

Inner Product (on $\mathbb{R}^n$)

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$$

$$\begin{aligned} \langle X, Y \rangle (= X \cdot Y) &= x_1 y_1 + x_2 y_2 + \dots + x_n y_n \\ &= X^T Y \underset{\substack{\text{"} \\ (x_1, \dots, x_n)}}{=} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = Y X^T \underset{\substack{\text{"} \\ (y_1, y_2, \dots, y_n)}}{=} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \end{aligned}$$

Properties (or Definition) of $\langle \cdot, \cdot \rangle$

- (1) Symmetry: $\langle X, Y \rangle = \langle Y, X \rangle$
- (2) Linearity: $\langle \alpha X + \beta Y, Z \rangle = \alpha \langle X, Z \rangle + \beta \langle Y, Z \rangle$
- (3) Positive Definiteness:
 $\langle X, X \rangle \geq 0, \quad = 0 \Leftrightarrow X = 0$

Length / Norm / Distance

$$\|X\| = 0 \Leftrightarrow X = 0$$

$$\begin{aligned} \|X\|^2 &= \langle X, X \rangle \geq 0, & \|X\| &= \sqrt{\langle X, X \rangle} \\ \|X - Y\|^2 &= \langle X - Y, X - Y \rangle & \|X - Y\| &= \sqrt{\langle X - Y, X - Y \rangle} \end{aligned}$$

Thm (Cauchy-Schwarz Inequality)

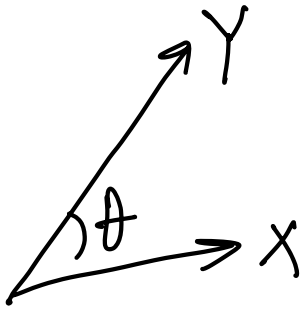
$$\|X - Y\| = 0 \Leftrightarrow X = Y$$

$$|\langle X, Y \rangle| \leq \|X\| \|Y\|$$

"=" hold iff $X \parallel Y$
i.e. $X = \lambda Y$ or $Y = \lambda X$

$$(x_1 y_1 + \dots + x_n y_n) \leq \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} \sqrt{y_1^2 + \dots + y_n^2}$$

Angle between X & Y ($X, Y \neq 0$)



$$\cos \theta = \frac{\langle X, Y \rangle}{\|X\| \|Y\|}, \quad \text{or} \quad \theta = \cos^{-1} \left(\frac{\langle X, Y \rangle}{\|X\| \|Y\|} \right)$$

will be defined as $-1 \leq \frac{\langle X, Y \rangle}{\|X\| \|Y\|} \leq 1$

$\uparrow$ iff $X = \lambda Y$ $\lambda < 0$

$\uparrow$ iff $X = \lambda Y$ $\lambda > 0$

Ex $X = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad Y = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

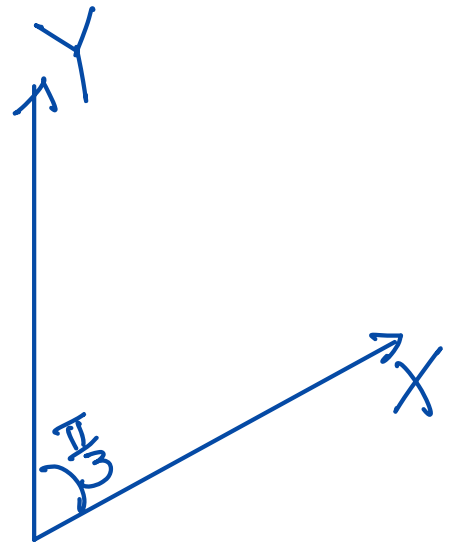
$$\|X\| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6},$$

$$\|Y\| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$$

$$\langle X, Y \rangle = 2 - 1 + 2 = 3$$

$$\cos \theta = \frac{\langle X, Y \rangle}{\|X\| \|Y\|} = \frac{3}{\sqrt{6} \sqrt{6}} = \frac{3}{6} = \frac{1}{2}$$

