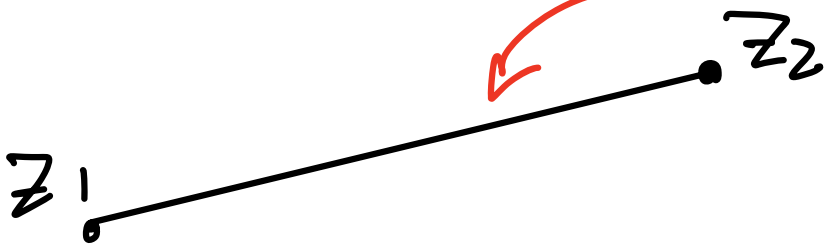
Convex Analysis [V] Chapter 10

Def 1 $S \subseteq \mathbb{R}^m$ is **convex** if

for any $z_1, z_2 \in S$, $\Rightarrow$ $t z_1 + (1-t) z_2 \in S$

$0 \le t \le 1$

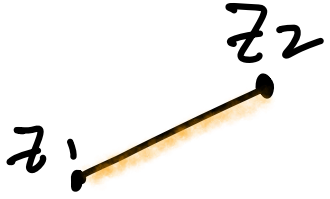


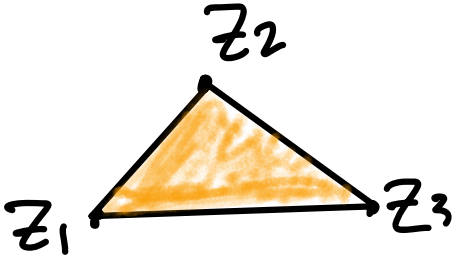
Def 2 Convex combination of $z_1, z_2, \cdots, z_n \in \mathbb{R}^m$

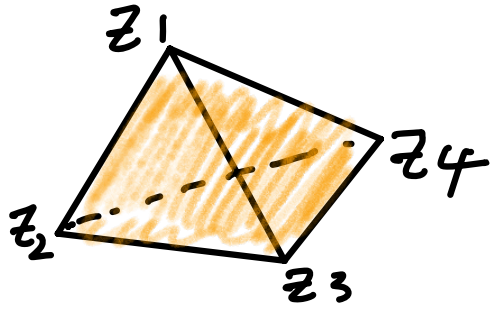
$t_1 z_1 + t_2 z_2 + \cdots + t_n z_n$

$t_i \ge 0$

$\sum_i t_i = 1$







Thm 10.1 S is convex iff it contains all conv. comb. of points in S

pf "$\Leftarrow$"

Suppose S contain all conv. comb. of pts in S,

then clearly, for any $z_1, z_2 \in S$

$$\underbrace{tz_1 + (1-t)z_2}_{\text{conv. comb. of } z_1, z_2} \in S$$

$\Rightarrow$ S is convex

Thm 10.1 S is convex iff it contains all conv. comb. of points in S

pf "$\Rightarrow$" Suppose S is convex.

$n=2$: $z_1, z_2 \in S \Rightarrow t_1 z_1 + t_2 z_2 \in S$, $t_1, t_2 \geq 0$, $t_1 + t_2 = 1$

$n=3$: $z_1, z_2, z_3 \in S \Rightarrow$

$$t_1 z_1 + t_2 z_2 + t_3 z_3 = (t_1 + t_2)\left(\underbrace{\frac{t_1}{t_1+t_2} z_1 + \frac{t_2}{t_1+t_2} z_2}_{\in S}\right) + t_3 z_3 \in S$$

$t_1 + t_2 + t_3 = 1$, $t_1 + t_2 \geq 0$, $t_3 \geq 0$

Thm 10.1 S is convex iff it contains all conv. comb. of points in S

Pf "$\Rightarrow$" Suppose S is convex.

