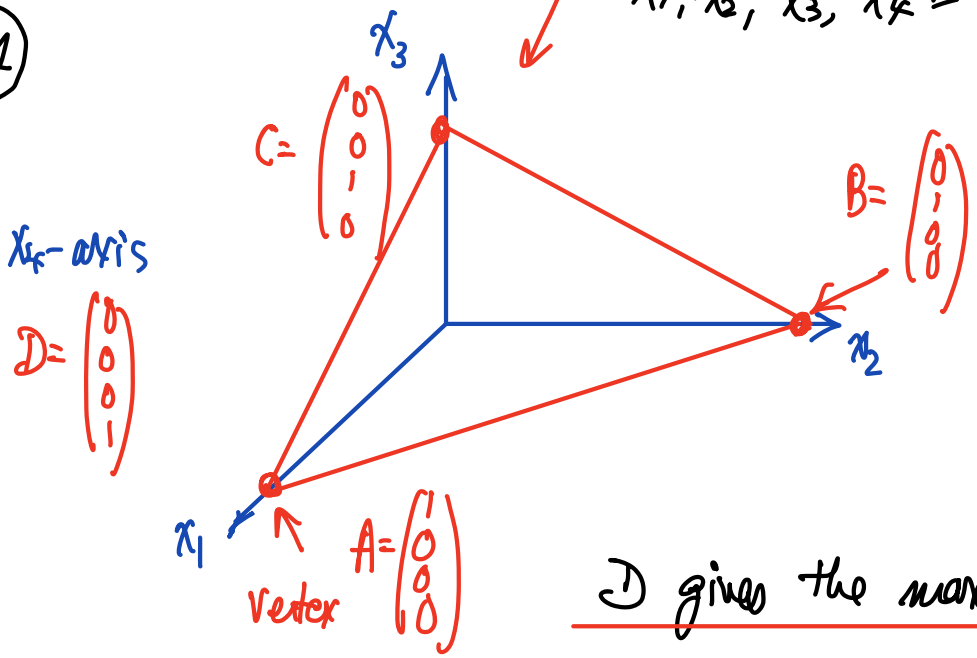
MA 421 Review Problems (Fall 2026)

[V] 2.10

$$\max \quad 6x_1 + 8x_2 + 5x_3 + 9x_4$$
$$\text{s.t.} \quad x_1 + x_2 + x_3 + x_4 = 1$$
$$x_1, x_2, x_3, x_4 \geq 0$$

M1



D gives the maximum value, 9.

M2

$$\max \quad 6x_1 + 8x_2 + 5x_3 + 9x_4$$
$$\text{s.t.} \quad x_1 + x_2 + x_3 + x_4 = 1$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Single out x_4: $\quad x_4 = 1 - x_1 - x_2 - x_3$

Objective fct becomes:

$$\zeta(x_1, x_2, x_3) = 6x_1 + 8x_2 + 5x_3 + 9(1 - x_1 - x_2 - x_3)$$
$$= 9 - 3x_1 - x_2 - 4x_3$$

s.t. $\quad x_1, x_2, x_3 \geq 0$

↑ opt. dictionary

(M3) Consider the dual problem:
(Note the primal problem has only 1 constraint.)

$$\begin{aligned} \max \quad & 6x_1 + 8x_2 + 5x_3 + 9x_4 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 + x_4 = 1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

$$\begin{aligned} \min_{y} \quad & 1 \cdot y \\ \text{s.t.} \quad & y \geq 6 \\ & y \geq 8 \\ & y \geq 5 \\ & y \geq 9 \end{aligned} \Rightarrow y \geq 9$$

Hence $y = 9$ is the min.

By complementary slackness $\Rightarrow$

- $x_1 = 0$ (since $9 > 6$)
- $x_2 = 0$ (since $9 > 8$)
- $x_3 = 0$ (since $9 > 5$)
- $x_4 = 1$ (since $x_1 + x_2 + x_3 + x_4 = 1$)

[C] 1.2

$$\min \quad -8x_1 + 9x_2 + 2x_3 - 6x_4 - 5x_5$$
$$\text{s.t.} \quad 6x_1 + 6x_2 - 10x_3 + 2x_4 - 8x_5 \geq 3$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

(M1) From the coeff. of x_1, x_4

$\Rightarrow$ x_1, x_2 can be as big as possible

$\Rightarrow$ $\min = -\infty$ (unbounded)

(M2) Consider dual problem:

($y \geq 0 \Rightarrow 3y \leq 6yx_1 + 6yx_2 - 10yx_3 + 2yx_4 - 8yx_5$
$\leq -8x_1 + 9x_2 + 2x_3 - 6x_4 - 5x_5$)

$$\max_{y \geq 0} \quad 3y$$
$$\text{s.t.} \quad 6y \leq -8 \quad \leftarrow \text{infeasible with } y \geq 0$$
$$6y \leq 9$$
$$-10y \leq 2$$
$$2y \leq -6 \quad \leftarrow \text{infeasible with } y \geq 0$$
$$-8y \leq -5$$

Hence primal problem is unbounded

(since primal problem is feasible to start with)

