Quadratic Programming

$$\min \quad f(x)$$
$$\text{s.t.} \quad \begin{cases} AX = b \\ X \geq 0 \end{cases} \quad \text{linear constraints}$$

$C \geq 0$, positive def.

$$f(x) = \frac{1}{2} X^T C X + p^T X$$

quadratic linear linear

In terms of coordinates:

$$A^{m \times n}, \; X^{n \times 1}, \; C^{n \times n}, \; b^{m \times 1}, \; p^{n \times 1}$$

$C = (c_{ij})$ (WLOG: $c_{ij} = c_{ji}$, i.e. $C = C^T$, sym)

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \qquad p = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$$

$$f(x) = \frac{1}{2} \sum_{ij} c_{ij} x_i x_j + \sum_i p_i x_i$$

pos. def.

$$[\nabla^2 f] = C \geq 0 \Rightarrow f \text{ is convex.}$$

Lagrangian function

$$L(x, \lambda, \mu) = \frac{1}{2} x^T C x + p^T x + x^T (Ax - b) - \mu^T x$$

$\lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix}$ $\mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} \geq 0$

KKT: find $x, \lambda, \mu \geq 0$ s.t.

$$D_x L = 0 \Rightarrow x^T C + p^T + x^T A - \mu^T = 0$$

$(\nabla_x L = 0)$ $\Leftrightarrow$

$$\begin{cases} Cx + p + A^T \lambda - \mu = 0 \\ Ax = b \\ x \geq 0 \\ \mu^T x = 0 \quad \leftarrow \text{C.S.} \\ \mu \geq 0 \end{cases}$$

(*)

$$\begin{cases} Cx + A^T \lambda - \mu = -p & (1) \\ Ax = b & (2) \\ x \geq 0 & (3) \\ \mu \geq 0 & (4) \\ \mu^T x = 0 & (5) \end{cases}$$

(1)–(4): LP; (5): nonlin.

Solution of (*) using LP/Simplex (Frank-Wolfe)

(1) WLOG, assume $p \leq 0$, $b \geq 0$
otherwise, multiply individual row by (-1)
eg
$$a_1 x_1 + a_2 x_2 = -3$$
$$\Leftrightarrow (-a_1) x_1 + (-a_2) x_2 = 3$$

(2) Introduce auxiliary variables $w, \tilde{w} \geq 0$ s.t.
$$Cx + A^T \lambda - \mu + w = -p$$
$$Ax + w = b$$
$w_1, \ldots, w_n$, $\tilde{w}_1, \ldots, \tilde{w}_m$

$$\Leftrightarrow \begin{cases} w = -p - Cx - A^T \lambda + \mu & \geq 0 \\ \tilde{w} = b - Ax & \geq 0 \end{cases}$$

$b - Ax \geq 0$

feasible dictionary

(3) Objective function (need: $w_1 = 0, w_2 = 0$)
$$\min \; w_1 + \cdots + w_n + \tilde{w}_1 + \cdots + \tilde{w}_m$$
$$(\max \; -w_1 - \cdots - w_n - \tilde{w}_1 - \cdots - \tilde{w}_m)$$

(4) Set $\lambda = \lambda_+ - \lambda_-$, $\lambda_+, \lambda_- \geq 0$

(5) C.S. $\mu^T X = 0$

i.e. $\mu_1 x_1 + \cdots + \mu_n x_n = 0$

$x_i \geq 0, \ \mu_i \geq 0$

i.e. x_i, μ_i cannot be simultaneously positive.

