Regression

Given $\{x_1, x_2, \cdots, x_m\}$ $x_i \in \mathbb{R}$

(1) $\min_x \frac{1}{m} \sum_{i=1}^{m} (x - x_i)^2 \Rightarrow x = \frac{x_1 + \cdots + x_m}{m}$

L^2-error

(mean)

(2) $\min_x \frac{1}{m} \sum_{i=1}^{m} |x - x_i| \Rightarrow x = \text{median}$

L^1-error

(3) $\min_x \left(\max_i |x - x_i| \right) \Rightarrow x = \frac{\min x_i + \max x_i}{2}$

"mid-range"

L^∞-error

Given attributes $\{a_1, a_2, \cdots, a_n\}$, b

find parameters $\{x_1, x_2, \cdots, x_n\}$ s.t.

exact

$$b = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

real

$$b = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n + \varepsilon$$

error

$$\varepsilon = b - (a_1 x_1 + a_2 x_2 + \cdots + a_n x_n)$$

Given $a_1, \cdots, a_n, b$, find $x_1, \cdots, x_n$

data parameter

So as to minimize the error ε

Given data points (m individuals)

$$b_i, \{a_{ij}\}_{j=1}^{n}, \quad i = 1, \cdots, m$$

Find $x_1, x_2, \cdots x_n$ s.t.

$$b_i = \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, \cdots, m$$

$$b_i = \sum_{j=1}^{n} a_{ij} x_j + \varepsilon_i \quad i = 1, \cdots, m$$

error

$$b = AX + \varepsilon \iff \varepsilon = AX - b$$

L^2 - Regression (Least Square)

$$\min_{(x_1,\dots,x_n)} \sum_i \left[\left(b_i - \sum_j a_{ij} x_j \right)^2 \right]$$

$$\min_x \| b - Ax \|_2^2$$

$Ax = b$ (might not be solvable)

$A^T A \hat{x} = A^T b$ (normal equation, always solvable)

$\hat{x} = (A^T A)^{-1} A^T b$ (if $(A^T A)^{-1}$ exists)

L^1 - regression

$$\min_{(x_1,\cdots,x_n)} \sum_i \left| b_i - \sum_j a_{ij} x_j \right|$$

$$\min_x \| b - Ax \|_1$$

$$\min_{i} \sum_{i=1}^{m} t_i$$

$$\text{s.t.} \quad -t_i \le b_i - \sum_j a_{ij} x_j \le t_i$$

$$\left| b_i - \sum_j a_{ij} x_j \right| \le t_i$$

L^∞ - Regression

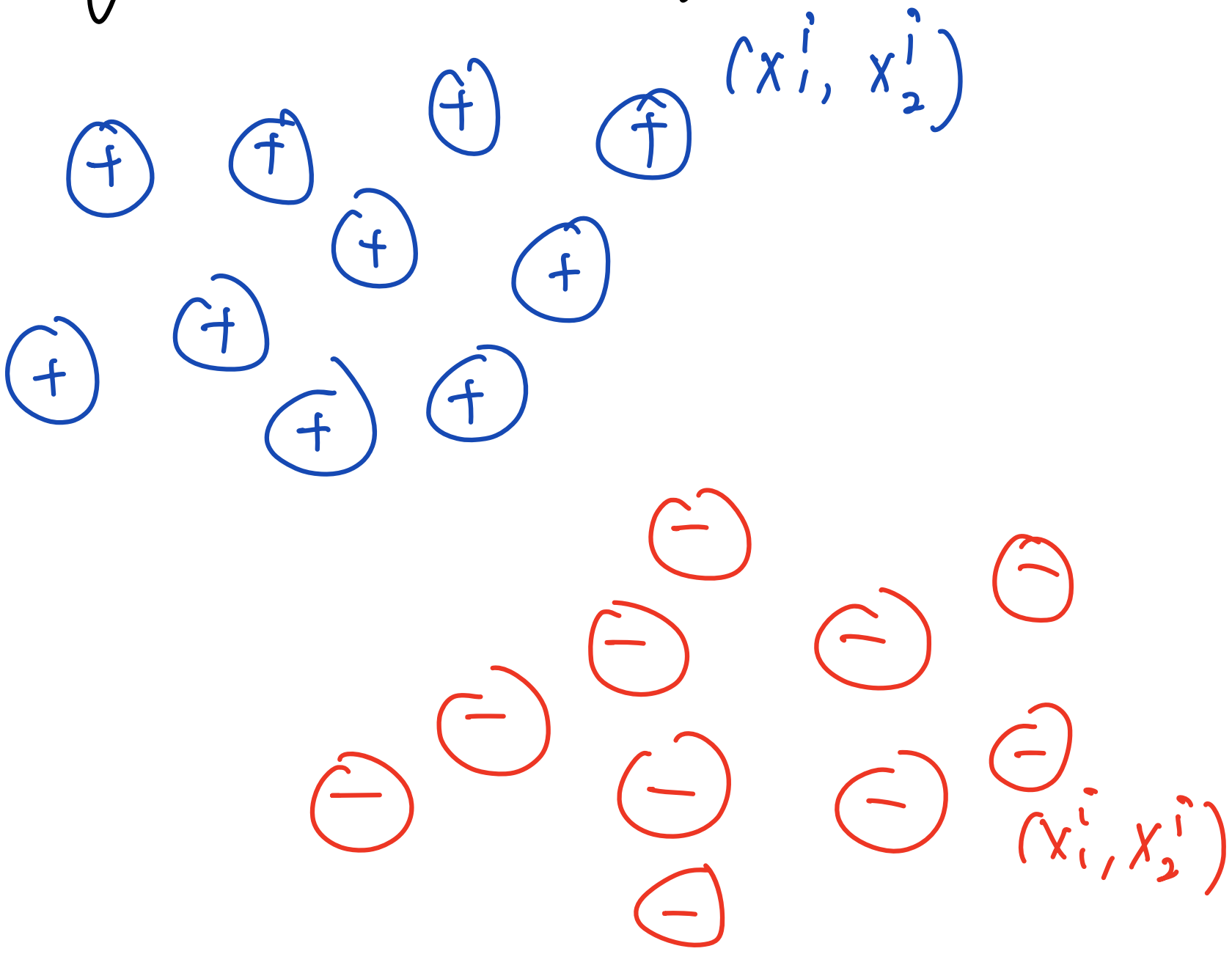
$$\min \left(\max_i |b_i - \sum_j a_{ij} x_j| \right)$$

$$\min_x \|b - Ax\|_\infty$$

$$\begin{aligned} &\min \quad \delta \\ &\text{s.t.} \quad -\delta \le b_i - \sum_j a_{ij} x_j \le \delta \end{aligned}$$