[V] **2.6**

change to e.g. $-x_1 - x_2$

maximize $x_1 + 3x_2$

subject to
$$\begin{aligned} -x_1 - x_2 &\le -3 \\ -x_1 + x_2 &\le -1 \\ x_1 + 2x_2 &\le 2 \\ x_1,\ x_2 &\ge 0. \end{aligned}$$

(M1) Consider dual:

$$\min\ -3y_1 - y_2 + 2y_3 \qquad \max\ 3y_1 + y_2 - 2y_3$$

$$\text{s.t.}\ \begin{aligned} -y_1 - y_2 + y_3 &\ge -1 \\ -y_1 + y_2 + 2y_3 &\ge -1 \\ y_1, y_2 &\ge 0 \end{aligned} \iff \text{s.t.}\ \begin{aligned} y_1 + y_2 - y_3 &\le 1 \\ y_1 - y_2 - 2y_3 &\le 1 \\ y_1, y_2, y_3 &\ge 0 \end{aligned}$$

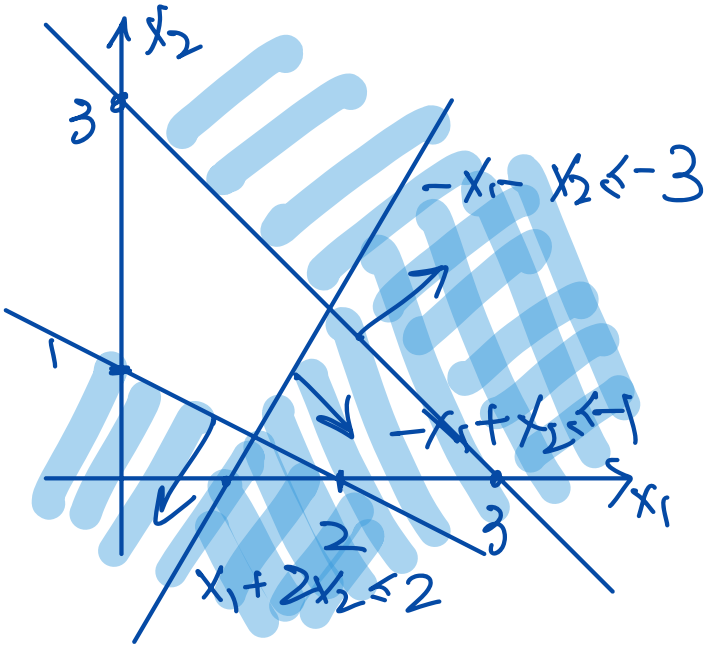
"Opt. dict"

$$\begin{aligned} \zeta &= 3 - 3z_1 - 2y_2 + y_3 \\ y_1 &= 1 - z_1 - y_2 + y_3 \\ z_2 &= 0 + z_1 + y_2 + y_3 \end{aligned}$$

$\Downarrow$

unbounded

$\Longleftarrow$ initial problem not feasible



2.6

$$\begin{aligned} \text{maximize} \quad & x_1 + 3x_2 \\ \text{subject to} \quad & -x_1 - x_2 \leq -3 \\ & -x_1 + x_2 \leq -1 \\ & x_1 + 2x_2 \leq 2 \\ & x_1,\ x_2 \geq 0. \end{aligned}$$

(M2) Alternatively, introduce auxiliary variable x_0:

$$\begin{aligned} \max \quad & -x_0 \quad (=0\ ?) \\ \text{s.t.} \quad & -x_1 - x_2 - x_0 \leq -3 \\ & -x_1 + x_2 - x_0 \leq -1 \\ & x_1 + 2x_2 - x_0 \leq 2 \\ & x_1, x_2, x_0 \geq 0 \end{aligned}$$

$$\begin{aligned} w_1 &= -3 + x_1 + x_2 + x_0 \quad \leftarrow \text{most infeasible} \\ w_2 &= -1 + x_1 - x_2 + x_0 \\ w_3 &= 2 - x_1 - 2x_2 + x_0 \end{aligned}$$

$= -x_0$

$$\zeta = -3 + x_1 + x_2 - w_1$$

$$\begin{aligned} x_0 &= 3 - x_1 - x_2 + w_1 \\ w_2 &= 2 - 2x_2 + w_1 \\ w_3 &= 5 - 2x_2 - 3x_2 + w_1 \end{aligned}$$

opt

$\neq 0$

$$\zeta = -\frac{1}{2} - \frac{1}{2}w_3 - \frac{1}{2}x_2 - \frac{1}{2}w_1$$

$$\begin{aligned} x_0 &= \frac{1}{2} + \frac{1}{2}w_3 + \frac{1}{2}x_2 + \frac{1}{2}w_1 \\ w_2 &= 2 - 2x_2 + w_1 \\ x_1 &= \frac{5}{2} - \frac{1}{2}w_3 - \frac{3}{2}x_2 + \frac{1}{2}w_1 \end{aligned}$$

$\Rightarrow$ original problem not feasible

M3

$$-x_1 - x_2 + w_1 = -3$$
$$-x_1 + x_2 + w_2 = -1$$
$$x_1 + x_2 + w_3 = 2$$

w_1, w_2, w_3 are slack variables

$$-x_1 - x_2 + w_1 - w_4 = -3$$
$$-x_1 + x_2 + w_2 - w_5 = -1$$
$$+x_1 + x_2 + w_3 = 2$$

w_4, w_5 are auxiliary var.

