

Sensitivity Analysis of Simplex Method

$$(P) \quad \max \quad \mathcal{J}(x) = c^T x$$

s.t. $AX \leq b$
 $x \geq 0$



$$W = b - AX$$
$$AX + W = b$$
$$\begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} X \\ W \end{bmatrix} = b$$

Simplex
 $\Rightarrow$

$$\mathcal{J} = \bar{\mathcal{J}} + \bar{c}^T \bar{x}$$

$$\bar{w} = \bar{b} - \bar{A} \bar{x}$$

$$\bar{A} \bar{x} + \bar{w} = \bar{b}$$

$$\begin{bmatrix} \bar{A} & I \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{w} \end{bmatrix} = \bar{b}$$

$\bar{w}$ -basic, $\bar{x}$ - non-basic

$$\begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} X \\ W \end{bmatrix} \rightarrow \begin{matrix} x_B = b \\ x_N \end{matrix}$$

choose corresponding columns

$$\begin{bmatrix} B & N \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} = b$$

$$Bx_B + Nx_N = b$$

$$\bar{W} = \bar{b} - \bar{A} \bar{X}$$

$$x_B = B^{-1}(b - Nx_N) = B^{-1}b - B^{-1}Nx_N$$

$$\begin{aligned}
\textcircled{J} &= C^T X = C^T X + 0^T W = \bar{C}^T \bar{X} \\
&= C_B^T X_B + C_N^T X_N \\
&= C_B^T (\bar{B}^{-1} b - \bar{B}^{-1} N X_N) + C_N^T X_N \\
&= C_B^T \bar{B}^{-1} b - (C_B^T \bar{B}^{-1} N - C_N^T) X_N \\
&= \underbrace{(\bar{B}^{-T} C_B)^T b}_{\textcircled{J}} - \underbrace{((\bar{B}^{-1} N)^T C_B - C_N^T)^T X_N}_{\bar{Z}_N}
\end{aligned}$$

$$\textcircled{X_B} = \bar{B}^{-1} b - (\bar{B}^{-1} N) X_N$$

Simplex Iterations in Matrix Form

$$\begin{aligned} \max \quad & f(x) = C^T x \\ \text{s.t.} \quad & W = b - AX \\ & x, W \geq 0 \end{aligned}$$

$$x \in \mathbb{R}^n,$$

$$W \in \mathbb{R}^m$$

Simplex iterations

$$\begin{aligned} \max \quad & f(x) = \underline{C_B^T \bar{B}^{-1} b} - \underline{((\bar{B}^{-1} N)^T C_B - C_N)^T} x_N \\ \text{s.t.} \quad & x_B = \underline{\bar{B}^{-1} b} - \underline{\bar{B}^{-1} N} x_N \\ & x_B, x_N \geq 0 \end{aligned}$$

B, N are basic and non-basic var. cols of $[A \ I]$

Simplex Iterations in Matrix Form

(P)

$C_B^T \bar{B}^{-1} b$	$-((\bar{B}^{-1} N)^T C_B - C_N)^T$
$\bar{B}^{-1} b$	$-\bar{B}^{-1} N$

negative
transpose

(D)

$-C_B^T \bar{B}^{-1} b$	$-(\bar{B}^{-1} b)^T$
$(\bar{B}^{-1} N)^T C_B - C_N$	$(\bar{B}^{-1} N)^T$

Simplex Iterations in Matrix Form

(P)

J^*	$-(Z_N^*)^T$
X_B^*	$-B^{-1}N$

(D)

$-J^*$	$-(X_B^*)^T$
Z_N^*	$(B^{-1}N)^T$

negative
transpose

Simplex Iterations in Matrix Form

(P)

$$\max \quad \hat{J}(X) = \hat{J}^* - (Z_N^*)^T X_N$$

$$\text{s.t.} \quad X_B = X_B^* - (B^{-1}N) X_N$$

$$X_N, X_B \geq 0$$

(D)

$$\max \quad -\hat{J}(\bar{Z}) = -\hat{J}^* - (X_B^*)^T Z_B$$

$$\text{s.t.} \quad Z_N = (Z_N^*) + (B^{-1}N)^T Z_B$$

$$Z_B, Z_N \geq 0$$

$$\underline{\hat{J}^* = C_B^T B^{-1} b}, \quad \underline{Z_N^* = (B^{-1}N)^T C_B - C_N}, \quad \underline{X_B^* = B^{-1} b}$$