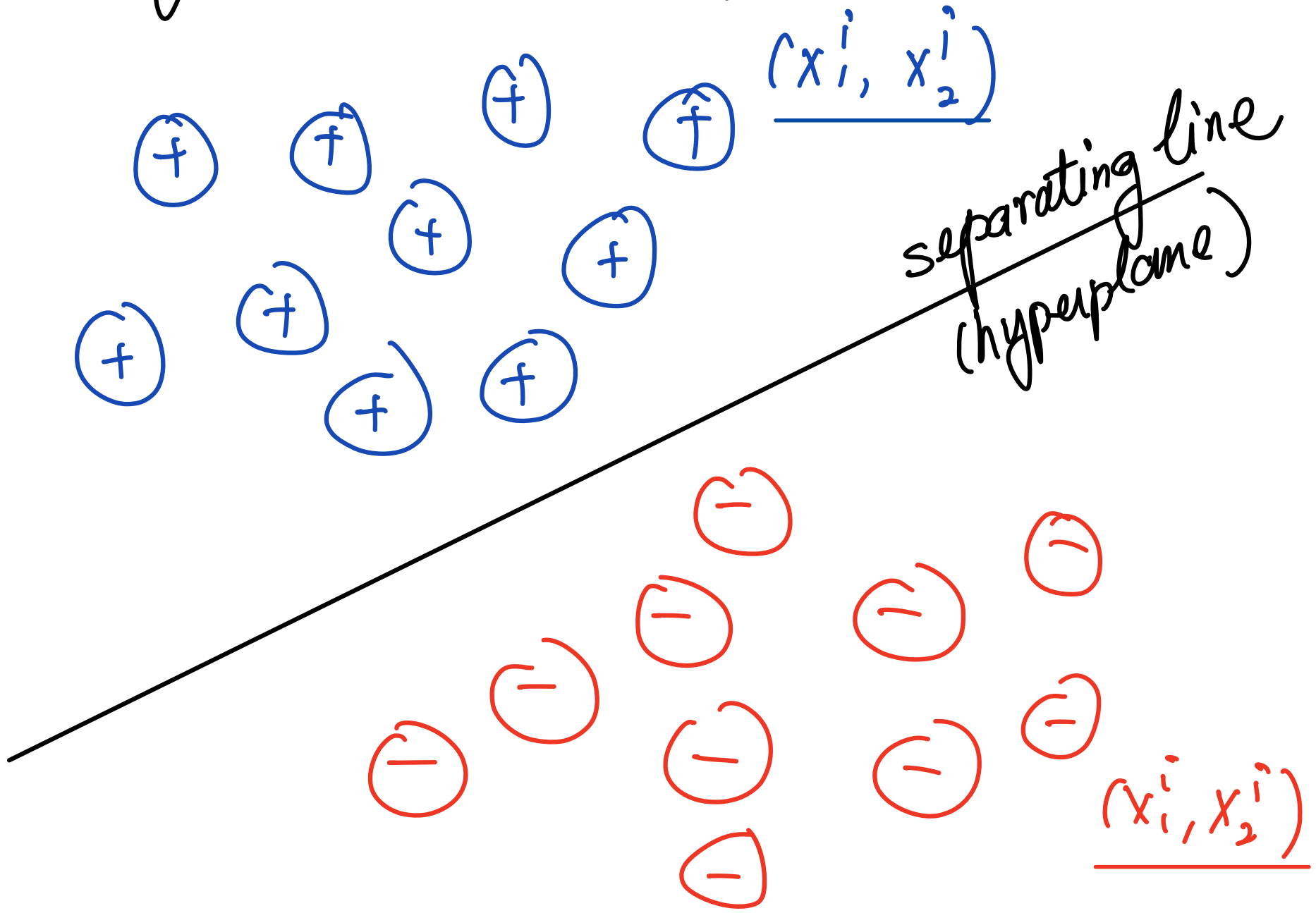
$|b_i - \sum_j a_{ij} x_j| \le \delta$ for all i

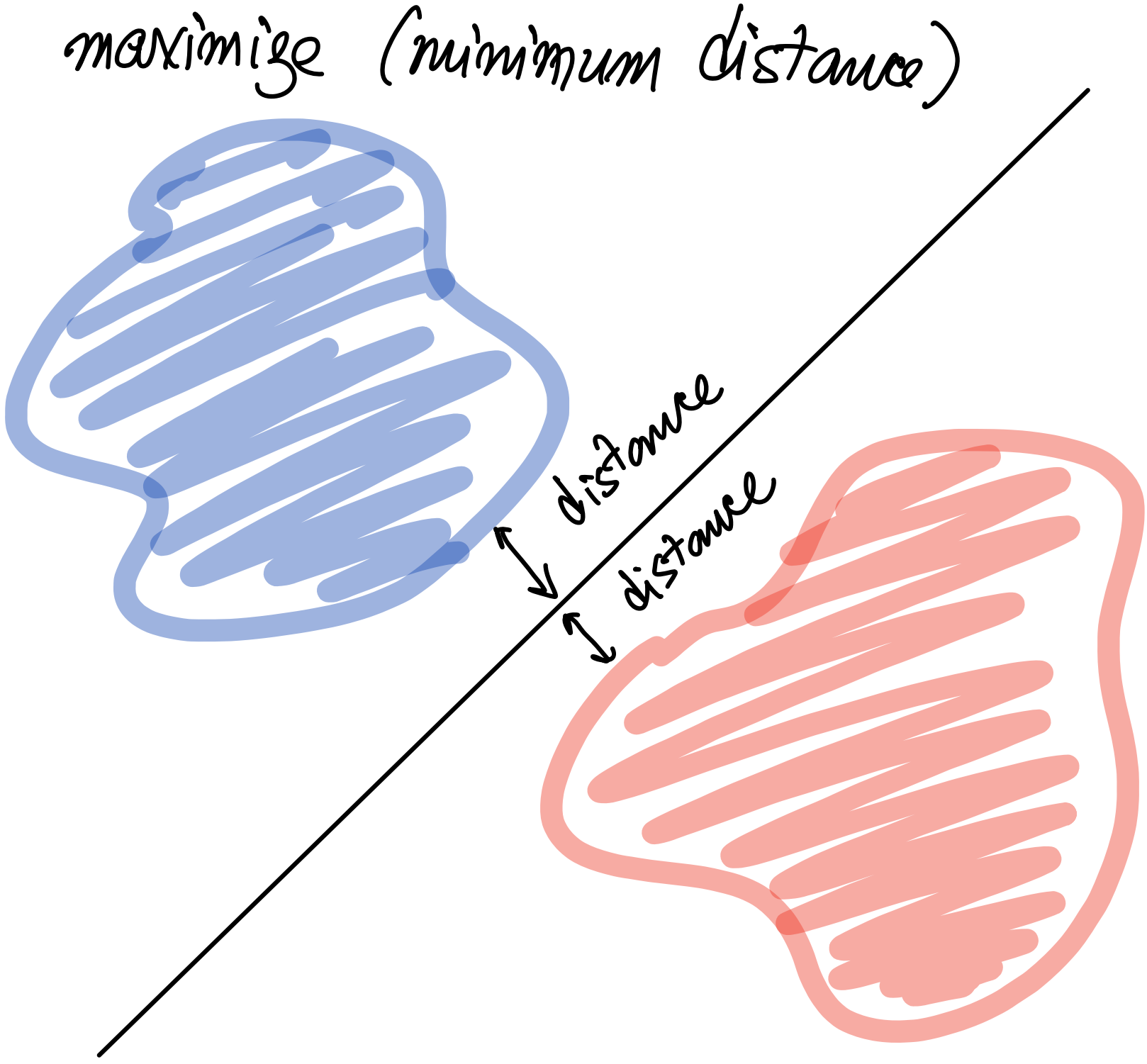
Binary Classification given labelled points