$$\theta = \cos^{-1} \frac{1}{2} = \frac{\pi}{3} \quad (= 60^\circ)$$

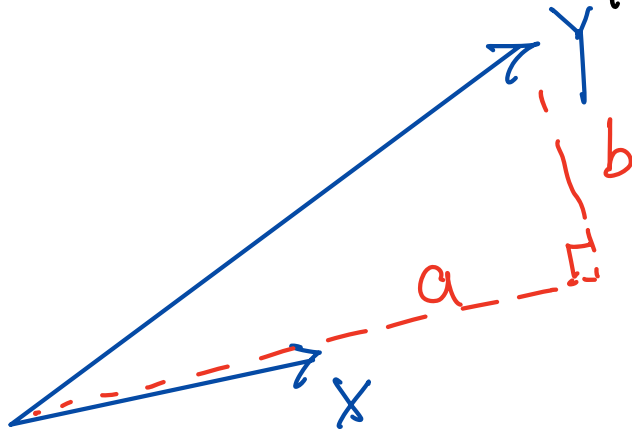


$$X \perp Y$$

orthogonal/perpendicular

$$\text{if } \langle X, Y \rangle = 0, \text{ i.e. } \theta = \frac{\pi}{2}$$

Projection of Y onto X ($\neq 0$)



Decompose $Y = a + b$ where $a \parallel X, b \perp X$

$$Y = \lambda X + b$$

$$a = \lambda X \quad \langle b, X \rangle = 0$$

$$\langle Y, X \rangle = \langle \lambda X + b, X \rangle = \lambda \langle X, X \rangle + \cancel{\langle b, X \rangle}$$

$$\lambda = \frac{\langle Y, X \rangle}{\langle X, X \rangle}$$

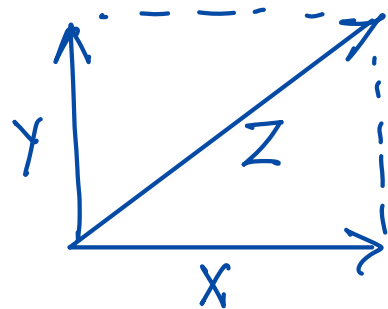
$$a = \frac{\langle Y, X \rangle}{\langle X, X \rangle} X, \quad b = Y - a = Y - \frac{\langle Y, X \rangle}{\langle X, X \rangle} X$$

$$Y = \text{Proj}_X Y + \text{Proj}_X^\perp Y$$

Pythagorean Theorem

if $Z = x + y, \quad x \perp y$

then $\|Z\|^2 = \|x\|^2 + \|y\|^2$



(pf $\|Z\|^2 = \|x+y\|^2 = \langle x+y, x+y \rangle$
 $= \langle x, x \rangle + \cancel{\langle x, y \rangle} + \cancel{\langle y, x \rangle} + \langle y, y \rangle$
 $= \|x\|^2 + \|y\|^2$)

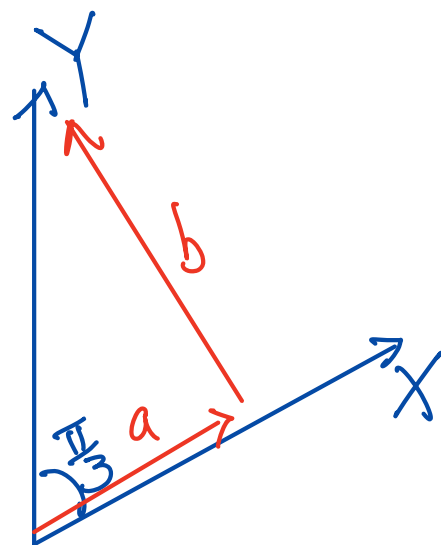
$$y = \text{Proj}_x y + \text{Proj}_x^\perp y$$