Simplex Iterations in Matrix Form

(P)

$$\begin{aligned} \max \quad & \mathcal{J}(X) = \mathcal{J}^* - (Z_N^*)^T X_N \\ \text{s.t.} \quad & X_B = X_B^* - (B^{-1}N) X_N \\ & X_N, X_B \geq 0 \end{aligned}$$

Optimal
Dictionary



$$Z_N^* \geq 0$$

(D)

$$\begin{aligned} \max \quad & -\mathcal{Z}(Z) = -\mathcal{J}^* - (X_B^*)^T Z_B \\ \text{s.t.} \quad & Z_N = (Z_N^*) + (B^{-1}N)^T Z_B \\ & Z_B, Z_N \geq 0 \end{aligned}$$

feasibility



$$X_B^* \geq 0$$

$$\underline{\mathcal{J}^* = C_B^T B^{-1} b}, \quad \underline{Z_N^* = (B^{-1}N)^T C_B - C_N}, \quad \underline{X_B^* = B^{-1} b}$$

Complementary Variables

$$\begin{array}{c} (w_1 \text{ --- } w_m) \\ \left(\underline{x_1, x_2, \dots, x_n, x_{n+1} \dots x_{n+m}} \right) \\ \updownarrow \\ \left(\underline{z_1, z_2, \dots, z_n, z_{n+1} \dots z_{n+m}} \right) \\ \updownarrow \\ (y_1 \text{ --- } y_m) \end{array}$$

① $x_j \sim z_j$
 $(x_j z_j = 0)$

$w_i \sim y_i$
 $(w_i y_i = 0)$

② x_j leaves/enters $\sim z_j$ enters/leaves
 w_i leaves/enters $\sim y_i$ enters/leaves

③ $x_N \sim z_N$, $x_B \sim z_B$

[C] p.67

$$\max. \quad J = 40x_1 + 70x_2$$

$$(P) \quad \text{s.t.} \quad \begin{cases} x_1 + x_2 \leq 100 \\ 10x_1 + 30x_2 \leq 4000 \\ x_1, x_2 \geq 0 \end{cases}$$

Opt. solution: $x_1^* = 25$, $x_2^* = 75$,

$$J^* = 40(25) + 70(75) = \underline{6250}$$

Q. (1) Check optimality.

(2) What if 4000 is changed to $4000 + t$
How would x_1^* , x_2^* , J^* be changed?
How large can t be?

A (1) find y_1^* , y_2^* and check validity of Strong duality.

$$(D) \quad \min \quad 100y_1 + 4000y_2$$
$$\text{s.t.} \quad \begin{cases} y_1 + 10y_2 \geq 40 \\ y_1 + 50y_2 \geq 70, \end{cases} \quad y_1, y_2 \geq 0$$

Use complementary slackness:

$$\begin{aligned} x_1^* > 0 &\Rightarrow z_1^* = 0 \Rightarrow y_1^* + 10y_2^* = 40 \\ x_2^* > 0 &\Rightarrow z_2^* = 0 \Rightarrow y_1^* + 50y_2^* = 70 \end{aligned}$$

$$y_1^* = 32.5, \quad y_2^* = 0.75$$

$\checkmark \quad \checkmark$
0 0

(Check:

$$J^* = 40(25) + 70(75) = 100(32.5) + 4000(0.75) = \underline{6250} = \bar{z}^*$$

A(2) Use B, N method

$$J = C_B^T \bar{B}^{-1} b - ((\bar{B}^{-1} N)^T C_B - C_N)^T X_N$$

$$X_B = \bar{B}^{-1} b - (\bar{B}^{-1} N) X_N$$

$$[A \quad I] = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 10 & 50 & 0 & 1 \end{bmatrix}$$