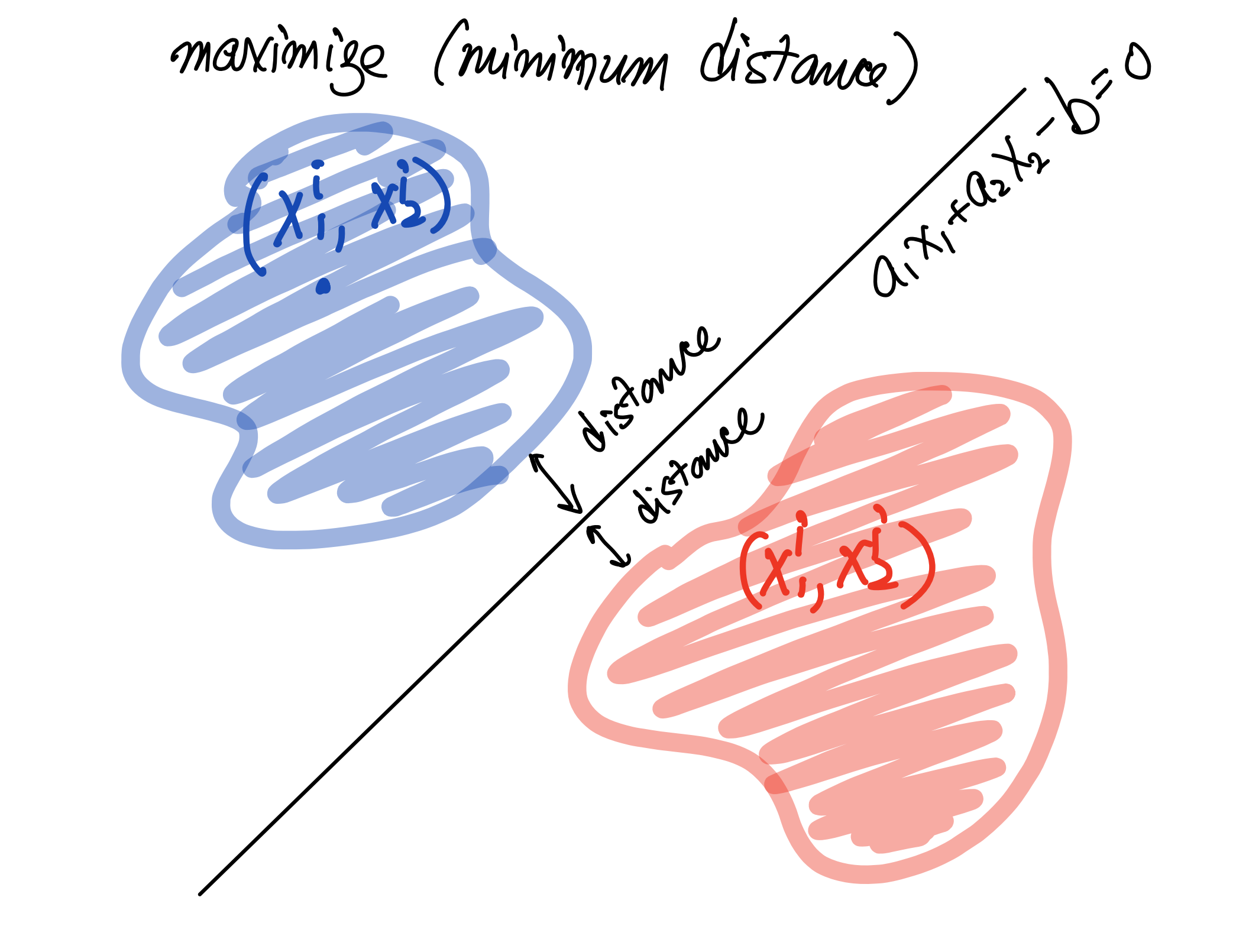


Binary Classification (supervised learning)