$$\|y\|^2 = \|\text{Proj}_x y\|^2 + \|\text{Proj}_x^\perp y\|^2$$

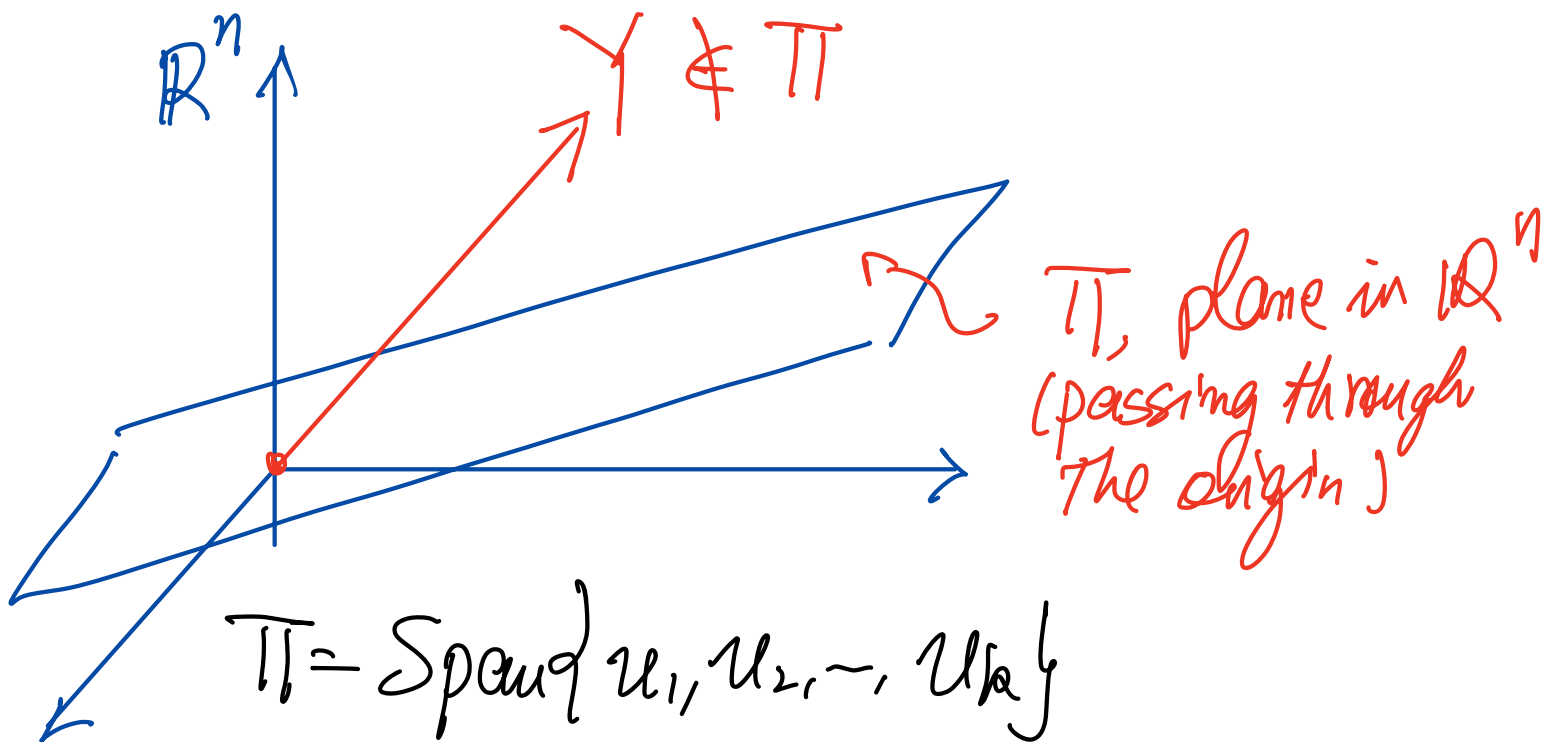
Ex $x = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad y = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

$$a = \text{Proj}_x y = \frac{\langle x, y \rangle}{\|x\|^2} x = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$b = \text{Proj}_x^\perp y = y - a = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}$$



Projection onto a 'plane' (A subspace)