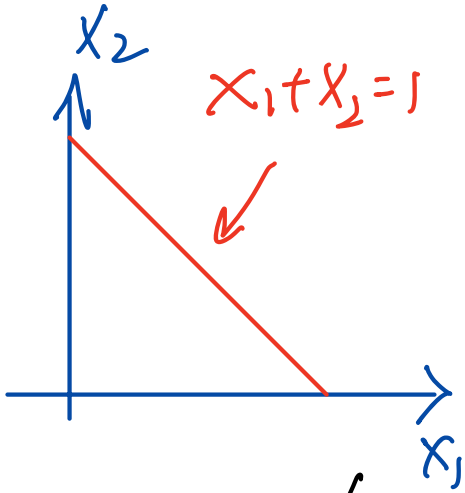
During simplex, while choosing entering and leaving variables, never let x_i, μ_i be basic variables at the same time.

Example

$$\min f(x) = x_1^2 + x_1 x_2 + x_2^2 - 5x_2$$

s.t. $x_1 + x_2 = 1$

$x_1, x_2 \geq 0$



$$f(x) = \frac{1}{2}(x_1 \ x_2)\underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}}_{C}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{(0 \ -5)}_{p^T}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = (\lambda-2)^2 - 1$$
$$= (\lambda-3)(\lambda-1)$$

$\Rightarrow \lambda = 3, 1 > 0$

$\Rightarrow C$ is pos. def

$\Rightarrow f$ is convex.

$$L(x_1, \lambda, \mu) = x_1^2 + x_1 x_2 + x_2^2 - 5x_2 + \lambda(x_1 + x_2 - 1) - \mu_1 x_1 - \mu_2 x_2$$

KKT $\Rightarrow$

$$\begin{aligned} 2x_1 + x_2 + \lambda - \mu_1 &= 0 \\ x_1 + 2x_2 - 5 + \lambda - \mu_2 &= 0 \\ x_1 + x_2 &= 1 \\ \mu_1, \mu_2 &\geq 0 \\ \mu_1 x_1 + \mu_2 x_2 &= 0 \end{aligned}$$

$$\begin{aligned} 2x_1 + x_2 + \lambda - \mu_1 &= 0 \\ x_1 + 2x_2 + \lambda - \mu_2 &= 5 \\ x_1 + x_2 &= 1 \end{aligned}$$

$\mu_1, \mu_2 \geq 0$

$\mu_1 x_1 + \mu_2 x_2 = 0$

(M1) Of course, solve for x_1: $x_1 = 1 - x_2$

$$f(x) = x_1^2 + x_1x_2 + x_2^2 - 5x_2$$
$$= \underbrace{(1-x_2)^2 + (1-x_2)x_2 + x_2^2 - 5x_2}$$

1D problem: $0 \le x_2 \le 1$

(M2) Using LP max $-w_1 - w_2 - w_3$

s.t. $w_1 = 0 - 2x_1 - x_2 - \lambda + \mu_1$

$w_2 = 5 - x_1 - 2x_2 - \mu + \mu_2$

$w_3 = 1 - x_1 - x_2$

(initial feasible dictionary)

$w_1, w_2, w_3, x_1, x_2, \mu_1, \mu_2 \ge 0$

$(\lambda = \lambda_+ - \lambda_-, \quad \lambda_+, \lambda_- \ge 0)$

x_1, μ_1 cannot be both basic vars.

Likewise for x_2, μ_2

(M3) Solving KKT directly:

$$2x_1 + x_2 + \lambda - \mu_1 = 0$$
$$x_1 + 2x_2 - 5 + \lambda - \mu_2 = 0$$
$$x_1 + x_2 = 1$$
$$\mu_1, \mu_2 \ge 0$$
$$\mu_1 x_1 + \mu_2 x_2 = 0$$

① $x_1, x_2 > 0 \Rightarrow \mu_1 = \mu_2 = 0$

$$\begin{aligned} 2x_1 + x_2 &= -\lambda \\ x_1 + 2x_2 &= 5 - \lambda \\ x_1 + x_2 &= 1 \end{aligned} \Rightarrow \begin{aligned} x_1 &= -1 - \lambda \\ x_2 &= 4 - \lambda \\ x_1 + x_2 &= 1 \end{aligned}$$

$\Rightarrow 1 = 3 - 2\lambda \Rightarrow \lambda = 1$

$\Rightarrow x_1 = -2$!!