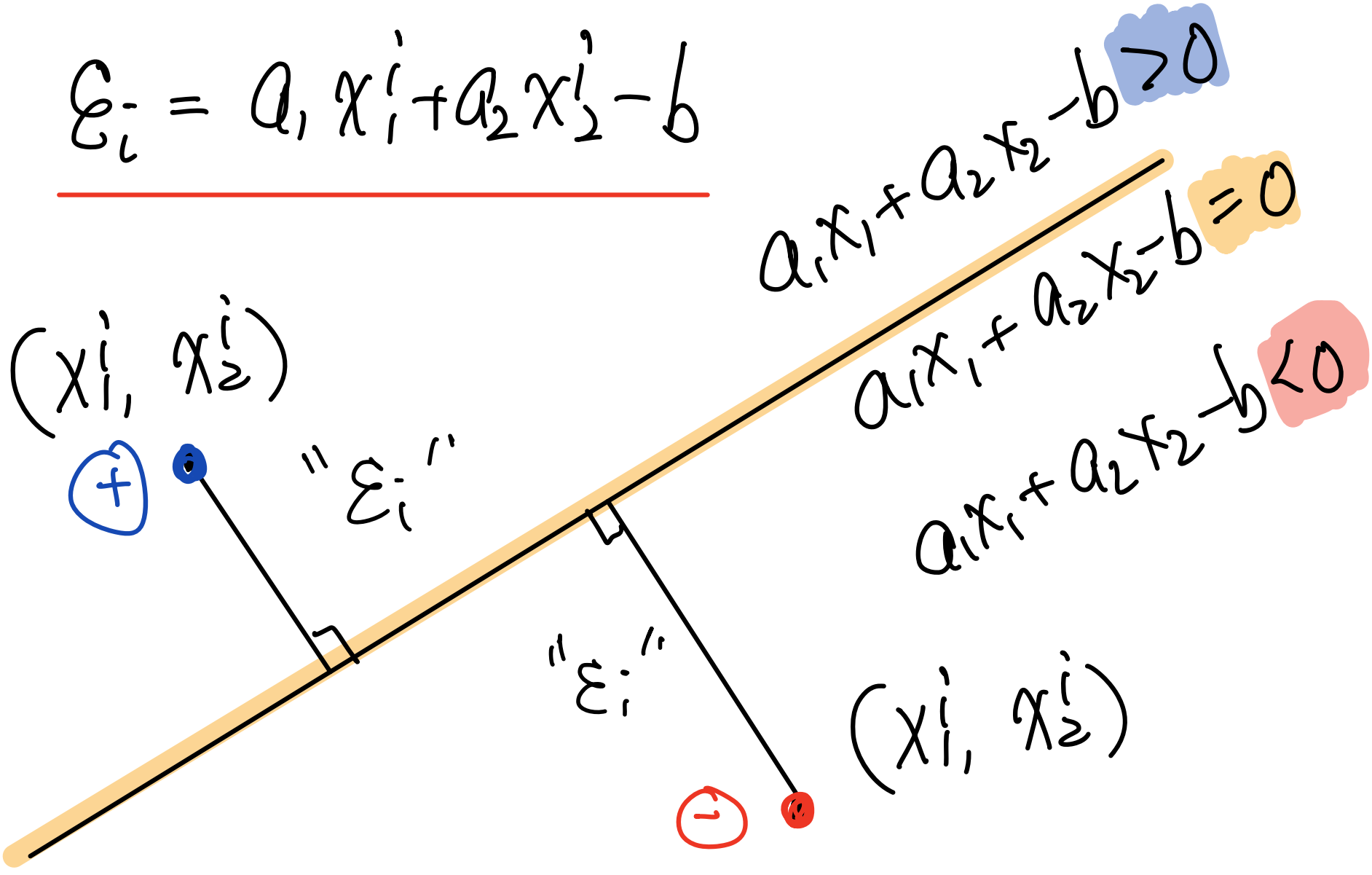
given labelled points




maximize (minimum distance)
distance
distance


maximize (minimum distance)
$a_1x_1+a_2x_2-b=0$
(x_1^i, x_2^i)
distance
distance
(x_1^j, x_2^j)

$$\varepsilon_i = a_1 x_1^i + a_2 x_2^i - b$$



$$\varepsilon_i = b - (a_1 x_1^i + a_2 x_2^i)$$

Distance of $\{\oplus\}$ from $a_1x_1 + a_2x_2 - b$

$$= \min_{i,\ \oplus} \left(a_1x_1^i + a_2x_2^i - b \right)$$

Distance of $\{\ominus\}$ from $a_1x_1 + a_2x_2 - b$

$$= \min_{i,\ \ominus} \left(b - \left(a_1x_1^i + a_2x_2^i \right) \right)$$

$$\max \delta$$

$$\delta, \varepsilon_i, a_1, a_2, b$$

s.t.

$$\varepsilon_i = a_1 x_1^i + a_2 x_2^i - b \quad \text{if Label}(i) = \oplus$$

$$\varepsilon_i = b - a_1 x_1^i - a_2 x_2^i \quad \text{if Label}(i) = \ominus$$

$$\delta \leq \varepsilon_i$$

$$\max \ \delta$$

$$\delta, \varepsilon_i, a_1, a_2, b$$

s.t.

$$\varepsilon_i = a_1 x_1^i + a_2 x_2^i - b \quad \text{if Label}(i) = \oplus$$

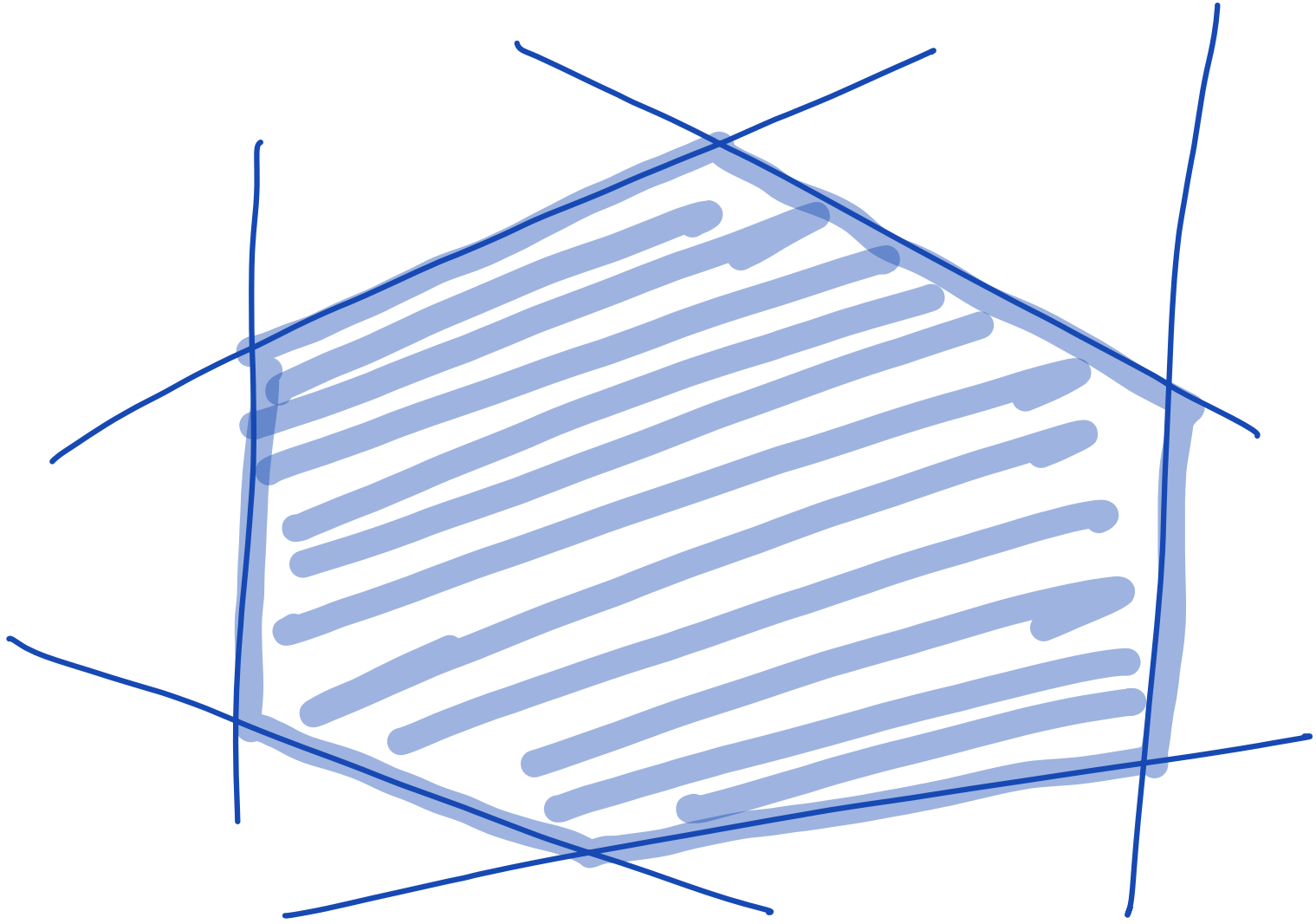
$$\varepsilon_i = b - a_1 x_1^i - a_2 x_2^i \quad \text{if Label}(i) = \ominus$$