= $\{$ all linear combinations of $\}$
the u_i 's

$$= \{ c_1 u_1 + c_2 u_2 + \dots + c_k u_k : c_i \text{'s} \in \mathbb{R} \}$$

Write / Decompose Y as

$$Y = a + b,$$

$$\begin{matrix} \downarrow & \downarrow \\ a \in \Pi & b \perp \Pi \end{matrix}$$

$$a = c_1 u_1 + c_2 u_2 + \dots + c_k u_k, \quad b \perp u_i \quad \forall i$$

$$c_1, c_2, \dots, c_k = ?$$

$$Y = a + b = C_1 u_1 + \dots + C_k u_k + b$$

$$\begin{aligned} \langle Y, u_1 \rangle &= \langle C_1 u_1 + C_2 u_2 + \dots + C_k u_k + b, u_1 \rangle \\ &= C_1 \langle u_1, u_1 \rangle + C_2 \langle u_2, u_1 \rangle \\ &\quad + \dots + C_k \langle u_k, u_1 \rangle + \langle b, u_1 \rangle \end{aligned}$$

$$\textcircled{1} \Rightarrow \langle Y, u_1 \rangle = C_1 \langle u_1, u_1 \rangle + C_2 \langle u_2, u_1 \rangle + \dots + C_k \langle u_k, u_1 \rangle$$

Similarly,

$$\textcircled{2} \langle Y, u_2 \rangle = C_1 \langle u_1, u_2 \rangle + C_2 \langle u_2, u_2 \rangle + \dots + C_k \langle u_k, u_2 \rangle$$

$$\vdots \quad \vdots \quad \vdots$$

$$\textcircled{k} \langle Y, u_k \rangle = C_1 \langle u_1, u_k \rangle + C_2 \langle u_2, u_k \rangle + \dots + C_k \langle u_k, u_k \rangle$$

k linear equations in k unknowns, $C_1, \dots, C_k$

$$(*) \begin{bmatrix} \langle Y, u_1 \rangle \\ \langle Y, u_2 \rangle \\ \vdots \\ \langle Y, u_k \rangle \end{bmatrix} = \begin{bmatrix} \langle u_1, u_1 \rangle & \langle u_2, u_1 \rangle & \dots & \langle u_k, u_1 \rangle \\ \langle u_2, u_1 \rangle & \langle u_2, u_2 \rangle & \dots & \langle u_k, u_2 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle u_1, u_k \rangle & \langle u_2, u_k \rangle & \dots & \langle u_k, u_k \rangle \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_k \end{bmatrix}$$

If $\{u_1, u_2, \dots, u_k\}$ are orthogonal

ie. $u_i \perp u_j$ ($\langle u_i, u_j \rangle = 0$) $\forall i \neq j$

then (*) becomes much simpler:

$$\langle Y, u_1 \rangle = c_1 \langle u_1, u_1 \rangle \Rightarrow c_1 = \frac{\langle Y, u_1 \rangle}{\langle u_1, u_1 \rangle}$$

$$\langle Y, u_2 \rangle = c_2 \langle u_2, u_2 \rangle \Rightarrow c_2 = \frac{\langle Y, u_2 \rangle}{\langle u_2, u_2 \rangle}$$

$$\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots$$

$$\langle Y, u_k \rangle = c_k \langle u_k, u_k \rangle \Rightarrow c_k = \frac{\langle Y, u_k \rangle}{\langle u_k, u_k \rangle}$$

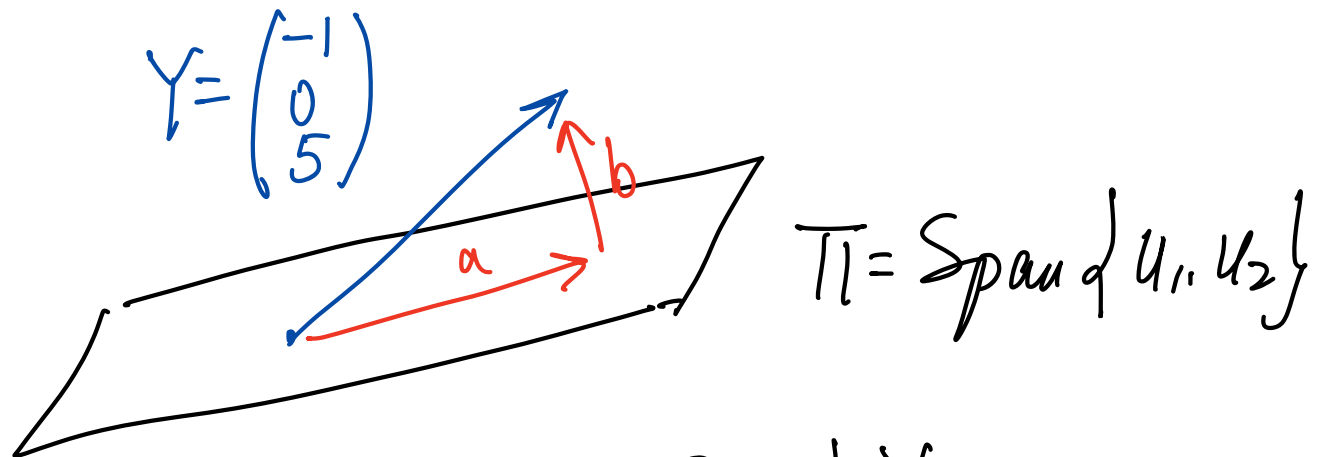
ie.