$x_1 \quad x_2 \quad x_3 \quad x_4$
 $(w_1) \quad (w_2)$

$$C = \begin{pmatrix} 40 \\ 70 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix}$$

$$b = \begin{pmatrix} 100 \\ 4000 \end{pmatrix}$$

Start

(P)				(D)		(Z ₃)		(Z ₄)	
		0		0		y ₁		y ₂	
		40		70		-100		-4000	
x ₃ (w ₁)		100		-1		-1		10	
x ₄ (w ₂)		4000		-10		-50		50	

→ (-ve)^T

End (Opt)

(P)				(D)		z ₁		z ₂	
		6250		-6250		-25		-75	
		32.5		-0.75		32.5		0.75	
x ₁		25							
x ₂		75							

$-B^{-1}N$

$(B^{-1}N)^T$

End(Opt)

$$B = \{1, 2\}, \quad N = \{3, 4\}$$

$$B = \begin{bmatrix} 1 & 1 \\ 10 & 50 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$B^{-1} = \frac{\begin{pmatrix} 50 & -1 \\ -10 & 1 \end{pmatrix}}{40} = \begin{pmatrix} 5/4 & -1/40 \\ -1/4 & 1/40 \end{pmatrix},$$

$$B^{-1}N = B^{-1}I = B^{-1}$$

$$x_B^* = B^{-1}b = \begin{pmatrix} 5/4 & -1/40 \\ -1/4 & 1/40 \end{pmatrix} \begin{pmatrix} 100 \\ 4000 \end{pmatrix} = \begin{pmatrix} 25 \\ 75 \end{pmatrix}$$

$$C_B = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 40 \\ 70 \end{pmatrix}, \quad C_N = \begin{pmatrix} c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$Z_N^* = (B^{-1}N)^T C_B - C_N = \begin{pmatrix} 5/4 & -1/4 \\ -1/40 & 1/40 \end{pmatrix} \begin{pmatrix} 40 \\ 70 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 32.5 \\ 0.75 \end{pmatrix}$$

$$J^* = C_B^T B^{-1} b = (40 \ 70) \begin{pmatrix} 5/4 & -1/40 \\ -1/4 & 1/40 \end{pmatrix} \begin{pmatrix} 100 \\ 4000 \end{pmatrix} = 6250$$

End (Opt)

Ⓐ		x_3	x_4	Ⓑ		z_1	z_2
	6250	-32.5	-0.75		-6250	-25	-75
x_1	25	$-5/4$	$1/40$	y_1	32.5	$5/4$	$-1/4$
x_2	75	$1/4$	$-1/40$	y_2	0.75	$-1/40$	$1/40$

$$J = C_B^T B^{-1} b - ((B^{-1}N)^T C_B - C_N)^T X_N$$

$$X_B = \bar{B}^{-1} b - (\bar{B}^{-1} N) X_N$$

$$-3000 \leq t \leq 1000$$

Changed to $B^{-1} \left(b + \begin{pmatrix} 0 \\ t \end{pmatrix} \right)$

$$= \begin{pmatrix} 3/4 & -1/40 \\ -1/4 & 1/40 \end{pmatrix} \begin{pmatrix} 100 \\ 4000 + t \end{pmatrix} = \begin{pmatrix} 25 - \frac{t}{40} \\ 75 + \frac{t}{40} \end{pmatrix} \geq 0$$

$$x_1^* \rightarrow 25 - \frac{t}{40}, \quad x_2^* \rightarrow 75 + \frac{t}{40}$$

$$\begin{aligned}
 \textcircled{M1} \quad J^* &= 40 x_1^* + 70 x_2^* \\
 &= 40 \left(25 - \frac{t}{40} \right) + 70 \left(75 + \frac{t}{40} \right) \\
 &= 6250 + \underline{0.75t}
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{M2} \quad J^* &= C_B^T B^{-1} \left(b + \begin{pmatrix} 0 \\ t \end{pmatrix} \right) \\
 &= (40 \quad 70) \begin{pmatrix} 5/4 & -1/40 \\ -1/4 & 1/40 \end{pmatrix} \begin{pmatrix} 100 \\ 4000 + t \end{pmatrix} \\
 &= \underline{(32.5 \quad 0.75)} \begin{pmatrix} 100 \\ 4000 + t \end{pmatrix} \\
 &= 6250 + \underline{0.75t}
 \end{aligned}$$

(y_1^*, y_2^*)

no accident.