$$\delta \leq \varepsilon_i$$

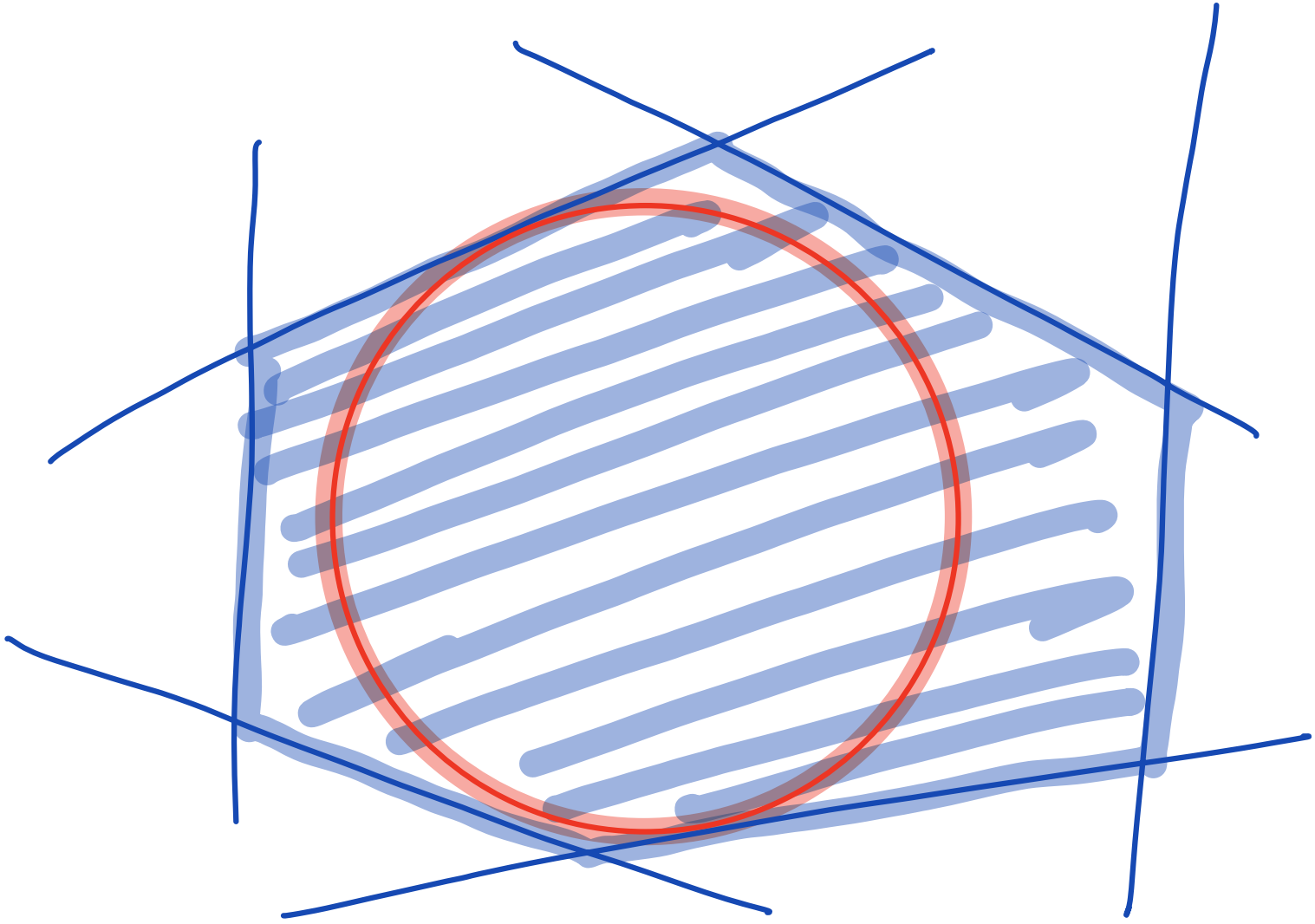
$$-1 \leq a_1 \leq 1, \quad -1 \leq a_2 \leq 1$$

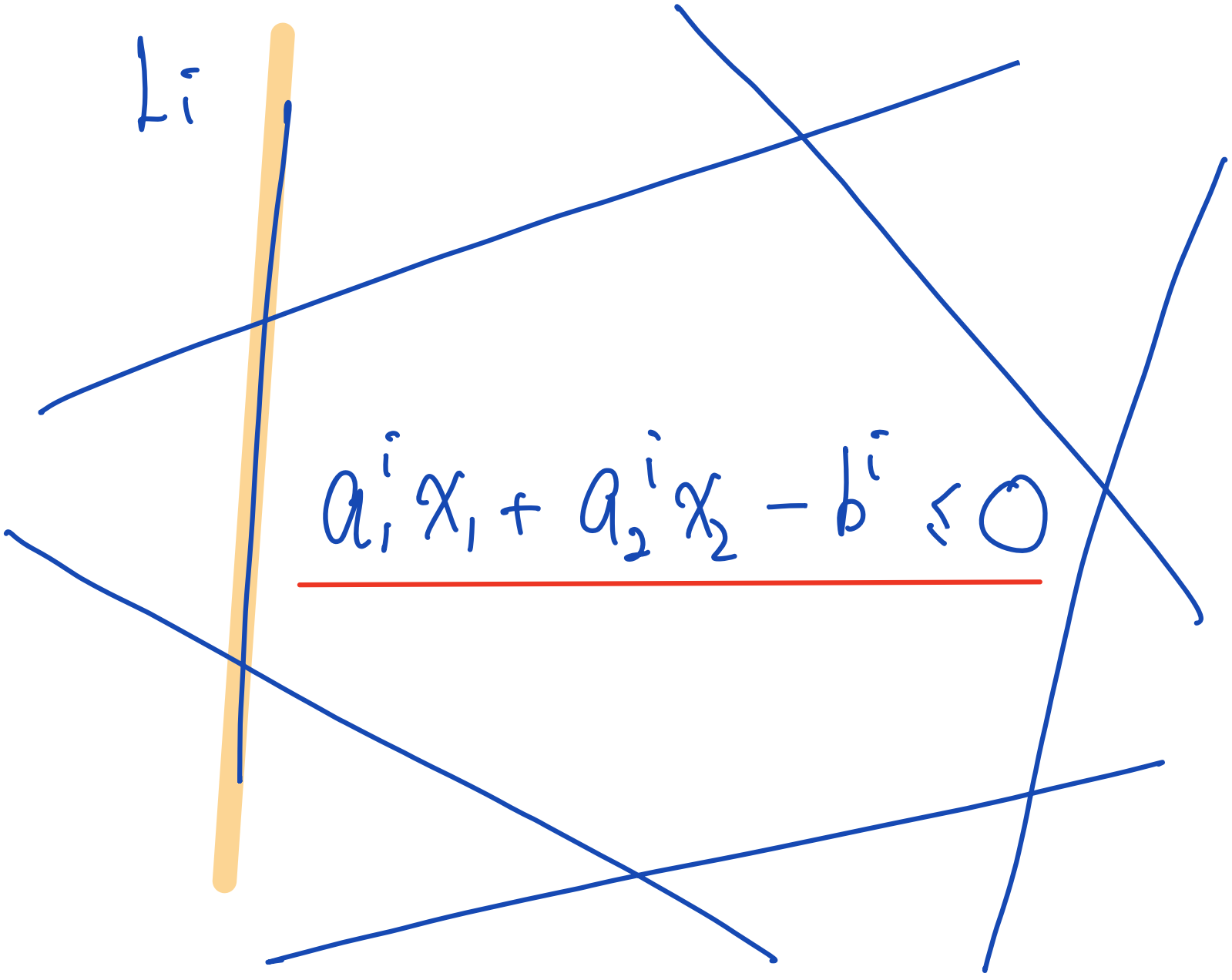
(normalization)

