar{B}^{-1} N} x_N \\ & x_B, x_N \geq 0 \end{aligned}$$

B, N are basic and non-basic var. cols of $[A \ I]$

Simplex Iterations in Matrix Form

(P)

$C_B^T \bar{B}^{-1} b$	$-\left((\bar{B}^{-1} N)^T C_B - C_N\right)^T$
$\bar{B}^{-1} b$	$-\bar{B}^{-1} N$

negative
transpose

(D)

$-C_B^T \bar{B}^{-1} b$	$-(\bar{B}^{-1} b)^T$
$(\bar{B}^{-1} N)^T C_B - C_N$	$(\bar{B}^{-1} N)^T$

Simplex Iterations in Matrix Form

(P)

J^*	$-(Z_N^*)^T$
X_B^*	$-B^{-1}N$

(D)

$-J^*$	$-(X_B^*)^T$
Z_N^*	$(B^{-1}N)^T$

negative
transpose

Simplex Iterations in Matrix Form

(P)

$$\begin{aligned} \max \quad & \hat{J}(X) = \hat{J}^* - (Z_N^*)^T X_N \\ \text{s.t.} \quad & X_B = X_B^* - (B^{-1}N) X_N \\ & X_N, X_B \geq 0 \end{aligned}$$

(D)

$$\begin{aligned} \max \quad & -\hat{J}(\bar{Z}) = -\hat{J}^* - (X_B^*)^T Z_B \\ \text{s.t.} \quad & Z_N = (Z_N^*) + (B^{-1}N)^T Z_B \\ & Z_B, Z_N \geq 0 \end{aligned}$$

$$\underline{\hat{J}^* = C_B^T B^{-1} b}, \quad \underline{Z_N^* = (B^{-1}N)^T C_B - C_N}, \quad \underline{X_B^* = B^{-1} b}$$

Simplex Iterations in Matrix Form

(P)

$$\begin{aligned} \max \quad & \mathcal{J}(X) = \mathcal{J}^* - (Z_N^*)^T X_N \\ \text{s.t.} \quad & X_B = X_B^* - (B^{-1}N) X_N \\ & X_N, X_B \geq 0 \end{aligned}$$

Optimal
Dictionary



(D)

$$\begin{aligned} \max \quad & -\mathcal{Z}(\bar{Z}) = -\mathcal{J}^* - (X_B^*)^T Z_B \\ \text{s.t.} \quad & Z_N = (Z_N^*) + (B^{-1}N)^T Z_B \\ & Z_B, Z_N \geq 0 \end{aligned}$$

$$Z_N^* \geq 0$$

$$(X_B^* \geq 0)$$

$$\underline{\mathcal{J}^* = C_B^T B^{-1} b}, \quad \underline{Z_N^* = (B^{-1}N)^T C_B - C_N}, \quad \underline{X_B^* = B^{-1} b}$$

Complementary Variables

$$\begin{array}{c} (w_1 \text{ --- } w_m) \\ \left(\underline{x_1, x_2, \dots, x_n, x_{n+1} \dots x_{n+m}} \right) \\ \updownarrow \\ \left(\underline{z_1, z_2, \dots, z_n, z_{n+1} \dots z_{n+m}} \right) \\ \updownarrow \\ (y_1 \text{ --- } y_m) \end{array}$$

① $x_j \sim z_j$
 $(x_j z_j = 0)$

$w_i \sim y_i$
 $(w_i y_i = 0)$

② x_j leaves/enters $\sim z_j$ enters/leaves
 w_i leaves/enters $\sim y_i$ enters/leaves

③ $x_N \sim z_N$, $x_B \sim z_B$

[v] p.95

①

maximize $4x_1 + 3x_2$

subject to $x_1 - x_2 \leq 1$

$2x_1 - x_2 \leq 3$

$x_2 \leq 5$

$x_1, x_2 \geq 0.$

$$\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ 2x_1 - x_2 + x_4 &= 3 \\ x_2 + x_5 &= 5 \end{aligned}$$

The matrix A is given by

$$\begin{array}{c} \text{1} \quad \text{2} \quad \text{3} \quad \text{4} \quad \text{5} \\ \begin{bmatrix} 1 & -1 & 1 & & \\ 2 & -1 & & 1 & \\ 0 & 1 & & & 1 \end{bmatrix} \end{array}.$$