$$w_4 = 3 - x_1 - x_2 + w_1$$
$$w_5 = 1 - x_1 + x_2 + w_2$$
$$w_3 = 2 - x_1 - x_2$$

?

$$\max \zeta = -w_4 - w_5 = -4 + 2x_1 - w_1 - w_2 \leq 0$$

$$w_4 = 3 - x_1 - x_2 + w_1$$
$$w_5 = 1 - x_1 + x_2 + w_2$$
$$w_3 = 2 - x_1 - x_2$$

opt. $\neq 0$ $\Rightarrow$ original problem infeasible

maximize	ζ =	-1	+	-1	w_5	+	-1	w_3	+	-1	w_1	+	0	w_2
subject to:	w_4 =	1	-	0	w_5	-	-1	w_3	-	-1	w_1	-	0	w_2
	x_1 =	3/2	-	1/2	w_5	-	1/2	w_3	-	0	w_1	-	-1/2	w_2
	x_2 =	1/2	-	-1/2	w_5	-	1/2	w_3	-	0	w_1	-	1/2	w_2

x_1 x_2 w_1 w_2 w_4 w_5 w_3 ≥ 0

HW3

2. Use simplex method to solve the following LP problem:

$$\begin{aligned} \max \quad & 4x_1 + 2x_2 - x_3 + 2x_4 \\ \text{s.t.} \quad & x_1 + x_2 + 3x_3 + 4x_4 = 8, \\ & x_1 + x_2 + x_3 + x_4 = 4, \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

introduce auxiliary var.

$$x_1 + x_2 + 3x_3 + 4x_4 + w_1 = 8$$
$$x_1 + x_2 + x_3 + x_4 + w_2 = 4$$

set to be basic var.

? 0 =

$$\max \zeta = -w_1 - w_2$$
$$w_1 = 8 - x_1 - x_2 - 3x_3 - 4x_4$$
$$w_2 = 4 - x_1 - x_2 - x_3 - x_4$$

HW3

3. Use simplex method to solve the following LP problem:

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 + 2x_3 \\ \text{s.t.} \quad & 2x_1 + x_2 - 3x_3 \geq -2, \\ & x_2 - x_3 \leq 3, \\ & 2x_1 + x_2 + x_3 = 4, \\ & x_1, x_3 \geq 0. \end{aligned}$$

$-2x_1 - x_2 + 3x_3 \leq 2$

introduce auxiliary var.

? 0 =

$$\max \zeta = -w_3$$
s.t.
$$w_1 = 2 + 2x_1 + x_2 - 3x_3$$
$$w_2 = 3 - x_2 + x_3$$
$$w_3 = 4 - 2x_1 - x_2 - x_3$$

[V] #3.4 $\max f(x) = \sum_{j}^{n} c_j x_j$

(M1) s.t. $\sum_j a_{ij} x_j \le 0$ $i = 1, \dots, m$

$x_j \ge 0$ $j = 1, \dots, n$

Must be feasible (take: $x_1 = x_2 = \dots = x_n = 0$)

Note: $f(0) = 0$

Either $f(0) = 0$ is optimal,

ie. $f(x) \le 0$ for all feasible x

or there is $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ s.t. $f(X) > 0$

If X is feasible, so is tX, $t > 0$.

then $f(tX) = t f(X) \longrightarrow +\infty$ as $t \to +\infty$

(M2) Consider dual problem

$\min g(y) = 0y_1 + 0y_2 + \dots + 0y_m \quad (= 0)$

s.t. $\sum_i a_{ij} y_i \ge c_j$ $j = 1, \dots, n$

$y_i \ge 0$ $i = 1, \dots, m$

Note that the dual objective function is always 0

Hence

if the dual problem is feasible, then $\xi^* = 0$

$$\Rightarrow \zeta^* = 0$$

if the dual problem is not feasible, then $\zeta^* = +\infty$ (unbounded)

(since the primal problem is always feasible.)

[C]

2.2 Use the simplex method to describe *all* the optimal solutions of the following problem:

$$\begin{aligned} \text{maximize} \quad & 2x_1 + 3x_2 + 5x_3 + 4x_4 \\ \text{subject to} \quad & x_1 + 2x_2 + 3x_3 + x_4 \le 5 \quad \leftarrow w_1 \\ & x_1 + x_2 + 2x_3 + 3x_4 \le 3 \quad \leftarrow w_2 \\ & x_1, x_2, x_3, x_4 \ge 0. \end{aligned}$$

$$\begin{cases} w_1 = 5 - x_1 - 2x_2 - 3x_3 - x_4 \\ w_2 = 3 - x_1 - x_2 - 2x_3 - 3x_4 \end{cases}$$

Use simplex $\Longrightarrow$ optimal dictionary:

$$\begin{aligned} \zeta &= 8 - w_1 - w_2 \\ x_2 &= 1 + x_1 - 2w_1 + 3w_2 + 7x_4 \\ x_3 &= 1 - x_1 + w_1 - 2w_2 - 5x_4 \end{aligned}$$