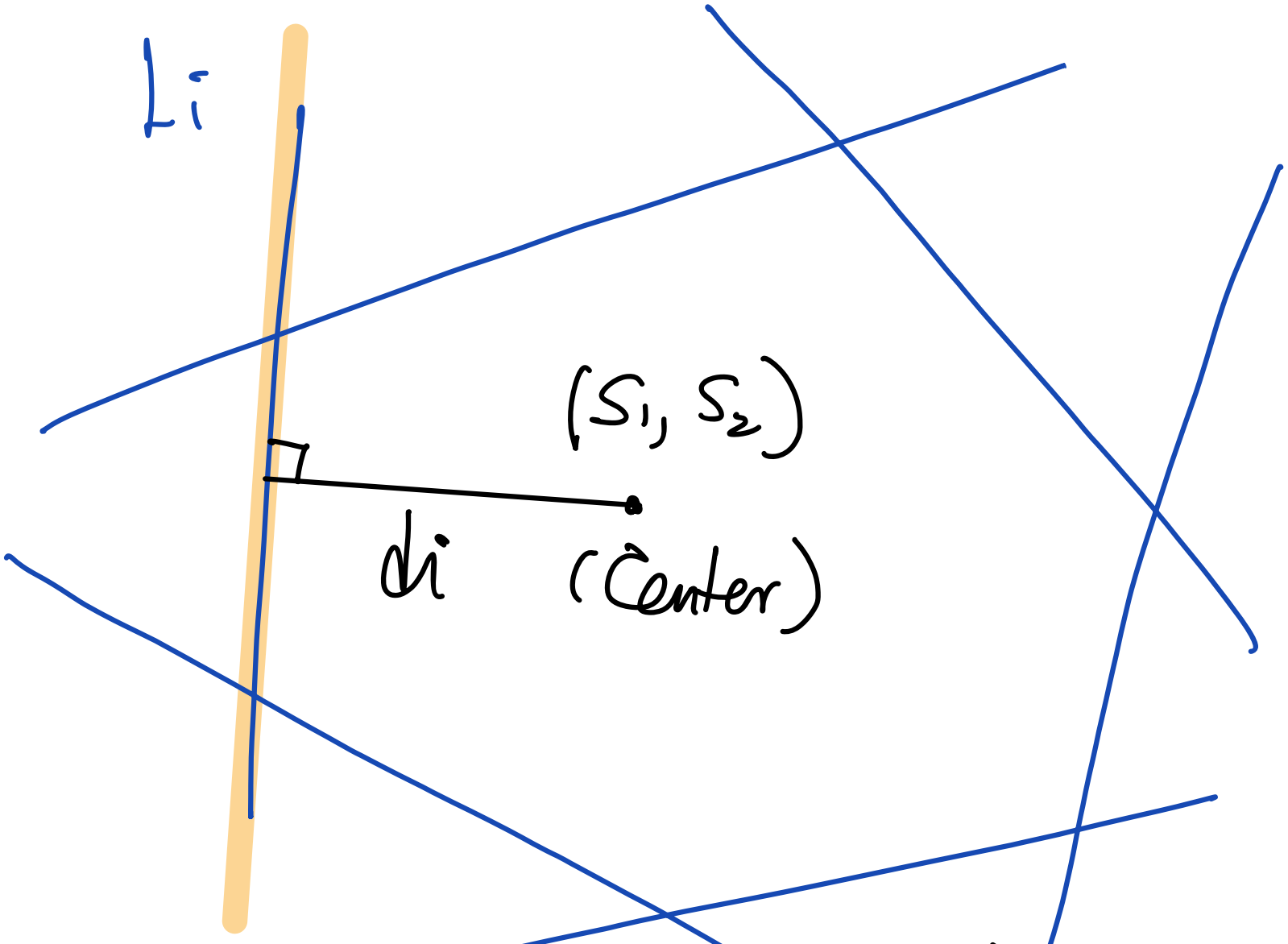
Maximum Inscribed Circle in a Convex Polyhedron [MG, Sec 2.6]



Maximum Inscribed Circle in a Convex Polyhedron [MG, Sec 2.6]




Li
$a_1^i x_1 + a_2^i x_2 - b^i \leq 0$



$$d_i = \left| \frac{a_1^i S_1 + a_2^i S_2 - b^i}{\sqrt{(a_1^i)^2 + (a_2^i)^2}} \right|$$

$$\text{maximize} \quad d$$

$$(s_1, s_2, d)$$

$$\text{s.t.} \quad d \leq d_i = \left| \frac{a_1^i s_1 + a_2^i s_2 - b^i}{\sqrt{(a_1^i)^2 + (a_2^i)^2}} \right|$$