(Note that some zeros have not been shown.) The initial sets of basic and nonbasic indices are

$$\underline{\mathcal{B} = \{3, 4, 5\}} \quad \text{and} \quad \underline{\mathcal{N} = \{1, 2\}}.$$

Corresponding to these sets, we have the submatrices of A :

$$B = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad N = \begin{bmatrix} 1 & -1 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}.$$

Hint

Undo

Number format: Fraction

Zero Visibility: Visible

Current Dictionary

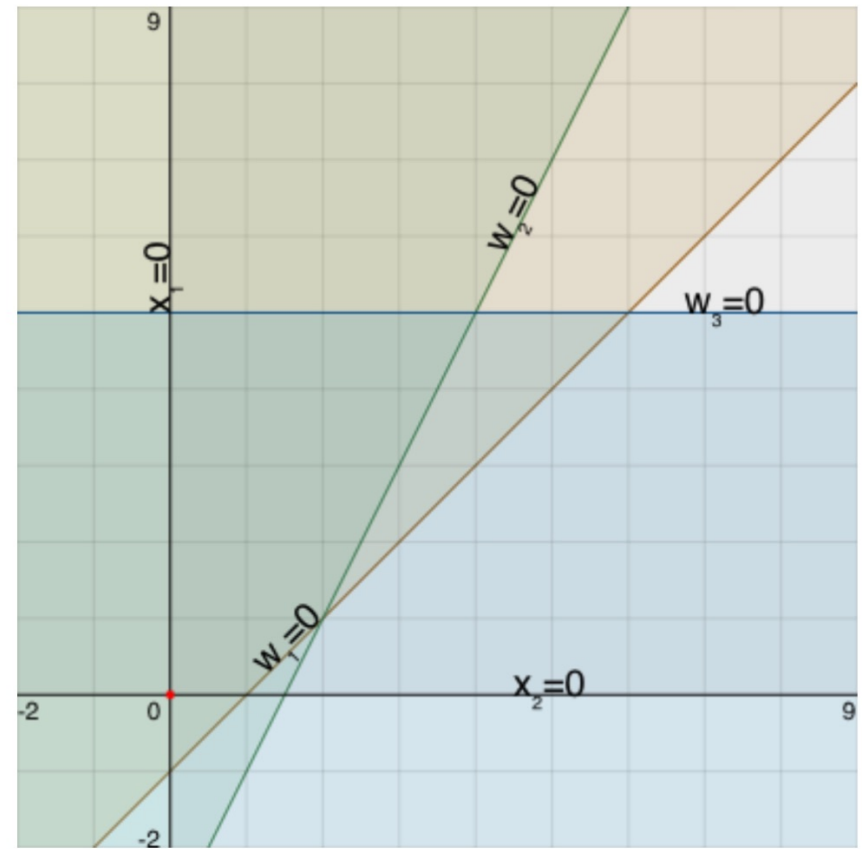
maximize $\zeta =$ 4 x_1 + 3 x_2 subject to: $w_1 =$ 1 - 1 x_1 - 1 x_2 $w_2 =$ 3 - 2 x_1 - 1 x_2 $w_3 =$ 5 - 0 x_1 - 1 x_2 $x_1 \ x_2 \ w_1 \ w_2 \ w_3 \geq 0$

Optimal

Infeasible

Unbounded

Pick a judge: Bart Simpson



$$B = I, \quad N = \begin{bmatrix} 1 & -1 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}$$

$$x_B^* = B^{-1}b = I^{-1}b = b = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix},$$

$$z_N^* = (\bar{B}^{-1}N)^T C_B - C_N = -\begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\bar{B}^{-1}N = N = \begin{bmatrix} 1 & -1 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}$$

$$(C_B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad C_N = \begin{pmatrix} 4 \\ 3 \end{pmatrix})$$

①

$$\begin{aligned} x_1 &\leq 1 \\ x_1 &\leq 3/2 \\ x_1 &\leq \infty \end{aligned}$$



$$\begin{aligned} &\text{maximize } 4x_1 + 3x_2 \\ &\text{subject to } x_1 - x_2 \leq 1 \\ &\quad \quad \quad 2x_1 - x_2 \leq 3 \\ &\quad \quad \quad x_2 \leq 5 \\ &\quad \quad \quad x_1, x_2 \geq 0. \end{aligned}$$

x_1 enters, largest coeff.