Solution: $\zeta^* = 8$, $w_1 = 0$, $w_2 = 0$

$$\begin{aligned} x_2 &= 1 + x_1 + 7x_4 \\ x_3 &= 1 - x_1 - 5x_4 \end{aligned}$$

All solutions:

$$\begin{aligned} & x_1, x_2, x_3, x_4 \ge 0 \\ & x_2 = 1 + x_1 + 7x_4 \\ & x_3 = 1 - x_1 - 5x_4 \end{aligned}$$

(x_1, x_4 are free var. as long as they are ≥ 0)

[V] #5.1

$$\max \quad x_1 - 2x_2$$
$$\text{s.t.} \quad -x_1 - 2x_2 + x_3 - x_4 \le 0$$
$$4x_1 + 3x_2 + 4x_3 - 2x_4 \le 3$$
$$-x_1 - x_2 + 2x_3 + x_4 = 1$$
$$x_2, x_3 \ge 0$$

$$-y_1 x_1 - 2y_1 x_2 + y_1 x_3 - y_1 x_4 \le 0 \qquad y_1 \ge 0$$
$$4y_2 x_1 + 3y_2 x_2 + 4y_2 x_3 - 2y_2 x_4 \le 3y_2 \qquad y_2 \ge 0$$
$$-y_3 x_1 - y_3 x_2 + 2y_3 x_3 + y_3 x_4 = y_3$$

$$(-y_1 + 4y_2 - y_3) x_1 + (-2y_1 + 3y_2 - y_3) x_2 + (y_1 + 4y_2 + 2y_3) x_3 + (-y_1 - 2y_2 + y_3) x_4 \le 3y_2 + y_3$$

$\le$ $x_1 - 2x_2$

Hence

$$x_1 \le (-y_1 + 4y_2 - y_3) x_1 \Rightarrow -y_1 + 4y_2 - y_3 = 1$$
$$-2x_2 \le (-2y_1 + 3y_2 - y_3) x_2 \overset{(x_2 \ge 0)}{\Rightarrow} -2y_1 + 3y_2 - y_3 \ge -2$$
$$0 \le (y_1 + 4y_2 + 2y_3) x_3 \overset{(x_3 \ge 0)}{\Rightarrow} y_1 + 4y_2 + 2y_3 \ge 0$$
$$0 \le (-y_1 - 2y_2 + y_3) x_4 \Rightarrow -y_1 - 2y_2 + y_3 = 0$$

Dual Problem:

$$\min \quad 3y_2 + y_3$$

$$\text{s.t.} \quad -y_1 + 4y_2 - y_3 = 1$$

$$-2y_1 + 3y_2 - y_3 \geq -2$$

$$y_1 + 4y_2 + 2y_3 \geq 0$$

$$-y_1 - 2y_2 + y_3 = 0$$

$$y_1, y_2 \geq 0$$

5.6 Solve the following linear program:

negative coeffs.

$$\begin{aligned}
\text{maximize} \quad & -x_1 - 2x_2 \\
\text{subject to} \quad & -2x_1 + 7x_2 \leq 6 \\
& -3x_1 + x_2 \leq -1 \\
& 9x_1 - 4x_2 \leq 6 \\
& x_1 - x_2 \leq 1 \\
& 7x_1 - 3x_2 \leq 6 \\
& -5x_1 + 2x_2 \leq -3 \\
& x_1, x_2 \geq 0.
\end{aligned}$$

Origin not feasible.

Consider dual problem:

$$\begin{aligned}
\min \quad & 6y_1 - y_2 + 6y_3 + y_4 + 6y_5 - 3y_6 \\
\text{s.t.} \quad & -2y_1 - 3y_2 + 9y_3 + y_4 + 7y_5 - 5y_6 \geq -1 \\
& 7y_1 + y_2 - 4y_3 - y_4 - 3y_5 + 2y_6 \geq -2 \\
& y_1, \dots, y_6 \geq 0
\end{aligned}$$

Now the origin is feasible for dual problem.

or

$$\begin{aligned}
\max \quad & -6y_1 + y_2 - 6y_3 - y_4 - 6y_5 + 3y_6 \\
\text{s.t.} \quad & 2y_1 + 3y_2 - 9y_3 - y_4 - 7y_5 + 5y_6 \leq 1 \\
& -7y_1 - y_2 + 4y_3 + y_4 + 3y_5 - 2y_6 \leq 2 \\
& y_1, \dots, y_6 \geq 0
\end{aligned}$$

Init Dict.

maximize	$-\xi$	=		-6	y_1	+	1	y_2	+	-6	y_3	+	-1	y_4	+	-6	y_5	+	3	y_6
subject to:	z_1	=	1	-	2	y_1	-	3	y_2	-	-9	y_3	-	-1	y_4	-	-7	y_5	-	5	y_6
	z_2	=	2	-	-7	y_1	-	-1	y_2	-	4	y_3	-	1	y_4	-	3	y_5	-	-2	y_6

$y_1\ y_2\ y_3\ y_4\ y_5\ y_6\ \ z_1\ z_2 \geq 0$

Opt. Dict. (after one step)

maximize	$-\xi$	=	3/5	+	-36/5	y_1	+	-4/5	y_2	+	-3/5	y_3	+	-2/5	y_4	+	-9/5	y_5	+	-3/5	z_1
subject to:	y_6	=	1/5	-	2/5	y_1	-	3/5	y_2	-	-9/5	y_3	-	-1/5	y_4	-	-7/5	y_5	-	1/5	z_1
	z_2	=	12/5	-	-31/5	y_1	-	1/5	y_2	-	2/5	y_3	-	3/5	y_4	-	1/5	y_5	-	2/5	z_1