② $x_1 > 0, \; x_2 = 0 \Rightarrow \mu_1 = 0, \; x_1 = 1$

$\Rightarrow 2 + \lambda = 0$

$1 - 5 + \lambda - \mu_2 = 0$

$\Rightarrow \lambda = -2$

$\mu_2 = 1 - 5 - 2 = -6$!!

③ $x_1 = 0, \; x_2 > 0 \Rightarrow \mu_2 = 0, \; x_2 = 1$

$\Rightarrow 1 + \lambda - \mu_1 = 0$

$2 - 5 + \lambda = 0$

$\Rightarrow \lambda = 3, \; \mu_1 = 4$

$$\boxed{x_1 = 0, \; x_2 = 1, \; \lambda = 3, \; \mu_1 = 4, \; \mu_2 = 0}$$

[F, p. 187]

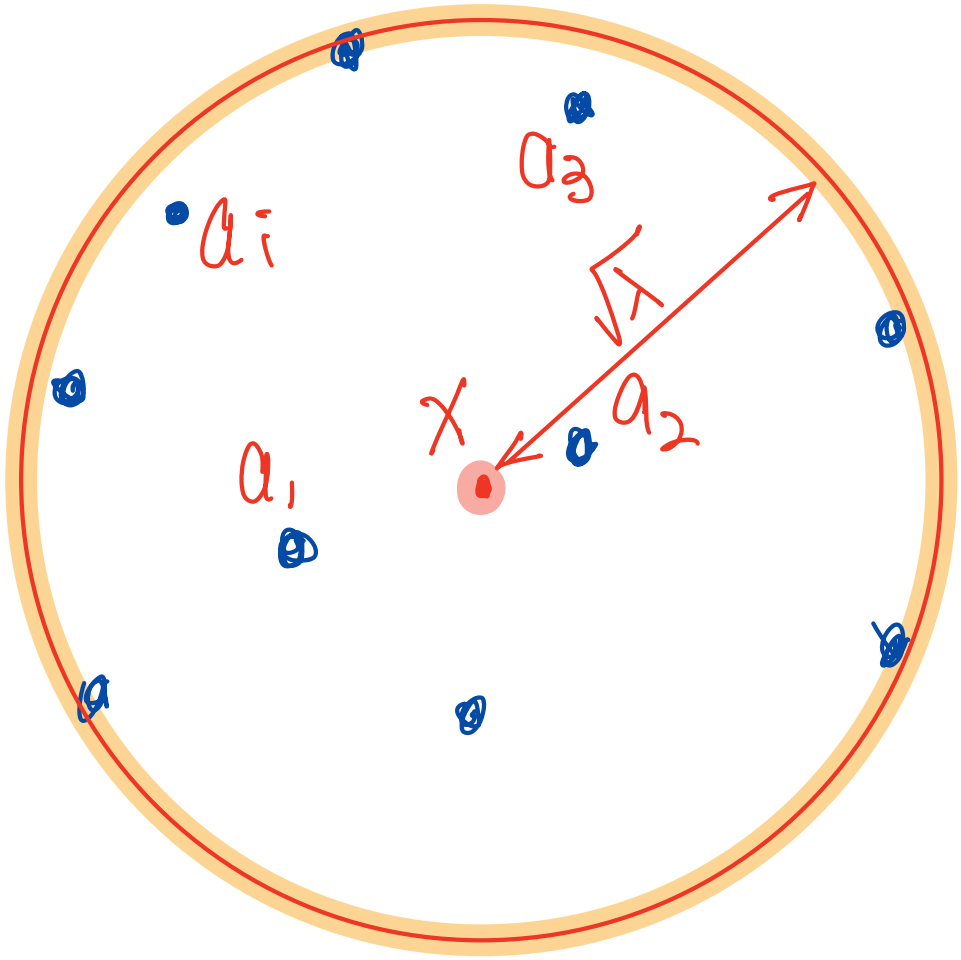
J. J. Sylvester, "A Question in the Geometry of Situation," Quarterly J. Pure and Appl. Math, vol. 1 (1857) p. 79.