$n=2$: $z_1, z_2 \in S \Rightarrow t_1 z_1 + t_2 z_2 \in S$, $t_1, t_2 \geq 0$, $t_1 + t_2 = 1$

$n=4$: $z_1, z_2, z_3, z_4 \in S$

$$t_1 z_1 + t_2 z_2 + t_3 z_3 + t_4 z_4 = (t_1 + t_2 + t_3)\left(\frac{t_1 z_1 + t_2 z_2 + t_3 z_3}{t_1 + t_2 + t_3}\right) + t_4 z_3 \in S$$

($\frac{t_1 z_1 + t_2 z_2 + t_3 z_3}{t_1 + t_2 + t_3} \in S$)

$(t_1 + t_2 + t_3) + t_4 = 1$

$t_1 + t_2 + t_3 \geq 0$, $t_4 \geq 0$

Def $S \subseteq \mathbb{R}^n$, Convex Hull of S,

$$\text{Conv}(S) = \bigcap_{G \text{ convex}, S \subseteq G} G$$

(= Smallest convex set containing S)

(Note intersection of convex set is convex.)

Thm 10.2 $S \subseteq \mathbb{R}^n$, $\text{Conv}(S)$ = collection of all conv. comb. of finitely many pts from S

Thm 10.2 $S \subseteq \mathbb{R}^m$, $\text{Conv}(S)$ = collection of all conv. comb. of finitely many pts from S

Pf. Let H = collection of all conv. comb. of finitely many pts from S

Need to show $\text{Conv}(S) = H$

(i) $\text{Conv}(S) \subseteq H$ ($\because H$ is conv.)

Thm 10.2 $S \subseteq \mathbb{R}^m$, $\text{Conv}(S)$ = collection of all conv. comb. of finitely many pts from S

Pf. Let H = collection of all conv. comb. of finitely many pts from S

Need to show $\text{Conv}(S) = H$

(2) $H \subseteq \text{Conv}(S)$

↖ Thm 10.1, contains all conv. comb. of pts from $\text{Conv}(S)$ and hence from S

Caratheodory Theorem

THEOREM 10.3. *The convex hull* $\text{conv}(S)$ *of a set* S *in* $\mathbb{R}^m$ *consists of all convex combinations of* $m+1$ *points from* S*:*

$$\text{conv}(S) = \left\{ z = \sum_{j=1}^{m+1} t_j z_j : z_j \in S \text{ and } t_j \geq 0 \text{ for all } j, \text{ and } \sum_j t_j = 1 \right\}.$$

Pf $\text{Conv}(S) = \left\{ z = \sum_{j=1}^{n} x_j z_j,\ z_j \in S,\ x_j \geq 0,\ \sum_j x_j = 1 \right\}$

Consider the LP: $\max\ c^T X$

$$\text{s.t. } AX = Z$$

$$x_1 + x_2 + \cdots + x_n = 1$$

$$x_1, x_2, \ldots, x_n \geq 0$$

(Z is given, fixed, $A = [z_1\ z_2\ \cdots\ z_n]^{m \times n}$)

Caratheodory Theorem

THEOREM 10.3. *The convex hull* $\text{conv}(S)$ *of a set* S *in* $\mathbb{R}^m$ *consists of all convex combinations of* $m+1$ *points from* S*:*

$$\text{conv}(S) = \left\{ z = \sum_{j=1}^{m+1} t_j z_j : z_j \in S \text{ and } t_j \geq 0 \text{ for all } j, \text{ and } \sum_j t_j = 1 \right\}.$$

Pf $\text{Conv}(S) = \left\{ z = \sum_{j=1}^{n} x_j z_j,\ z_j \in S,\ x_j \geq 0,\ \sum_j x_j = 1 \right\}$

Consider the LP: max $c^T X$

s.t. $AX = Z$

$x_1 + x_2 + \cdots + x_n = 1$ } $m+1$ constraints.