Note: $\int^* = C_B^T B^{-1} b = b^T B^{-T} C_B$

$$\Delta \int^* = (\Delta b)^T (\underbrace{B^{-T} C_B}_{y^* = B^{-T} C_B})$$

In fact

$$\int_{\text{new}}^* = \int_{\text{old}}^* + (\Delta b)^T y_{\text{old}}^*$$



[C] p. 68 Thm 5.5

(if non-degenerate, &
 $\|\Delta b\| \ll 1$)

$$(\Delta b)^T y_{\text{old}}^* = (\Delta b)_1 \underbrace{y_1^*}_{\text{shadow price}} + (\Delta b)_2 \underbrace{y_2^*}_{\text{shadow price}} + \dots + (\Delta b)_m \underbrace{y_m^*}_{\text{shadow price}}$$

shadow prices

$$\begin{aligned}
 (P) \quad & \max \quad J = c^T X \\
 & \text{s.t.} \quad AX \leq b \\
 & \quad \quad X \geq 0
 \end{aligned}$$

At Opt: $(P) \quad J = J^* - Z_N^* X_N$

$$X_B = X_B^* - (B^{-1}N) X_N$$

$$\underline{Z_N^* = (B^{-1}N)^T C_B - C_N \geq 0}$$

$$\underline{X_B^* = B^{-1}b \geq 0}$$

$$J^* = C_B^T B^{-1}b = C_B^T X_B^* = b^T B^{-T} C_B$$

$$(D) \quad \min \quad \xi = b^T y$$

$$\text{s.t.} \quad A^T y \geq c$$

$$y \geq 0$$

At Opt.

$$-\xi = -\xi^* + (x_B^*)^T Z_B$$

$$Z_N = Z_N^* + (\bar{B}^T N)^T Z_B$$

Compare:

$$c^T x^* = b^T B^{-T} C_B = \xi^* = b^T y^*$$


Guess/Prove

$$y^* = \bar{B}^{-T} C_B$$

(1) $\bar{B}^T(\bar{B}^{-T} C_B)$ does gives opt. objective value ✓

(2) Check for feasibility of $\bar{B}^{-T} C_B$

$$A^T(\bar{B}^{-T} C_B) \geq C \quad ?$$

$$\bar{B}^{-T} C_B \geq 0 \quad ?$$

We need

$$A^T y \geq c$$

$$\iff$$

$$y \geq 0$$

$$\iff$$

$$y^T A \geq c^T$$

$$y^T \geq 0$$

$$y^T [A \quad I]$$

$$\geq$$

$$[c^T \quad 0^T]$$

$$y^T [B \quad N]$$

$$\geq$$

$$[c_B^T \quad c_N^T]$$

$$y^T B \geq c_B^T$$

$$y^T N \geq c_N^T$$

$$y = B^{-T} C_B,$$

$$y^T B = C_B^T B^{-1} B = C_B^T \geq C_B^T$$

$$y^T N = \underbrace{C_B^T B^{-1} N}_{\geq C_N}$$

Optimality condition

$$Z_N^* = (B^{-1} N)^T C_B - C_N \geq 0$$

Sensitivity and Parametric Analysis

[v] p. 111

$$\text{maximize } 5x_1 + 4x_2 + 3x_3$$

$$\text{subject to } 2x_1 + 3x_2 + x_3 \leq 5$$

$$4x_1 + x_2 + 2x_3 \leq 11$$

$$3x_1 + 4x_2 + 2x_3 \leq 8$$

$$x_1, x_2, x_3 \geq 0 .$$

Sensitivity and Parametric Analysis

Opt.
Dict.