$y_1\ y_2\ y_3\ y_4\ y_5\ y_6\ \ z_1\ z_2 \geq 0$

$y_6 = \frac{1}{5} \Rightarrow -5x_1 + 2x_2 = -3$

$z_2 = \frac{12}{5} \Rightarrow x_2 = 0$

$\Rightarrow (x_1 = 3/5,\ x_2 = 0)$

$\xi^* = -\frac{3}{5}, \iff \zeta^* = -\frac{3}{5} - 2(0) = -\frac{3}{5}$

(Note that $(x_1 = \frac{3}{5},\ x_2 = 0)$ satisfies all the constraints:

$$
\begin{aligned}
-2x_1 + 7x_2 &\leq 6 && < && y_1 = 0 \\
-3x_1 + x_2 &\leq -1 && < && y_2 = 0 \\
9x_1 - 4x_2 &\leq 6 && < && y_3 = 0 \\
x_1 - x_2 &\leq 1 && < && y_4 = 0 \\
7x_1 - 3x_2 &\leq 6 && < && y_5 = 0 \\
-5x_1 + 2x_2 &\leq -3 && = && y_6 > 0 \\
x_1,\ x_2 &\geq 0. && && \qquad)
\end{aligned}
$$

[C] p.69 #5.3

a. Maximize $7x_1 + 6x_2 + 5x_3 - 2x_4 + 3x_5$

subject to

$$\begin{aligned} x_1 + 3x_2 + 5x_3 - 2x_4 + 2x_5 &\le 4 \quad — y_1 \\ 4x_1 + 2x_2 - 2x_3 + x_4 + x_5 &\le 3 \quad — y_2 \\ 2x_1 + 4x_2 + 4x_3 - 2x_4 + 5x_5 &\le 5 \quad — y_3 \\ 3x_1 + x_2 + 2x_3 - x_4 - 2x_5 &\le 1 \quad — y_4 \\ x_1, x_2, x_3, x_4, x_5 &\ge 0. \end{aligned}$$

Proposed solution: $x_1^* = 0, \quad x_2^* = \frac{4}{3}, \quad x_3^* = \frac{2}{3}, \quad x_4^* = \frac{5}{3}, \quad x_5^* = 0.$

$x_2, x_3, x_4 > 0 \Rightarrow z_2 = z_3 = z_4 = 0$

y_1: $3\left(\frac{4}{3}\right) + 5\left(\frac{2}{3}\right) - 2\left(\frac{5}{3}\right) = 4$

y_2: $2\left(\frac{4}{3}\right) - 2\left(\frac{2}{3}\right) + \left(\frac{5}{3}\right) = 3$

y_3: $4\left(\frac{4}{3}\right) + 4\left(\frac{2}{3}\right) - 2\left(\frac{5}{3}\right) = \frac{14}{3} < 5 \Rightarrow y_3 = 0$

y_4: $\left(\frac{4}{3}\right) + 2\left(\frac{2}{3}\right) - \frac{5}{3} = 1$

Dual:

$$\begin{aligned} y_1 + 4y_2 + 2y_3 + 3y_4 &\ge 7 && z_1 \\ 3y_1 + 2y_2 + 4y_3 + y_4 &= 6 && z_2 = 0 \\ 5y_1 - 2y_2 + 4y_3 + 2y_4 &= 5 && z_3 = 0 \\ -2y_1 + y_2 - 2y_3 - y_4 &= -2 && z_4 = 0 \\ 2y_1 + y_2 + 5y_3 - 2y_4 &\ge 3 && z_5 \end{aligned}$$

$$3y_1+2y_2+y_4=6$$
$$5y_1-2y_2+2y_4=5 \qquad (y_3=0)$$
$$-2y_1+y_2-y_4=-2$$

$$\Rightarrow \begin{cases} y_1=1,\ y_2=1,\ y_4=1 \geq 0 \ \checkmark \\ y_1+4y_2+2y_3+3y_4 \geq 7 \ \checkmark \\ 2y_1+y_2+5y_3-2y_4 \geq 3 \ \checkmark \end{cases}$$

Opt!

[Note: $\zeta = 7x_1+6x_2+5x_3-2x_4+3x_5$

$= 7(0)+6\left(\frac{4}{3}\right)+5\left(\frac{2}{3}\right)-2\left(\frac{5}{3}\right)+3(0)$

$= 8$

$\xi = 4y_1+3y_2+5y_3+y_4$

$= 4(1)+3(1)+5(0)+(1)$

$= 8.$)

[C] p.69 #5.3

b. Maximize $4x_1 + 5x_2 + x_3 + 3x_4 - 5x_5 + 8x_6$

subject to

$$\begin{aligned}
x_1 - 4x_3 + 3x_4 + x_5 + x_6 &\le 1 \quad y_1\\
5x_1 + 3x_2 + x_3 - 5x_5 + 3x_6 &\le 4 \quad y_2\\
4x_1 + 5x_2 - 3x_3 + 3x_4 - 4x_5 + x_6 &\le 4 \quad y_3\\
-x_2 + 2x_4 + x_5 - 5x_6 &\le 5 \quad y_4\\
-2x_1 + x_2 + x_3 + x_4 + 2x_5 + 2x_6 &\le 7 \quad y_5\\
2x_1 - 3x_2 + 2x_3 - x_4 + 4x_5 + 5x_6 &\le 5 \quad y_6\\
x_1, x_2, x_3, x_4, x_5, x_6 &\ge 0.
\end{aligned}$$