Sylvester's problem (1857): "It is required to find the least circle which shall contain a given set of points in the plane." Show that Sylvester's problem for finite point sets in R^n is equivalent to the following quadratic program: *If the points* $\mathbf{a}_1, \ldots, \mathbf{a}_m$ *are given, find a point* $\mathbf{x}$ *and a number* λ *solving*

$$\mathbf{a}_i \cdot \mathbf{x} + \lambda \geq \tfrac{1}{2}|\mathbf{a}_i|^2 \qquad (i = 1, \ldots, m),$$
$$\tfrac{1}{2}\mathbf{x} \cdot \mathbf{x} + \lambda = \min.$$

Then the required least sphere has center $\mathbf{x}$ and has radius $r = \sqrt{(\mathbf{x} \cdot \mathbf{x} + 2\lambda)}$.

(See also [M&T] p. 188)



$$\min_{x,\lambda} \quad \lambda \quad \text{s.t.} \quad \|a_i - x\|^2 \leq \lambda, \; i = 1, 2 \ldots$$

$$\|a_i - x\|^2 = \|a_i\|^2 - 2\langle a_i, x\rangle + \|x\|^2 \le \lambda$$

$$\Rightarrow \|a_i\|^2 - 2\langle a_i, x\rangle \le \underbrace{\lambda - \|x\|^2}_{\text{new var.}} = \tilde{\lambda}$$

$$\lambda = \lambda - \|x\|^2 + \|x\|^2 = \tilde{\lambda} + \|x\|^2$$

New problem:

$$\min_{x,\,\tilde{\lambda}} \ \tilde{\lambda} + \|x\|^2 \qquad \text{quadratic}$$

$$\text{s.t.} \quad \|a_i\|^2 - 2\langle a_i, x\rangle \le \tilde{\lambda}, \quad i = 1, 2, \cdots$$

$$\Leftrightarrow \quad 2\langle a_i, x\rangle + \tilde{\lambda} \ge \|a_i\|^2$$

lin constraints

$$\min_{x,\,\tilde{\lambda}} \ \tilde{\lambda} + \|x\|^2$$

$$\text{s.t.} \quad 2\langle a_i, x\rangle + \tilde{\lambda} \ge \|a_i\|^2$$

Markowitz Portfolio Selection Model [Vr p. 415]

$\{R_j\}_{j=1,\dots}$ price of portfolio j

$$\max \quad R = \sum_j x_j R_j$$

$$\text{s.t.} \quad x_j \geq 0, \quad \sum_j x_j = 1$$

In general, R_j is random

$\Rightarrow$

Goal is to $\begin{cases} \max \text{ average of } R \\ \min \text{ fluctuation of } R \end{cases}$

$$ER = E\left[\sum_j x_j R_j\right] = \sum_j x_j \underbrace{ER_j}_{r_j}$$

r_j mean of R_j

$$\text{Var}(R) = E\left[(R - ER)^2\right]$$

$$= E\left[\left(\sum_j x_j R_j - \sum_j x_j ER_j\right)^2\right]$$

$$= E\left[\left(\sum_i x_i (R_i - ER_i)\right)^2\right]$$

$$= E \sum_{i,j} x_i x_j (R_i - ER_i)(R_j - ER_j)$$

$$= \sum_{i,j} x_i x_j \underbrace{E[(R_i - ER_i)(R_j - ER_j)]}$$

C_{ij} covariance between R_i, R_j

$$\max_{x_i} \quad \sum_j x_j r_j - \mu \sum_{i,j} x_i x_j C_{ij}$$

$$\text{s.t.} \quad \sum_j x_j = 1, \quad x_j \geq 0$$

linear

quadratic

$\mu > 0$ is user defined parameter.

Risk averse investor will choose a larger μ