$$\xi = 13 - 3x_2 - 1x_4 - 1x_6$$

$x_3 =$	1	+	x_2	+	3	x_4	-	2	x_6
$x_1 =$	2	-	2	x_2	-	2	x_4	+	x_6
$x_5 =$	1	+	5	x_2	+	2	x_4		

Initial
Step

$$A = \begin{matrix} & x_1 & x_2 & x_3 \\ \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix}, \end{matrix}$$

$$I = \begin{matrix} & x_4 & x_5 & x_6 \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

$$C^T = (5 \ 4 \ 3 \ 0 \ 0 \ 0), \quad b = \begin{pmatrix} 5 \\ 11 \\ 8 \end{pmatrix}$$

At Opt $B = \{3, 1, 5\}, \quad N = \{2, 4, 6\}$

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 2 & 3 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 0 \\ 4 & 0 & 1 \end{bmatrix}, \quad B^{-1} = \begin{pmatrix} -3 & 0 & 2 \\ 2 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}$$

$$C_B = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix},$$

$$C_N = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$$

Sensitivity and Parametric Analysis

Opt. Dict. $z_N^* = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = (\bar{B}^{-1}N)^T C_B - C_N, \quad x_N = \begin{pmatrix} x_2 \\ x_4 \\ x_6 \end{pmatrix}$

$C_B^T \bar{B}^{-1} b \rightarrow \xi \geq \boxed{13} \quad \boxed{-3x_2 - 1x_4 - 1x_6}$

$x_B = \begin{pmatrix} x_3 \\ x_1 \\ x_5 \end{pmatrix} = \begin{array}{l} \boxed{1} + x_2 + 3x_4 - 2x_6 \\ 2 - 2x_2 - 2x_4 + x_6 \\ \boxed{1} + 5x_2 + 2x_4 \end{array}$

$x_B^* = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \bar{B}^{-1}b$

$\bar{B}^{-1}N = \begin{pmatrix} -1 & -3 & 2 \\ 2 & 2 & -1 \\ -5 & -2 & 0 \end{pmatrix}$

Sensitivity and Parametric Analysis

What if

$$\text{maximize } 5x_1 + 4x_2 + 3x_3$$

$$\text{subject to } 2x_1 + 3x_2 + x_3 \leq 5 + \Delta b_1$$

$$4x_1 + x_2 + 2x_3 \leq 11 + \Delta b_2$$

$$3x_1 + 4x_2 + 2x_3 \leq 8 + \Delta b_3$$

$$x_1, x_2, x_3 \geq 0 .$$

Sensitivity and Parametric Analysis

Still optimal?

$$\xi = 13 - 3x_2 - 1x_4 - 1x_6$$

$x_3 =$	1	+		x_2	+	3	x_4	-	2	x_6
$x_1 =$	2	-	2	x_2	-	2	x_4	+		x_6
$x_5 =$	1	+	5	x_2	+	2	x_4			

Sensitivity and Parametric Analysis

Still optimal?

$$\xi = 13 - 3x_2 - 1x_4 - 1x_6$$

$x_3 =$	1	+		x_2	+	3	x_4	-	2	x_6
$x_1 =$	2	-	2	x_2	-	2	x_4	+		x_6
$x_5 =$	1	+	5	x_2	+	2	x_4			

Sensitivity and Parametric Analysis

$$Z_N^* = (\bar{B}^T N)^T C_B - C_N \geq 0 \text{ not changed}$$

$$X_B^* = \bar{B}^{-1} b, \quad \Delta X_B^* = \bar{B}^{-1} \Delta b \quad \vec{\Delta b} = \begin{pmatrix} \Delta b_1 \\ \Delta b_2 \\ \Delta b_3 \end{pmatrix}$$

$$\text{We need } X_B^* + \Delta X_B^* \geq 0$$

$$\begin{pmatrix} -3 & 0 & 2 \\ 2 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} b_1^{\text{new}} \\ b_2^{\text{new}} \\ b_3^{\text{new}} \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Sensitivity and Parametric Analysis