Proposed solution: $x_1 = 0,\ x_2 = 0,\ x_3 = \frac{5}{2},\ x_4 = \frac{7}{2},\ x_5 = 0,\ x_6 = \frac{1}{2}.$

$x_3, x_4, x_6 > 0 \Rightarrow z_3, z_4, z_6 = 0$

y_1: $-4\left(\frac{5}{2}\right) + 3\left(\frac{7}{2}\right) + \frac{1}{2} = 1$

y_2: $\left(\frac{5}{2}\right) + 3\left(\frac{1}{2}\right) = 4$

y_3: $-3\left(\frac{5}{2}\right) + 3\left(\frac{7}{2}\right) + \frac{1}{2} = \frac{7}{2} < 4 \Rightarrow y_3 = 0$

y_4: $2\left(\frac{7}{2}\right) - 5\left(\frac{1}{2}\right) = \frac{9}{2} < 5 \Rightarrow y_4 = 0$

y_5: $\left(\frac{5}{2}\right) + \left(\frac{7}{2}\right) + 2\left(\frac{1}{2}\right) = 7$

y_6: $2\left(\frac{5}{2}\right) - \frac{7}{2} + 5\left(\frac{1}{2}\right) = 4 < 5 \Rightarrow y_6 = 0$

$z_3 = 0 \Rightarrow -4y_1 + y_2 - 3y_3 + y_5 + 2y_6 = 1$

$z_4 = 0 \Rightarrow 3y_1 + 3y_3 + 2y_4 + y_5 - y_6 = 3$

$z_6 = 0 \Rightarrow y_1 + 3y_2 + y_3 - 5y_4 + 2y_5 + 5y_6 = 8$

$y_3 = y_4 = y_6 = 0 \Rightarrow$

$$\begin{cases} -4y_1 + y_2 + y_5 = 1 \\ 3y_1 + y_5 = 3 \\ y_1 + 3y_2 + 2y_5 = 8 \end{cases}$$

$y_1 = \frac{1}{2},\ y_2 = \frac{3}{2},\ y_5 = \frac{3}{2} \geq 0$

Z_1: $y_1 + 5y_2 + 4y_3 - 2y_5 + 2y_6 \geq 4$ ✓

$\left(\frac{1}{2} + 5\left(\frac{3}{2}\right) - 2\left(\frac{3}{2}\right) = 5 > 4\right)$

Z_2: $3y_2 + 5y_3 - y_4 + y_5 - 3y_6 \geq 5$ ✓

$\left(3\left(\frac{3}{2}\right) + \frac{3}{2} = 6 > 5\right)$

Z_5: $y_1 - 5y_2 - 4y_3 + y_4 + 2y_5 + 4y_6 \geq -5$ ✓

$\left(\frac{1}{2} - 5\left(\frac{3}{2}\right) + 2\left(\frac{3}{2}\right) = -4 \geq -5\right)$

Opt.!

Note: $\zeta = 4x_1 + 5x_2 + x_3 + 3x_4 - 5x_5 + 8x_6$

$= \frac{5}{2} + 3\left(\frac{7}{2}\right) + 8\left(\frac{1}{2}\right) = 17$

$\zeta = y_1 + 4y_2 + 4y_3 + 5y_4 + 7y_5 + 5y_6$

$= \left(\frac{1}{2}\right) + 4\left(\frac{3}{2}\right) + 7\left(\frac{3}{2}\right) = 17$ #

6.1 Consider the following linear programming problem:

$$\begin{aligned} \text{maximize} \quad & -6x_1 + 32x_2 - 9x_3 \\ \text{subject to} \quad & -2x_1 + 10x_2 - 3x_3 \leq -6 \\ & x_1 - 7x_2 + 2x_3 \leq 4 \\ & x_1,\ x_2,\ x_3 \geq 0. \end{aligned}$$

Suppose that, in solving this problem, you have arrived at the following dictionary:

$$\begin{aligned} \zeta &= -18 - 3x_4 + 2x_2 \\ \hline x_3 &= 2 - x_4 + 4x_2 - 2x_5 \\ x_1 &= 0 + 2x_4 - x_2 + 3x_5 \end{aligned}$$

(a) Which variables are basic? Which are nonbasic?
(b) Write down the vector, x_B^*, of current primal basic solution values.
(c) Write down the vector, z_N^*, of current dual nonbasic solution values.
(d) Write down $B^{-1}N$.
(e) Is the primal solution associated with this dictionary feasible?
(f) Is it optimal?
(g) Is it degenerate?