$$\text{Proj}_{\Pi} Y = a = c_1 u_1 + c_2 u_2 + \dots + c_k u_k$$

$$= \frac{\langle Y, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 + \frac{\langle Y, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 + \dots + \frac{\langle Y, u_k \rangle}{\langle u_k, u_k \rangle} u_k$$

$$= \text{Proj}_{u_1} Y + \text{Proj}_{u_2} Y + \dots + \text{Proj}_{u_k} Y$$

Ex $u_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ $u_2 = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$ $u_1 \perp u_2$



$$Y = \underbrace{\text{Proj}_{\Pi} Y}_{\text{red bracket}} + \text{Proj}_{\Pi}^{\perp} Y$$

$$= \underbrace{\text{Proj}_{u_1} Y + \text{Proj}_{u_2} Y}_{\text{red bracket}} + \text{Proj}_{\Pi}^{\perp} Y$$

$$\underline{\text{Proj}_{\Pi} Y} = \frac{\left\langle \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\rangle}{1^2 + 1^2 + 2^2} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \frac{\left\langle \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} \right\rangle}{3^2 + 1^2 + 1^2} \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

$$= \frac{3}{2} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \frac{8}{11} \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{22} \begin{pmatrix} -15 \\ 49 \\ 82 \end{pmatrix}$$

$$\underline{\text{Proj}_{\Pi}^{\perp} Y} = Y - \text{Proj}_{\Pi} Y = \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix} - \frac{1}{22} \begin{pmatrix} -15 \\ 49 \\ 82 \end{pmatrix} = \frac{7}{22} \begin{pmatrix} -1 \\ -7 \\ 4 \end{pmatrix}$$

⊥ to u_1, u_2

Higher Dimensional Pythagorean Theorem

If $x_1, x_2, \dots, x_k$ are orthogonal
($x_i \perp x_j \quad \forall i \neq j$)

and $Y = x_1 + x_2 + \dots + x_k$

then

$$\underline{\|Y\|^2 = \|x_1\|^2 + \|x_2\|^2 + \dots + \|x_k\|^2}$$

If $u_1, u_2, \dots, u_k$ orthogonal, and

$$\underline{Y = \text{Proj}_{u_1} Y + \text{Proj}_{u_2} Y + \dots + \text{Proj}_{u_k} Y}$$

then

$$\underline{\|Y\|^2 = \|\text{Proj}_{u_1} Y\|^2 + \|\text{Proj}_{u_2} Y\|^2 + \dots + \|\text{Proj}_{u_k} Y\|^2}$$

Example of orthogonal vectors

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 7 \\ -4 \end{pmatrix}$$

Write $Y = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ as a lin comb. of u_1, u_2, u_3
i.e. $Y = c_1 u_1 + c_2 u_2 + c_3 u_3$

$$\langle Y, u_1 \rangle = \langle c_1 u_1 + c_2 u_2 + c_3 u_3, u_1 \rangle = c_1 \langle u_1, u_1 \rangle$$

$$c_1 = \frac{\langle Y, u_1 \rangle}{\langle u_1, u_1 \rangle} = \frac{\left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\rangle}{1^2 + 1^2 + 2^2} = \frac{2}{3}$$