$$a_1^i s_1 + a_2^i s_2 - b_i \leq 0$$

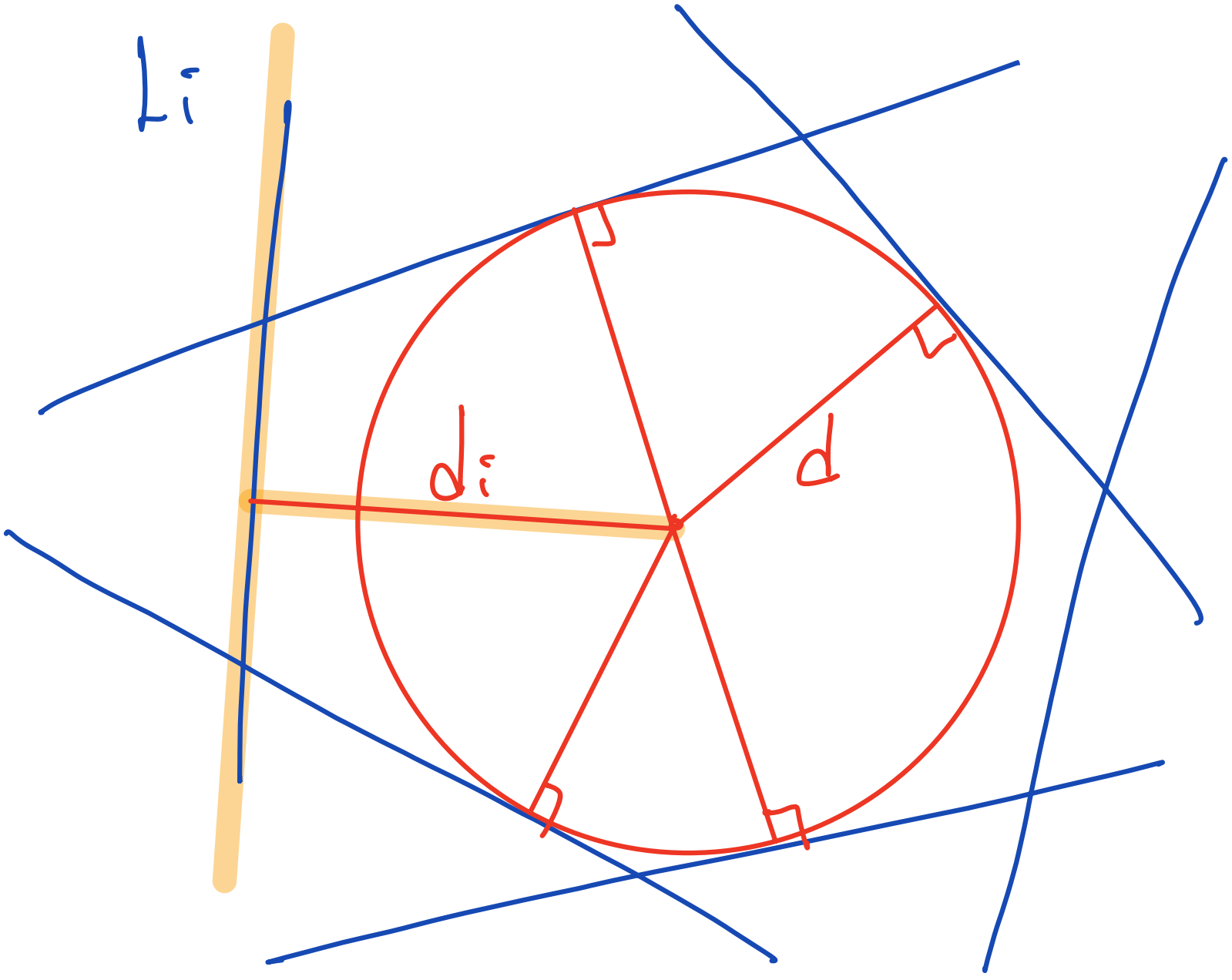
maximize d

(s_1, s_2, d)

s.t. $$d \leq \frac{b^i - a_1^i s_1 - a_2^i s_2}{\sqrt{(a_1^i)^2 + (a_2^i)^2}}$$

~~$a_1^i s_1 + a_2^i s_2 - b_i \leq 0$~~

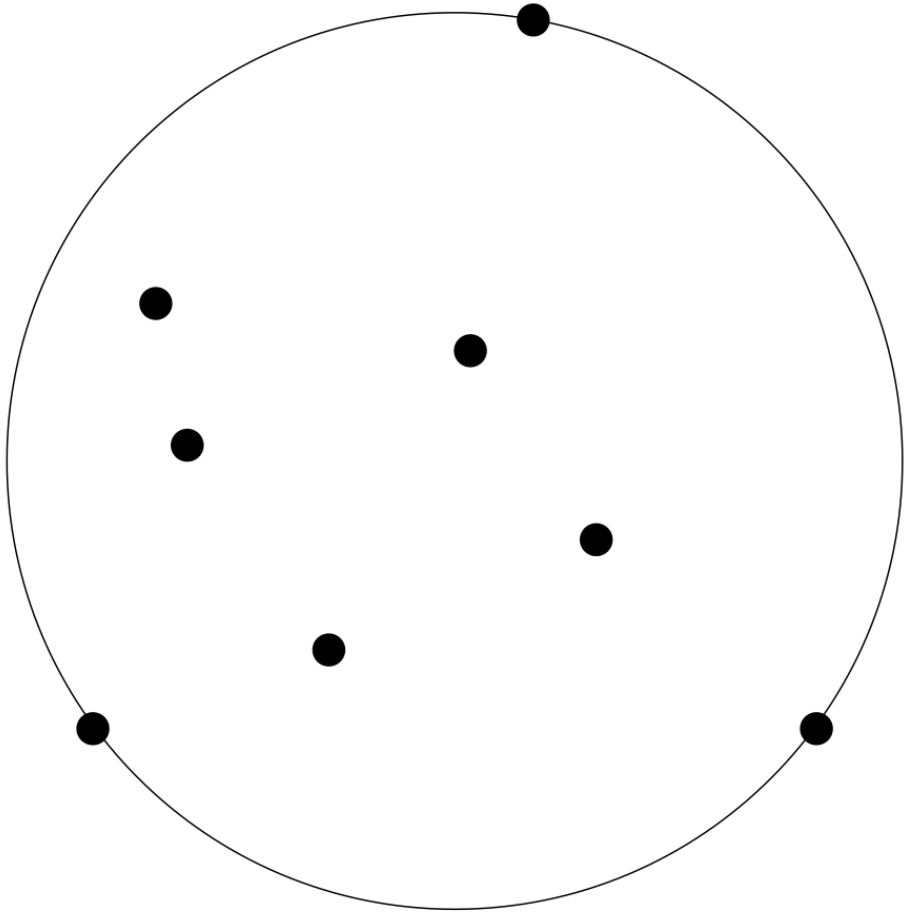
redundant


L_i
d_i
d

[MG, p. 184]

8.7 Smallest Balls and Convex Programming

The smallest ball problem. We are given points $\mathbf{p}_1, \ldots, \mathbf{p}_n \in \mathbb{R}^d$, and we want to find a ball of the smallest radius that contains all the points.[8]



Sparse Solutions of Linear System [MG, p.167]

Sparse solution of underdetermined system of linear equations

Given an $m \times n$ matrix A with $m < n$, a vector $\mathbf{b} \in \mathbb{R}^m$, and an integer r, find an $\mathbf{x} \in \mathbb{R}^n$ such that

$$A\mathbf{x} = \mathbf{b} \quad \text{and} \quad |\text{supp}(\mathbf{x})| \leq r \tag{8.10}$$

if one exists.

$\#\{i : x_i \neq 0\}$

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if one exists.

$\#\{i : x_i \neq 0\}$

Recovery of signals with noise:

$w \in R^k \Longrightarrow$ encode $z = Qw$

$\Longrightarrow$ transmit $\tilde{z} = z + x$ ← noise: only a few zeros

Given $\tilde{z}$, how to recover z?