$$\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ 2x_1 - x_2 + x_4 &= 3 \\ x_2 + x_5 &= 5 \end{aligned}$$

The matrix A is given by

x_3 leaves

$$\begin{array}{ccccc} & 1 & 2 & 3 & 4 & 5 \\ \begin{bmatrix} 1 & -1 & 1 & & \\ 2 & -1 & & 1 & \\ 0 & 1 & & & 1 \end{bmatrix} \end{array}$$

(Note that some zeros have not been shown.) The initial sets of basic and nonbasic indices are

$$\underline{\mathcal{B} = \{3, 4, 5\}} \quad \text{and} \quad \underline{\mathcal{N} = \{1, 2\}}.$$

Corresponding to these sets, we have the submatrices of A :

$$B = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad N = \begin{bmatrix} 1 & -1 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}.$$

Step 9. The new sets of basic and nonbasic indices are

$$\mathcal{B} = \{1, 4, 5\} \quad \text{and} \quad \mathcal{N} = \{3, 2\}.$$

Corresponding to these sets, we have the new basic and nonbasic submatrices of A ,

$$B = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 0 & & 1 \end{bmatrix} \quad N = \begin{bmatrix} 1 & -1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix},$$

and the new basic primal variables and nonbasic dual variables:

$$x_{\mathcal{B}}^* = \begin{bmatrix} x_1^* \\ x_4^* \\ x_5^* \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} \quad z_{\mathcal{N}}^* = \begin{bmatrix} z_3^* \\ z_2^* \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}.$$

x_2 enters
 x_4 leaves

$$\begin{bmatrix} 1 & -1 & 1 & & \\ 2 & -1 & & 1 & \\ 0 & 1 & & & 1 \end{bmatrix}$$

maximize $\zeta = 4 + (-4)w_1 + 7x_2$

subject to:

x_1	=	1	-	1	w_1	-	-1	x_2
w_2	=	1	-	-2	w_1	-	1	x_2
w_3	=	5	-	0	w_1	-	1	x_2

$$x_1 \quad x_2 \quad w_1 \quad w_2 \quad w_3 \geq 0$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{pmatrix} 4 \\ -7 \end{pmatrix} = \underline{(B^{-1}N)^T} C_B - C_N$$

$$= \left(\underline{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & -1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \right)^T \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} = \underline{B^{-1}} b = \underline{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$$

3

Step 9. The new sets of basic and nonbasic indices are

$$\underline{\mathcal{B} = \{1, 2, 5\}} \quad \text{and} \quad \underline{\mathcal{N} = \{3, 4\}}.$$

Corresponding to these sets, we have the new basic and nonbasic submatrices of A ,

$$B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and the new basic primal variables and nonbasic dual variables:

$$x_{\mathcal{B}}^* = \begin{bmatrix} x_1^* \\ x_2^* \\ x_5^* \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \quad z_{\mathcal{N}}^* = \begin{bmatrix} z_3^* \\ z_4^* \end{bmatrix} = \begin{bmatrix} -10 \\ 7 \end{bmatrix}.$$

x_3 enters
 x_5 leaves

$$\begin{bmatrix} 1 & -1 & 1 & & \\ 2 & -1 & & 1 & \\ 0 & 1 & & & 1 \end{bmatrix}$$

Current Dictionary

maximize	ζ	=	11	+	10	w_3	+	-7	w_4
subject to:	x_1	=	2	-	-1	w_1	-	1	w_2
	x_2	=	1	-	-2	w_1	-	1	w_2
	w_3	=	4	-	2	w_1	-	-1	w_2

$$x_1 \quad x_2 \quad w_1 \quad w_2 \quad w_3 \geq 0$$

$$B = \{1, 2, 5\}, \quad N = \{3, 4\}$$

$$B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

$$(\underline{B^{-1}N})^T C_B - C_N = \left(\begin{bmatrix} -1 & 1 & 0 \\ -2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{bmatrix} \right)^T \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -10 \\ 7 \end{pmatrix}$$

$$\underline{B^{-1}b} = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

Opt.

Step 9. The new sets of basic and nonbasic indices are

$\mathcal{B} = \{1, 2, 3\}$ and $\mathcal{N} = \{5, 4\}$.