Similarly $c_2 = \frac{\langle Y, u_2 \rangle}{\langle u_2, u_2 \rangle} = \frac{\left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} \right\rangle}{9 + 1 + 1} = -\frac{1}{11}$

$$c_3 = \frac{\langle Y, u_3 \rangle}{\langle u_3, u_3 \rangle} = \frac{\left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 7 \\ -4 \end{pmatrix} \right\rangle}{1 + 49 + 16} = \frac{2}{33}$$

i.e. $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{2}{3} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \frac{1}{11} \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} + \frac{2}{33} \begin{pmatrix} 1 \\ 7 \\ -4 \end{pmatrix}$

How to make orthogonal vectors?

Gram-Schmidt Process

Given $\{u_1, u_2, u_3, \dots, u_n\}$ not orthogonal

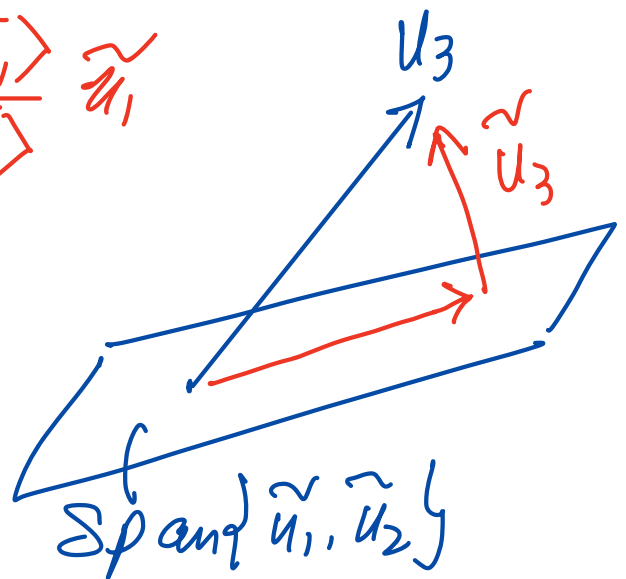
$\Downarrow$
 $\{\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \dots, \tilde{u}_n\}$ orthogonal

$$\tilde{u}_1 = u_1$$

$$\tilde{u}_2 = u_2 - \text{Proj}_{\tilde{u}_1} u_2$$

$$= \frac{\langle u_2, \tilde{u}_1 \rangle}{\langle \tilde{u}_1, \tilde{u}_1 \rangle} \tilde{u}_1$$

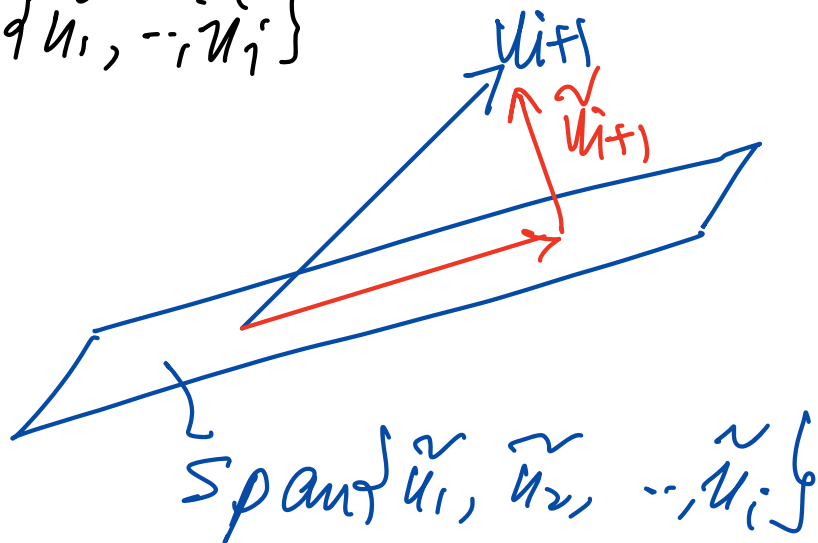
$$\tilde{u}_3 = u_3 - \text{Proj}_{\{\tilde{u}_1, \tilde{u}_2\}} u_3$$



$$= u_3 - \frac{\langle u_3, \tilde{u}_1 \rangle}{\langle \tilde{u}_1, \tilde{u}_1 \rangle} \tilde{u}_1 - \frac{\langle u_3, \tilde{u}_2 \rangle}{\langle \tilde{u}_2, \tilde{u}_2 \rangle} \tilde{u}_2$$