(a) Basic vars: x_3, x_1; Non-basic vars: x_4, x_2, x_5

(b) $x_B^* = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ (c) $z_N^* = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$

(d) $B^{-1}N = \begin{pmatrix} 1 & -4 & 2 \\ -2 & 1 & -3 \end{pmatrix}$

(e) Yes as $x_B^* \geq 0$

(f) No as $z_N^* \not\geq 0$

(g) Yes as 2nd entry of $x_B^* = 0$

6.2 Consider the following linear programming problem:

$$\begin{aligned}
\text{maximize} \quad & x_1 + 2x_2 + 4x_3 + 8x_4 + 16x_5 \\
\text{subject to} \quad & x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 \leq 2 \\
& 7x_1 + 5x_2 - 3x_3 - 2x_4 \leq 0 \\
& x_1,\ x_2,\ x_3,\ x_4,\ x_5 \geq 0.
\end{aligned}$$

Consider the situation in which x_3 and x_5 are basic and all other variables are nonbasic. Write down:

(a) B,
(b) N,
(c) b,
(d) $c_{\mathcal{B}}$,
(e) $c_{\mathcal{N}}$,
(f) $B^{-1}N$,
(g) $x^*_{\mathcal{B}} = B^{-1}b$,
(h) $\zeta^* = c^T_{\mathcal{B}} B^{-1} b$,
(i) $z^*_{\mathcal{N}} = (B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}}$,
(j) the dictionary corresponding to this basis.

$$\begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{matrix}$$
$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 1 & 0 \\ 7 & 5 & -3 & -2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

} slacks

$$B = \{3, 5\} \qquad N = \{1, 2, 4, 6, 7\}$$

(a) $$B = \begin{bmatrix} 3 & 5 \\ -3 & 0 \end{bmatrix}$$

(b) $N = \begin{bmatrix} 1 & 2 & 4 & 1 & 0 \\ 7 & 5 & -2 & 0 & 1 \end{bmatrix}$

(c) $b = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

(d) $\zeta = x_1 + 2x_2 + 4x_3 + 8x_4 + 16x_5 + 0x_6 + 0x_7$

$C_B = [4 \quad 16]^T$

(e) $C_N = [1 \quad 2 \quad 8 \quad 0 \quad 0]^T$

(f) $B^{-1}N = \frac{\begin{bmatrix} 0 & -5 \\ 3 & 3 \end{bmatrix}}{15} \begin{bmatrix} 1 & 2 & 4 & 1 & 0 \\ 7 & 5 & -2 & 0 & 1 \end{bmatrix}$

$= \frac{1}{15} \begin{bmatrix} -35 & -25 & 10 & 0 & -5 \\ 24 & 21 & 6 & 3 & 3 \end{bmatrix}$

(g) $B^{-1}b = \frac{1}{15} \begin{bmatrix} 0 & -5 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{1}{15} \begin{bmatrix} 0 \\ 6 \end{bmatrix}$

(h) $\zeta^* = C_B^T B^{-1} b$

$= [4 \quad 16] \begin{bmatrix} 0 \\ 6 \end{bmatrix} \frac{1}{15} = \frac{96}{15} = \frac{32}{5}$

(i) $(\Sigma_N^*) = (B^{-1}N)^T C_B - C_N$

$$= \frac{1}{15}\begin{bmatrix} -35 & 24 \\ -25 & 21 \\ 10 & 6 \\ 0 & 3 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 16 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 8 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{15}\begin{bmatrix} 244 \\ 236 \\ 136 \\ 48 \\ 28 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 8 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 229 \\ 206 \\ 16 \\ 48 \\ 28 \end{bmatrix}$$

(j)

		x_1	x_2	x_4	x_6	x_7
	$\frac{22}{5}$	$-\frac{229}{15}$	$-\frac{206}{15}$	$-\frac{16}{15}$	$-\frac{48}{15}$	$-\frac{28}{15}$
x_3	0	$\frac{7}{3}$	$\frac{5}{3}$	$-\frac{2}{3}$	0	$\frac{1}{3}$
x_5	$\frac{2}{5}$	$-\frac{8}{5}$	$-\frac{7}{5}$	$-\frac{2}{5}$	$-\frac{1}{5}$	$-\frac{1}{5}$

Σ^T

$-(\Sigma_N^*)^T$

$B^{-1}b$

$-B^{-1}N$

(Opt!)

[V] 6.7

$$\max_{x \geq 0} \left[\min_{y \geq 0} \; c^T x - y^T A x + b^T y \right]$$

($b^T y = y^T b$)

$$\min_{y \geq 0} \left[c^T x - y^T (Ax - b) \right]$$

$$= \begin{cases} -\infty & \text{if } Ax \geq b \\ c^T x & \text{if } Ax \leq b \end{cases}$$

$\Rightarrow$

$$\boxed{\begin{aligned} \max_{x \geq 0} \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \end{aligned}}$$

$$\min_{y \geq 0} \left[\max_{x \geq 0} \; c^T x - y^T A x + y^T b \right]$$

$$\max_{x \geq 0} \; (c^T - y^T A) x + y^T b$$

$$= \begin{cases} +\infty & \text{if } c^T \geq y^T A \\ y^T b & \text{if } c^T \leq y^T A \end{cases}$$

$\Rightarrow$

$$\boxed{\begin{aligned} \min_{y \geq 0} \quad & y^T b \\ & y^T A \geq c^T \end{aligned}}$$