Corresponding to these sets, we have the new basic and nonbasic submatrices of A ,

$B = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $N = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$,

and the new basic primal variables and nonbasic dual variables:

$x_{\mathcal{B}}^* = \begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix}$

$z_{\mathcal{N}}^* = \begin{bmatrix} z_5^* \\ z_4^* \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

Optimal as

$z_N^* \geq 0$

$(x_B^* \geq 0)$

$\begin{bmatrix} 1 & -1 & 1 & & & \\ 2 & -1 & & 1 & & \\ 0 & 1 & & & 1 & \end{bmatrix}$

$\delta^* = 31$

Current Dictionary

maximize	ζ	=	31	+	-5	w_1	+	-2	w_2
subject to:	x_1	=	4	-	1/2	w_3	-	1/2	w_2
	x_2	=	5	-	1	w_3	-	0	w_2
	w_1	=	2	-	1/2	w_3	-	-1/2	w_2

$x_1 \ x_2 \ w_1 \ w_2 \ w_3 \geq 0$

$$B = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

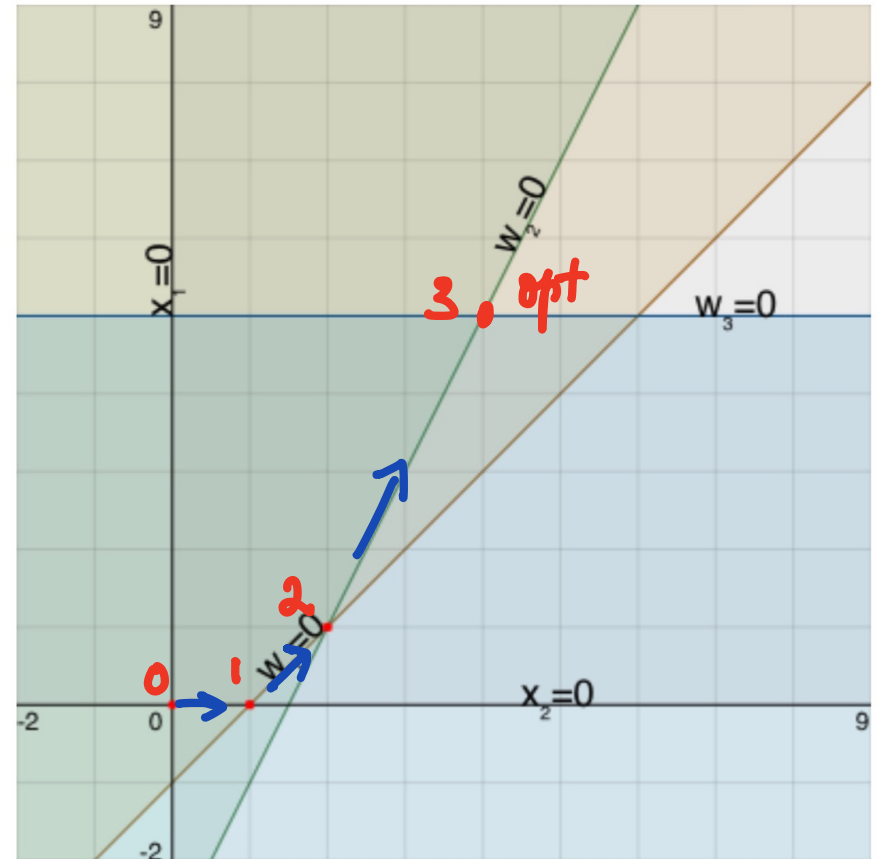
$$B^{-1} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ 1 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$(B^{-1}N)^T C_B - C_N = \left(\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ 1 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \right)^T \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$B^{-1}b = B^{-1} \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ 1 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix}$$

Hint

Undo

Number format: Fraction Zero Visibility: Visible**Current Dictionary**maximize $\zeta = 31 + (-5) w_1 + (-2) w_2$ subject to: $x_1 = 4 - \frac{1}{2} w_3 - \frac{1}{2} w_2$ $x_2 = 5 - 1 w_3 - 0 w_2$ $w_1 = 2 - \frac{1}{2} w_3 - \frac{1}{2} w_2$ $x_1 \ x_2 \ w_1 \ w_2 \ w_3 \geq 0$ OptimalInfeasibleUnboundedPick a judge: Bart Simpson

$$C_B^T B^{-1} b = (4 \ 3 \ 0) \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} = 31$$

$$\zeta^* = 31 \quad (= 4 x_1^* + 3 x_2^* = 4(4) + 3(5))$$