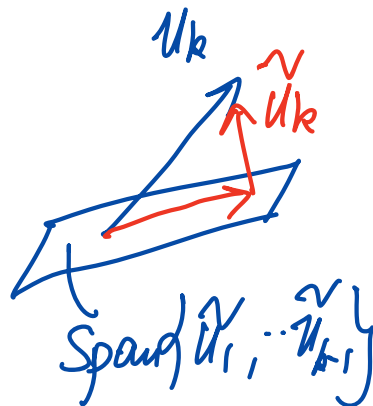
$\vdots$
 $\vdots$
 $\vdots$
 $\{\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_i\}$ found

$$\tilde{u}_{i+1} = u_{i+1} - \text{Proj}_{\{\tilde{u}_1, \dots, \tilde{u}_i\}} u_{i+1}$$



$$= u_{i+1} - \frac{\langle u_{i+1}, \tilde{u}_1 \rangle}{\langle \tilde{u}_1, \tilde{u}_1 \rangle} \tilde{u}_1 - \dots - \frac{\langle u_{i+1}, \tilde{u}_i \rangle}{\langle \tilde{u}_i, \tilde{u}_i \rangle} \tilde{u}_i$$

$\vdots$
 $\vdots$
 $\vdots$
 $\vdots$



$$\tilde{u}_k = u_k - \text{Proj}_{\{\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_{k-1}\}} u_k$$

$$= u_k - \frac{\langle u_k, \tilde{u}_1 \rangle}{\langle \tilde{u}_1, \tilde{u}_1 \rangle} \tilde{u}_1 - \dots - \frac{\langle u_k, \tilde{u}_{k-1} \rangle}{\langle \tilde{u}_{k-1}, \tilde{u}_{k-1} \rangle} \tilde{u}_{k-1}$$

Note : The $\{\tilde{u}_i\}$ so produced has an additional property:

$$\text{Span}\{u_1, \dots, u_i\} = \text{Span}\{\tilde{u}_1, \dots, \tilde{u}_i\}$$

for $i=1, 2, \dots, k$

i.e. $\text{Span}\{u_1\} = \text{Span}\{\tilde{u}_1\}$

$$\text{Span}\{u_1, u_2\} = \text{Span}\{\tilde{u}_1, \tilde{u}_2\}$$

$$\text{Span}\{u_1, u_2, u_3\} = \text{Span}\{\tilde{u}_1, \tilde{u}_2, \tilde{u}_3\}$$

$$\text{Span}\{u_1, u_2, \dots, u_k\} = \text{Span}\{\tilde{u}_1, \dots, \tilde{u}_k\}$$

Example

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\tilde{u}_1 = u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\tilde{u}_2 = u_2 - \text{Proj}_{\tilde{u}_1} u_2$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \rangle}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$\tilde{u}_2 \perp \tilde{u}_1$

$$\tilde{u}_3 = u_3 - \text{Proj}_{\{\tilde{u}_1, \tilde{u}_2\}} u_3$$

orthogonal

$$= u_3 - \frac{\langle u_3, \tilde{u}_1 \rangle}{\langle \tilde{u}_1, \tilde{u}_1 \rangle} \tilde{u}_1 - \frac{\langle u_3, \tilde{u}_2 \rangle}{\langle \tilde{u}_2, \tilde{u}_2 \rangle} \tilde{u}_2$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{\langle \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \rangle}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{\langle \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \rangle}{6} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} -4 \\ 4 \\ 4 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} -4 \\ 4 \\ 4 \end{pmatrix}$$

$\tilde{u}_3 = \begin{pmatrix} -4 \\ 4 \\ 4 \end{pmatrix}$

(Note: Changing the length does not affect the angle)

$$\tilde{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$

$$\tilde{u}_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix},$$

$$\tilde{u}_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