9.5 Generalize Theorems 5.2 and 5.3 to the context of general LP problems. Use your result to find out if

$$x_1^* = 3, \quad x_2^* = -1, \quad x_3^* = 0, \quad x_4^* = 2$$

is an optimal solution of the problem

$$\begin{aligned} \text{maximize} \quad & 6x_1 + x_2 - x_3 - x_4 \\ \text{subject to} \quad & x_1 + 2x_2 + x_3 + x_4 \le 5 \\ & 3x_1 + x_2 - x_3 \le 8 \\ & x_2 + x_3 + x_4 = 1 \\ & x_3, x_4 \ge 0. \end{aligned}$$

$6x_1 + x_2 - x_3 - x_4 \qquad x_3, x_4 \ge 0$

(*) $\le (y_1+3y_2)x_1 + (2y_1+y_2+y_3)x_2 + (y_1-y_2+y_3)x_3 + (y_1+y_3)x_4$

$\le 5y_1 + 8y_2 + y_3 \qquad y_1 \ge 0,\ y_2 \ge 0.$

(D) $\min \ 5y_1 + 8y_2 + y_3$

s.t. $y_1 + 3y_2 = 6$

$2y_1 + y_2 + y_3 = 1$

(**) $\rightarrow$ $y_1 - y_2 + y_3 \ge -1$

$y_1 + y_3 \ge -1$

$y_1, y_2 \ge 0$

(x_1 has no sign)

(x_2 has no sign)

($x_3 \ge 0$)

($x_4 \ge 0$)

$x_1^* = 3$ ($\neq 0$), $x_2^* = -1$ ($\neq 0$), $x_3^* = 0$, $x_4^* = 2$ (> 0)

To find y^*, we need tightness of (*):

$6x_1 + x_2 - x_3 - x_4$ $\quad x_3, x_4 \geq 0$

(*) $\leq (y_1+3y_2)x_1 + (2y_1+y_2+y_3)x_2 + (y_1-y_2+y_3)x_3 + (y_1 + y_3)x_4$ $\quad = $

$\leq 5y_1 + 8y_2 + y_3$ $\quad y_1 \geq 0,\ y_2 \geq 0.$

i.e.

$$\begin{cases} y_1 + 3y_2 = 6 \\ 2y_1 + y_2 + y_3 = 1 \\ y_1 + y_3 = -1 \end{cases} \Rightarrow \begin{aligned} y_1 + 3y_2 &= 6 \\ y_1 + y_2 &= 2 \end{aligned}$$

$$\Rightarrow \quad y_2 = 2,\quad y_1 = 0,\quad y_3 = -1$$

check (**) $\quad y_1 - y_2 + y_3 = 0 - 2 - 1 = -3 \not\geq -1$

Hence x^* is not optimal

HW5 3. As a newly hired data scientist, your very first assignment was to solve problem (2.1) in [V, p.11].

This was such a difficult problem that you needed to use one week to solve it. You submitted its solution (2.4) in [V, p.13] to your boss.

Your boss said, "Oops. I forgot to tell you an extra constraint, $3x_1 + x_2 - x_3 \le 3$. Can you tell me the new solution, *real soon*?"

Your inner-self tells you that "*real soon*" cannot mean another week of work. What would you do[1]?

$$
\begin{array}{lrcrcrcrcr}
\text{maximize} & \zeta & = & & +5 & x_1 & +4 & x_2 & +3 & x_3 \\
\text{subject to} & w_1 & = & 5 & -2 & x_1 & -3 & x_2 & - & x_3 \\
& w_2 & = & 11 & -4 & x_1 & - & x_2 & -2 & x_3 \\
& w_3 & = & 8 & -3 & x_1 & -4 & x_2 & -2 & x_3
\end{array}
$$

$$x_1,\ x_2,\ x_3,\ w_1,\ w_2,\ w_3 \ge 0.$$

$\Rightarrow$

$$
\begin{array}{rcrcrcrcr}
\zeta & = & 13 & -1 & w_1 & -3 & x_2 & -1 & w_3 \\
\hline
x_1 & = & 2 & -2 & w_1 & -2 & x_2 & + & w_3 \\
w_2 & = & 1 & +2 & w_1 & +5 & x_2 & & \\
x_3 & = & 1 & +3 & w_1 & + & x_2 & -2 & w_3
\end{array}
$$

Add $w_4 = 3 - 3x_1 - x_2 + x_3$

$= 3 - 3(2 - 2w_1 - 2x_2 + w_3) - x_2$

$+ (1 + 3w_1 + x_2 - 2w_3)$

$= -2 + 9w_1 + 6x_2 - 5w_3$

← all neg. coeff

$$
\begin{array}{rcrcrcrcr}
\zeta & = & 13 & -1 & w_1 & -3 & x_2 & -1 & w_3 \\
\hline
x_1 & = & 2 & -2 & w_1 & -2 & x_2 & + & w_3 \\
w_2 & = & 1 & +2 & w_1 & +5 & x_2 & & \\
x_3 & = & 1 & +3 & w_1 & + & x_2 & -2 & w_3
\end{array}
$$

$w_4 = -3 + 9w_1 + 6x_2 - 5w_3$

not feasible

$\Rightarrow$ dual

-13	-2	-1	-1	3
1	2	-2	-3	-9
3	2	-5	-1	-6
1	-1	0	2	5