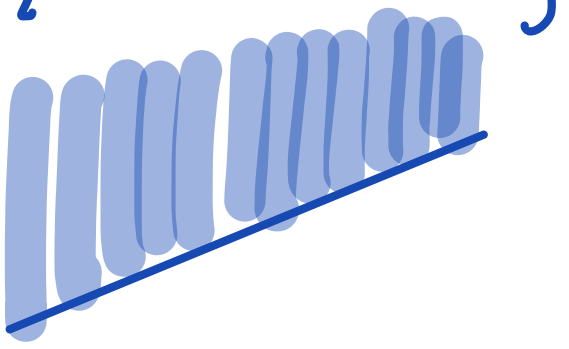
$x_1, x_2, \ldots, x_n \geq 0$

This is feasible, and hence has a basic opt. soln with $m+1$ basic variables.

Separation Theorem

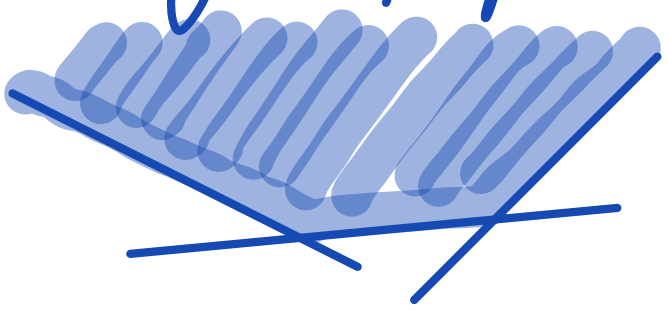
THEOREM 10.4. *Let P and $\tilde{P}$ be two disjoint nonempty polyhedra in $\mathbb{R}^n$. Then there exist disjoint half-spaces H and $\tilde{H}$ such that $P \subset H$ and $\tilde{P} \subset \tilde{H}$.*