$\tilde{z} = Qw + x$

$A\tilde{z} = AQw + Ax$

Sparse Solutions of Linear System [MG, p.167]

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$$A\mathbf{x} = \mathbf{b} \quad \text{and} \quad |\text{supp}(\mathbf{x})| \leq r \tag{8.10}$$

if one exists.

$\#\{i : x_i \neq 0\}$

$\tilde{A}\tilde{z} = AQW + AX \Longrightarrow AX = b$

$\tilde{A}\tilde{z}$: b given

find X with as few non-zeros as possible

$\tilde{z} = z + X \Longrightarrow z = \tilde{z} - X$

Sparse Solutions of Linear System [MG, p.167]

Sparse solution of underdetermined system of linear equations

Given an $m \times n$ matrix A with $m < n$, a vector $\mathbf{b} \in \mathbb{R}^m$, and an integer r, find an $\mathbf{x} \in \mathbb{R}^n$ such that

$$A\mathbf{x} = \mathbf{b} \quad \text{and} \quad |\text{supp}(\mathbf{x})| \leq r \tag{8.10}$$

if one exists.

$\#\{i : x_i \neq 0\}$

Data Compression:

$$AX = b \qquad A^{m \times n},\ b \in \mathbb{R}^m$$

representation ↗ ↖ data

$$x_1 u_1 + x_2 u_2 + \cdots + x_n u_n = b$$

coordinates of b w.r.t. cols of A

Sparse Solutions of Linear System [MG, p.167]

Sparse solution of underdetermined system of linear equations

Given an $m \times n$ matrix A with $m < n$, a vector $\mathbf{b} \in \mathbb{R}^m$, and an integer r, find an $\mathbf{x} \in \mathbb{R}^n$ such that

$$A\mathbf{x} = \mathbf{b} \quad \text{and} \quad |\text{supp}(\mathbf{x})| \leq r \tag{8.10}$$

if one exists.

$\#\{i : x_i \neq 0\}$

$$\min \|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$
$$\text{s.t.} \quad Ax = b$$

Sparse Solutions of Linear System [MG, p.167]

Sparse solution of underdetermined system of linear equations

Given an $m \times n$ matrix A with $m < n$, a vector $\mathbf{b} \in \mathbb{R}^m$, and an integer r, find an $\mathbf{x} \in \mathbb{R}^n$ such that

$$A\mathbf{x} = \mathbf{b} \quad \text{and} \quad |\text{supp}(\mathbf{x})| \leq r \tag{8.10}$$

if one exists.

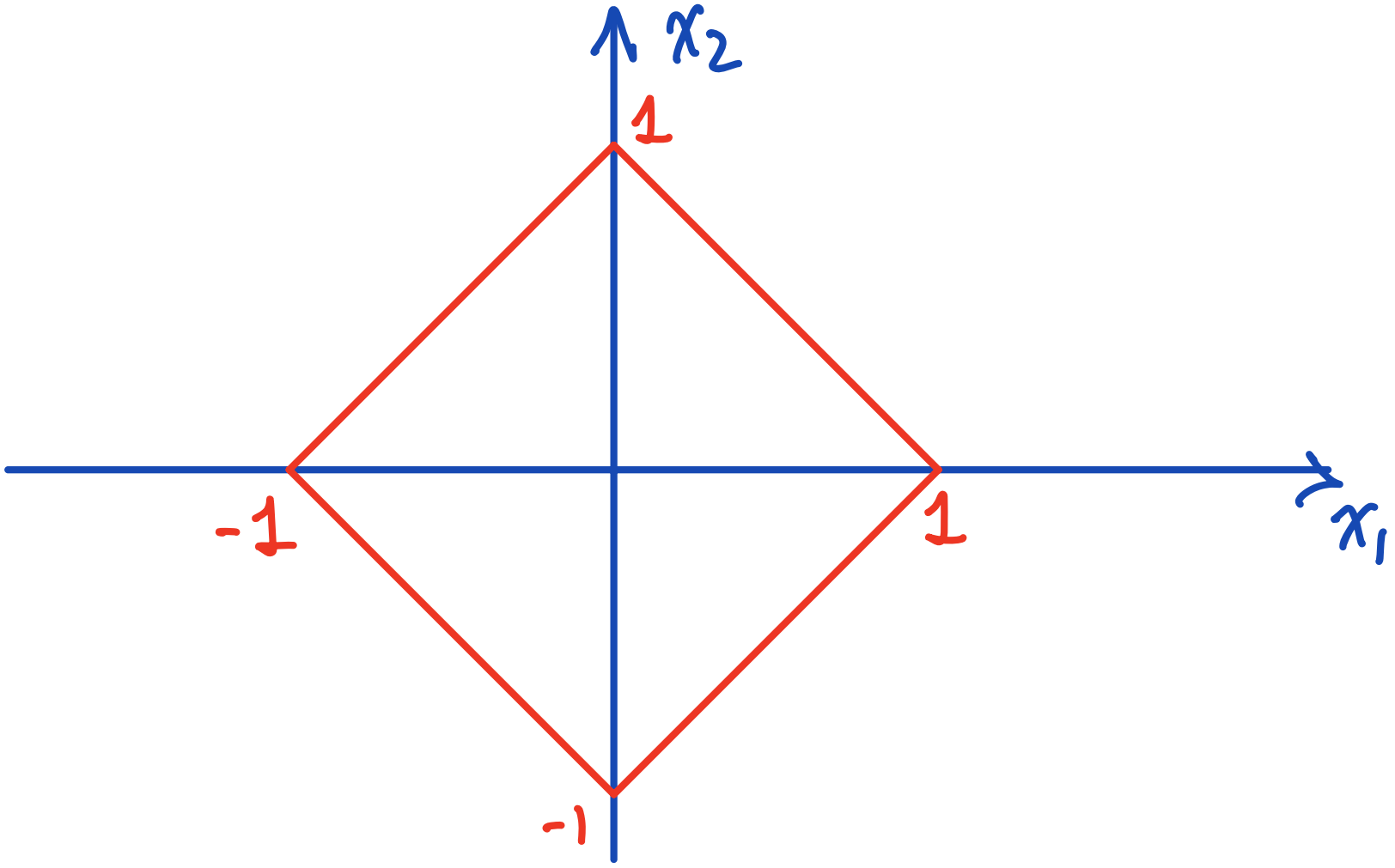
$\#\{i : x_i \neq 0\}$

$$\min \ \|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$
$$\text{s.t.} \ Ax = b$$

$$\min \ u_1 + u_2 + \cdots + u_n$$
$$\text{s.t.} \ -u_i \leq x_i \leq u_i, \ i = 1, 2, \cdots, n$$
$$Ax = b$$

Sparse Solutions of Linear System [MG, p.167]

$n=2$ $\quad \|x\|_1 = |x_1| + |x_2| = 1$



Sparse Solutions of Linear System [MG, p.167]

$$\min \|x\|_1 \quad \text{s.t.} \quad Ax = b$$

