Inequality Constrained Opt [E, ch.4] [CZ ch.21]

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad \vec{h}(x) = \vec{0} \in \mathbb{R}^k, \quad \vec{g}(x) \leq \vec{0} \in \mathbb{R}^l$$

[CZ]

Theorem 21.1 ***Karush-Kuhn-Tucker (KKT) Theorem.*** *Let $f, \boldsymbol{h}, \boldsymbol{g} \in C^1$. Let $\boldsymbol{x}^*$ be a regular point and a local minimizer for the problem of minimizing f subject to $\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}$, $\boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0}$. Then, there exist $\boldsymbol{\lambda}^* \in \mathbb{R}^m$ and $\boldsymbol{\mu}^* \in \mathbb{R}^p$ such that:*

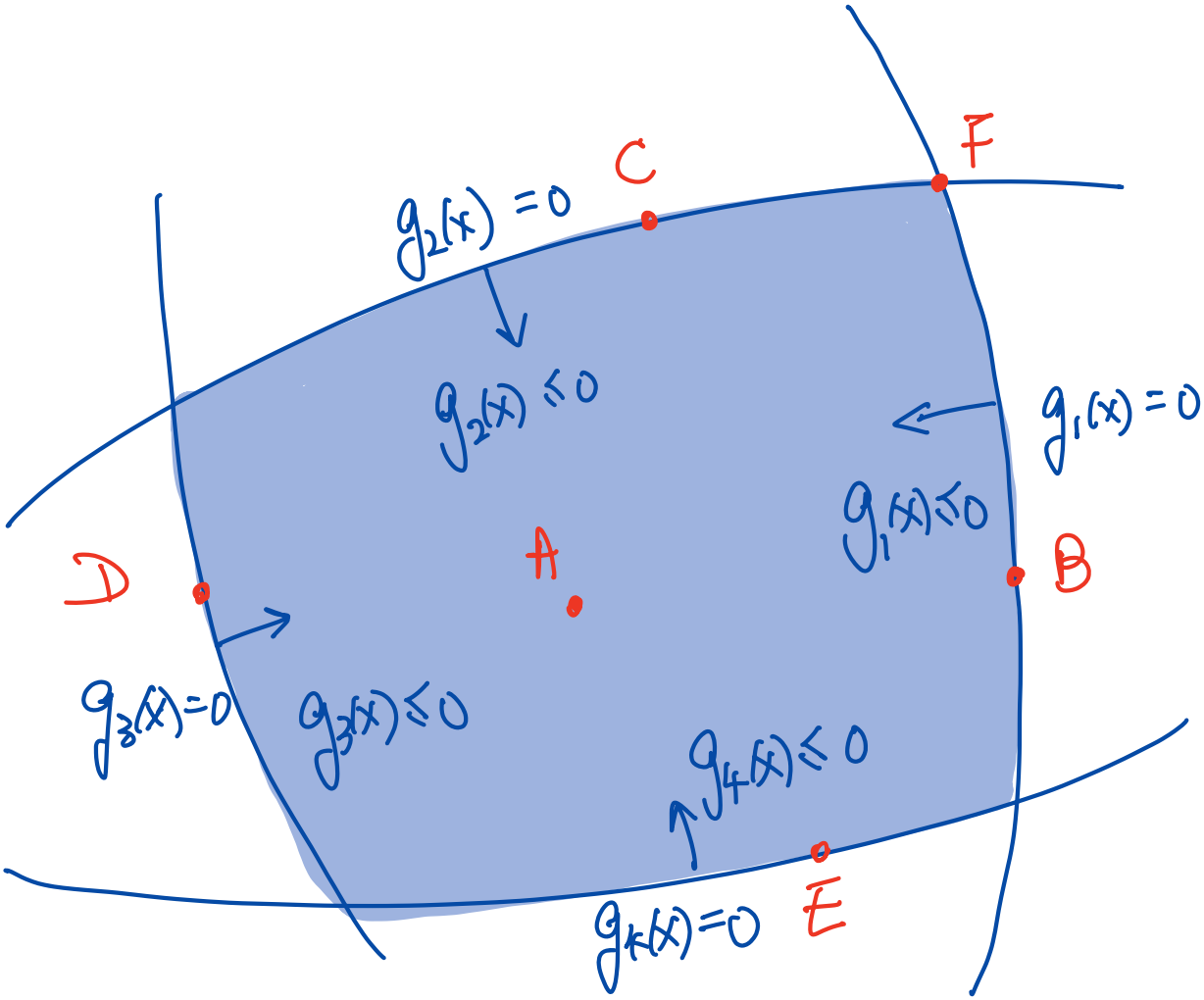
1. $\boldsymbol{\mu}^* \geq \boldsymbol{0}$.
2. $D\boldsymbol{f}(\boldsymbol{x}^*) + \boldsymbol{\lambda}^{*\top} D\boldsymbol{h}(\boldsymbol{x}^*) + \boldsymbol{\mu}^{*\top} D\boldsymbol{g}(\boldsymbol{x}^*) = \boldsymbol{0}^\top$.
3. $\boldsymbol{\mu}^{*\top} \boldsymbol{g}(\boldsymbol{x}^*) = 0$.
4. $\vec{h}(x) = 0$ ← always active
5. $\vec{g}(x) \leq 0$

3. $\mu^\top g(x) = 0$

$$0 = \mu_1 g_1(x) + \mu_2 g_2(x) + \cdots + \mu_l g_l(x) \leq 0$$

$$\mu_i g_i(x) = 0 \Rightarrow \begin{cases} g_i(x) = 0 & \leftarrow \text{active constraint} \\ g_i(x) < 0 & \leftarrow g_i(x) = 0 \text{ not active} \\ & \mu_i = 0 \end{cases}$$

$h(x) = 0$ always active



at A, interior point:

$g_1, g_2, g_3, g_4 < 0 \Rightarrow \mu_1, \mu_2, \mu_3, \mu_4 = 0$

$g_1(x) = 0$
$g_2(x) = 0$
$g_3(x) = 0$
$g_4(x) = 0$
} not active

$$\nabla f(x) + \sum_i \lambda_i \nabla h_i(x) = 0$$

at B : $g_1(x) = 0$ is active. $\mu_1 \geq 0$

$g_2, g_3, g_4 < 0 \Rightarrow \mu_2 = \mu_3 = \mu_4 = 0$

$$\nabla f(x) + \sum_i \lambda_i \nabla h_i(x) + \mu_1 \nabla g_1 = 0$$

(Similar consideration for C, D, E)

at F: $g_1(x) = 0$ and $g_2(x) = 0$ are active

$\Rightarrow \mu_1, \mu_2 \geq 0$

$g_2, g_3, g_4 < 0 \Rightarrow \mu_2 = \mu_3 = \mu_4 = 0$

$$\nabla f + \sum_i \lambda_i \nabla h_i(x) + \mu_1 \nabla g_1 + \mu_2 \nabla g_2 = 0$$