$\{x: v^T x \leq c\}$



intersection of finitely many half spaces

$\{x: Ax \leq b\}$



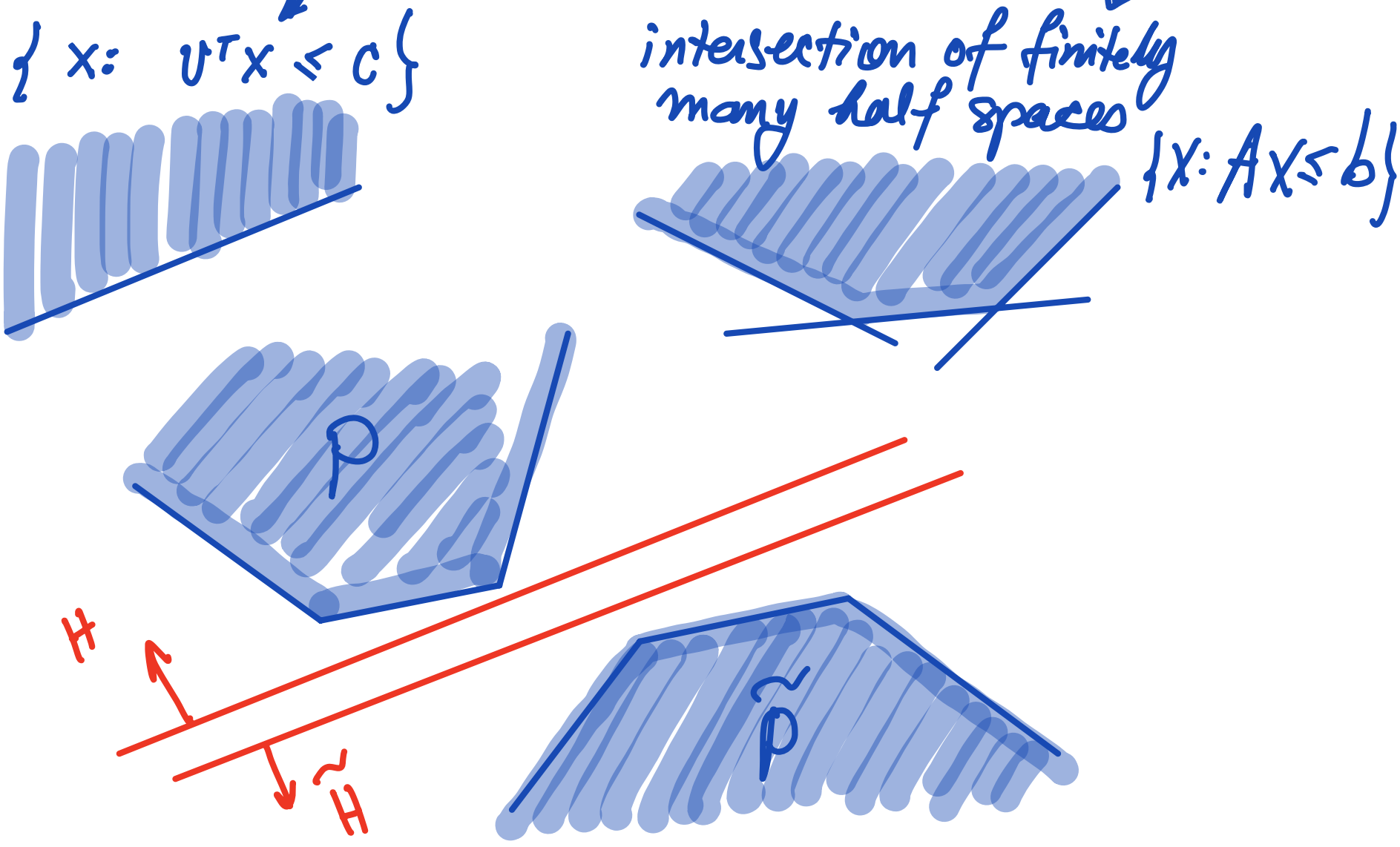
Separation Theorem

THEOREM 10.4. *Let P and $\tilde{P}$ be two disjoint nonempty polyhedra in $\mathbb{R}^n$. Then there exist disjoint half-spaces H and $\tilde{H}$ such that $P \subset H$ and $\tilde{P} \subset \tilde{H}$.*

$\{x: v^T x \leq c\}$

intersection of finitely many half spaces

$\{x: Ax \leq b\}$



Separation Theorem

THEOREM 10.4. *Let P and $\tilde{P}$ be two disjoint nonempty polyhedra in $\mathbb{R}^n$. Then there exist disjoint half-spaces H and $\tilde{H}$ such that $P \subset H$ and $\tilde{P} \subset \tilde{H}$.*

Pf $P = \{x : Ax \leq b\}$, $\tilde{P} = \{x : \tilde{A}x \leq \tilde{b}\}$

$P \cap \tilde{P} = \phi \iff$ $\begin{cases} Ax \leq b \\ \tilde{A}x \leq \tilde{b} \end{cases}$ has no solution

$\iff$ (Farka's Lemma)

There is a $y, \tilde{y}$ s.t.

$y^T A + \tilde{y}^T \tilde{A} = 0$

$y^T b + \tilde{y}^T \tilde{b} < 0$

$y, \tilde{y} \geq 0$

Separation Theorem

THEOREM 10.4. *Let P and $\tilde{P}$ be two disjoint nonempty polyhedra in $\mathbb{R}^n$. Then there exist disjoint half-spaces H and $\tilde{H}$ such that $P \subset H$ and $\tilde{P} \subset \tilde{H}$.*

Pf $P = \{x : Ax \leq b\}$, $\tilde{P} = \{x : \tilde{A}x \leq \tilde{b}\}$

$P \cap \tilde{P} = \emptyset \iff$ $\begin{cases} Ax \leq b \\ \tilde{A}x \leq \tilde{b} \end{cases}$ has no solution

Farkas' Lemma

LEMMA 10.5. *The system $Ax \leq b$ has no solutions if and only if there is a y such that*

$$\begin{aligned} A^T y &= 0 \\ y &\geq 0 \\ b^T y &< 0. \end{aligned} \tag{10.8}$$

$\left(\iff \begin{array}{c} y^T A = 0 \\ y \geqslant 0 \\ y^T b < 0 \end{array} \right)$