What if

$$\text{maximize } 5x_1 + 4x_2 + 3x_3$$

$$\text{subject to } 2x_1 + 3x_2 + x_3 \leq 5$$

$$4x_1 + x_2 + 2x_3 \leq 11$$

$$3x_1 + 4x_2 + 2x_3 \leq 8$$

$$x_1, x_2, x_3 \geq 0$$

Sensitivity and Parametric Analysis

Still optimal?

$$\xi = 13 - 3x_2 - 1x_4 - 1x_6$$

$x_3 =$	1	+		x_2	+	3	x_4	-	2	x_6
$x_1 =$	2	-	2	x_2	-	2	x_4	+		x_6
$x_5 =$	1	+	5	x_2	+	2	x_4			

Sensitivity and Parametric Analysis

Still optimal?

$$\xi = 13 - 3x_2 - 1x_4 - 1x_6$$

$$x_3 = 1 + x_2 + 3x_4 - 2x_6$$

$$x_1 = 2 - 2x_2 - 2x_4 + x_6$$

$$x_5 = 1 + 5x_2 + 2x_4$$

Sensitivity and Parametric Analysis

$$x_B^* = \bar{B}^{-1} b \geq 0 \quad \text{not changed}$$

$$z_N^* = (\bar{B}^{-1} N)^T C_B - C_N$$

$$\Delta z_N^* = (\bar{B}^{-1} N)^T \Delta C_B - \Delta C_N$$

We need $z_N^* + \Delta z_N^* \geq 0$

Sensitivity and Parametric Analysis

$$\begin{pmatrix} -1 & -3 & 2 \\ 2 & 2 & -1 \\ -5 & -2 & 0 \end{pmatrix}^T \begin{pmatrix} C_B^{\text{new}} \\ C_3 \\ C_1 \\ C_5 \end{pmatrix} - \begin{pmatrix} C_N^{\text{new}} \\ C_2 \\ C_4 \\ C_6 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 & -5 \\ -3 & 2 & -2 \\ 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 5+t \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3+2t \\ 1+2t \\ 1-t \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{i.e.} \quad -\frac{1}{2} \leq t \leq 1$$

What if $A \rightarrow A + \Delta A$?

$$B \rightarrow B + \Delta B \leftarrow \text{new } B$$

$$N \rightarrow N + \Delta N \leftarrow \text{new } N$$

Need ① $X_B^* = (B + \Delta B)^{-1} b \geq 0$

② $Z_N^* = ((B + \Delta B)^{-1} (N + \Delta N))^T C_B - C_N \geq 0$

New ③ $J^* = C_B^T (B + \Delta B)^{-1} b$

In general $(M + \Delta M)^{-1} = M^{-1} + \boxed{?}$
 $\hat{=} M^{-1} + \underbrace{\Delta(M^{-1})}_{=?}$

(if ΔM is small)

$$M M^{-1} = I$$

$$\Delta(M M^{-1}) = \Delta I = 0$$

$$(\Delta M) M^{-1} + M (\Delta(M^{-1})) = 0$$

$$\underline{\Delta(M^{-1}) = -M^{-1}(\Delta M)M^{-1}}$$

Hence $\underline{(M + \Delta M)^{-1} \hat{=} M^{-1} - M^{-1}(\Delta M)M^{-1}}$

$$\begin{aligned}
x_B^* &= (B + \Delta B)^{-1} b \\
&\approx (B^{-1} - B^{-1}(\Delta B)B^{-1})b \\
&= B^{-1}b - B^{-1}(\Delta B)B^{-1}b \quad (\geq 0)
\end{aligned}$$

$$\begin{aligned}
z_N^* &= ((B + \Delta B)^{-1}(N + \Delta N))^T C_B - C_N \\
&\approx [(B^{-1} - B^{-1}(\Delta B)B^{-1})(N + \Delta N)]^T C_B - C_N
\end{aligned}$$

$$s^* = C_B^T (B + \Delta B)^{-1} b \approx \underline{C_B^T (B - B^{-1}(\Delta B)B^{-1})b}$$