"Rayleigh Quotient"

– min/max quadratic form with quadratic constraints

$$\min/\max \quad f(x,y) = 18x^2 - 8xy + 12y^2$$
$$\text{s.t.} \quad x^2 + y^2 = 1$$

$$L(x,y) = 18x^2 - 8xy + 12y^2 + \lambda(x^2+y^2-1)$$

$$\partial_x L = 0 \Rightarrow 36x - 8y + 2\lambda x = 0$$
$$\partial_y L = 0 \Rightarrow -8x + 24y + 2\lambda y = 0$$
$$\partial_\lambda L = 0 \Rightarrow x^2 + y^2 = 1$$

$$18x - 4y = -\lambda x$$
$$-4x + 12y = -\lambda y$$

$$\Leftrightarrow \begin{bmatrix} 18 & -4 \\ -4 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (-\lambda) \begin{bmatrix} x \\ y \end{bmatrix}$$

$(-)$ eigenvalue of $\begin{bmatrix} 18 & -4 \\ -4 & 12 \end{bmatrix}$

$$AX = -\lambda X \iff (A+\lambda I)X = 0$$

$$\det(A+\lambda I) = 0$$

$$\det \begin{bmatrix} 18+\lambda & -4 \\ -4 & 12+\lambda \end{bmatrix} = 0$$

$$(\lambda+18)(\lambda+12) - 16 = 0$$

$$\lambda^2 + 30\lambda + 200 = 0$$

$$(\lambda+10)(\lambda+20) = 0$$

$$\lambda = -10 \Rightarrow \begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$x^2 + y^2 = 1 \Rightarrow \alpha^2 + 4\alpha^2 = 1 \Rightarrow \alpha = \pm\frac{1}{\sqrt{5}}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \pm\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\lambda = -20 \Rightarrow \begin{bmatrix} -2 & -4 \\ -4 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \beta \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$x^2 + y^2 = 1 \Rightarrow 4\beta^2 + \beta^2 = 1 \Rightarrow \beta = \pm\frac{1}{\sqrt{5}}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \pm\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

What are the min/max values of

$f(x,y) = 18x^2 - 8xy + 12y^2$?

$$f(x,y) = \begin{pmatrix} x & y \end{pmatrix} \begin{bmatrix} 18 & -4 \\ -4 & 12 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$= \pm\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

at $\lambda = -10$

$$\begin{bmatrix} 18 & -4 \\ -4 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 10 \begin{bmatrix} x \\ y \end{bmatrix}$$

$=$

$$f(x,y) = \begin{pmatrix} x & y \end{pmatrix} 10 \begin{pmatrix} x \\ y \end{pmatrix} = 10\underbrace{(x^2+y^2)}_{1} = 10$$

$= \lambda_{min}$

at $\lambda = -20$

$$\begin{bmatrix} 18 & -4 \\ -4 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 20 \begin{bmatrix} x \\ y \end{bmatrix}$$

$= \pm\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$=$

$$f(x,y) = \begin{pmatrix} x & y \end{pmatrix} 20 \begin{pmatrix} x \\ y \end{pmatrix} = 20(x^2+y^2) = 20$$

$= \max$

$\max f = 20 \leftarrow$ max eigenvalue

$\min f = 10 \leftarrow$ min eigenvalue

In fact the original problem can also be stated as:

$$\max/\min_{x,y\neq 0} \; g(x,y) = \frac{18x^2 - 8xy + 12y^2}{x^2+y^2}$$

Rayleigh Quotient

Let $x^2+y^2 = t^2$. Introduce $x = t\tilde{x}$, $y = t\tilde{y}$

Then $t^2\tilde{x}^2 + t^2\tilde{y}^2 = t^2 \Rightarrow \tilde{x}^2+\tilde{y}^2 = 1$

$$g(x,y) = \frac{18x^2 - 8xy + 12y^2}{x^2+y^2}$$

$$= \frac{18(t\tilde{x})^2 - 8(t\tilde{x})(t\tilde{y}) + 12(t\tilde{y})^2}{(t\tilde{x})^2+(t\tilde{y})^2}$$

$$= \frac{18\tilde{x}^2 - 8\tilde{x}\tilde{y} + 12\tilde{y}^2}{\tilde{x}^2+\tilde{y}^2} \quad (\tilde{x}^2+\tilde{y}^2 = 1)$$

$$= 18\tilde{x}^2 - 8\tilde{x}\tilde{y} + 12\tilde{y}^2$$

$$\min/\max_{x,y\neq 0} \frac{18x^2 - 8xy + 12y^2}{x^2+y^2} \iff \min/\max_{\text{s.t.}} \; 18x^2 - 8xy + 12y^2, \quad x^2+y^2 = 1$$