Separation Theorem

THEOREM 10.4. *Let P and $\tilde{P}$ be two disjoint nonempty polyhedra in $\mathbb{R}^n$. Then there exist disjoint half-spaces H and $\tilde{H}$ such that $P \subset H$ and $\tilde{P} \subset \tilde{H}$.*

Pf $P = \{x : Ax \leq b\}$, $\tilde{P} = \{x : \tilde{A}x \leq \tilde{b}\}$

There is a $y, \tilde{y}$ s.t.

$$\begin{cases} y^T A + \tilde{y}^T \tilde{A} = 0 \\ y^T b + \tilde{y}^T \tilde{b} < 0 \\ y, \tilde{y} \geq 0 \end{cases}$$

Define

$$H := \{x : y^T A x \leq y^T b\}, \quad \tilde{H} := \{x : \tilde{y}^T \tilde{A} x \leq \tilde{y}^T \tilde{b}\}$$

(textbook) $\tilde{y}^T A = -y^T A$ $\rightarrow$ $= \{x : y^T A x \geq -\tilde{y}^T \tilde{b}\}$

Separation Theorem

THEOREM 10.4. *Let P and $\tilde{P}$ be two disjoint nonempty polyhedra in $\mathbb{R}^n$. Then there exist disjoint half-spaces H and $\tilde{H}$ such that $P \subset H$ and $\tilde{P} \subset \tilde{H}$.*

Pf $P = \{x : Ax \leq b\}$, $\tilde{P} = \{x : \tilde{A}x \leq \tilde{b}\}$

There is a $y, \tilde{y}$ s.t.

$$\begin{cases} y^T A + \tilde{y}^T \tilde{A} = 0 \\ y^T b + \tilde{y}^T \tilde{b} < 0 \\ y, \tilde{y} \geq 0 \end{cases}$$

Define

$$H := \{x : y^T A x \leq y^T b\}, \quad \tilde{H} := \{x : \tilde{y}^T \tilde{A} x \leq \tilde{y}^T \tilde{b}\}$$

$H \cap \tilde{H} = \emptyset$, $P \subseteq H$, $\tilde{P} \subseteq \tilde{H}$

Separation Theorem

$$P = \{x : Ax \leq b\}, \quad \tilde{P} = \{x : \tilde{A}x \leq \tilde{b}\}$$

$$H := \{x : y^T A x \leq y^T b\}, \quad \tilde{H} := \{x : \tilde{y}^T \tilde{A} x \leq \tilde{y}^T \tilde{b}\}$$

$P \subseteq H$: $\quad Ax \leq b \Rightarrow y^T A x \leq y^T b \quad (\because y \geq 0)$

$\tilde{P} \subseteq \tilde{H}$: $\quad \tilde{A}x \leq \tilde{b} \Rightarrow \tilde{y}^T A x \leq \tilde{y}^T b \quad (\because \tilde{y} \geq 0)$

$H \cap \tilde{H} = \phi$:

$$\begin{array}{ll} & y^T A x \leq y^T b \\ + & \tilde{y}^T \tilde{A} x \leq \tilde{y}^T b \\ \hline \end{array}$$

$$0 = \underbrace{(y^T A + \tilde{y}^T \tilde{A})}_{0} x \leq y^T b + \tilde{y}^T b < 0 \quad !!$$

Farkas' Lemma

LEMMA 10.5. *The system $Ax \leq b$ has no solutions if and only if there is a y such that*

$$\begin{aligned} A^T y &= 0 \\ y &\geq 0 \\ b^T y &< 0. \end{aligned} \tag{10.8}$$

PROOF. Consider the linear program

(P)
$$\begin{aligned} \text{maximize} \quad & 0 \\ \text{subject to} \quad & Ax \leq b \end{aligned}$$

and its dual

(D)
$$\begin{aligned} \text{minimize} \quad & b^T y \\ \text{subject to} \quad & A^T y = 0 \\ & y \geq 0. \end{aligned}$$