Solutions on Nonlinear Programming

6.8 Consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f(\boldsymbol{x}) = \boldsymbol{x}^\top \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} \boldsymbol{x} + \boldsymbol{x}^\top \begin{bmatrix} 3 \\ 5 \end{bmatrix} + 6.$$

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$f(x_1, x_2) = (x_1, x_2) \begin{pmatrix} 1 & 2 \\ 4 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (x_1, x_2) \begin{pmatrix} 3 \\ 5 \end{pmatrix} + 6$$

$$= (x_1, x_2) \begin{pmatrix} x_1 + 2x_2 \\ 4x_1 + 7x_2 \end{pmatrix} + 3x_1 + 5x_2 + 6$$

$$= x_1^2 + 6x_1x_2 + 7x_2^2 + 3x_1 + 5x_2 + 6$$

$$\nabla f = (\partial_{x_1} f, \partial_{x_2} f)^T$$

$$= (2x_1 + 6x_2 + 3, \; 6x_1 + 14x_2 + 5)^T$$

Set $\nabla f = 0 \Longrightarrow$

$$\begin{aligned} 2x_1 + 6x_2 + 3 &= 0 \\ 6x_1 + 14x_2 + 5 &= 0 \end{aligned}$$

$$\Longrightarrow \begin{pmatrix} x_1 = 3/2 \\ x_2 = -1 \end{pmatrix}$$

$$f(x_1,x_2) = x_1^2 + 6x_1x_2 + 7x_2^2 + 3x_1 + 5x_2 + 6$$

$$= x_1^2 + 6x_1x_2 + 9x_2 - 9x_2^2 + 7x_2^2 + 3x_1 + 5x_2 + 6$$

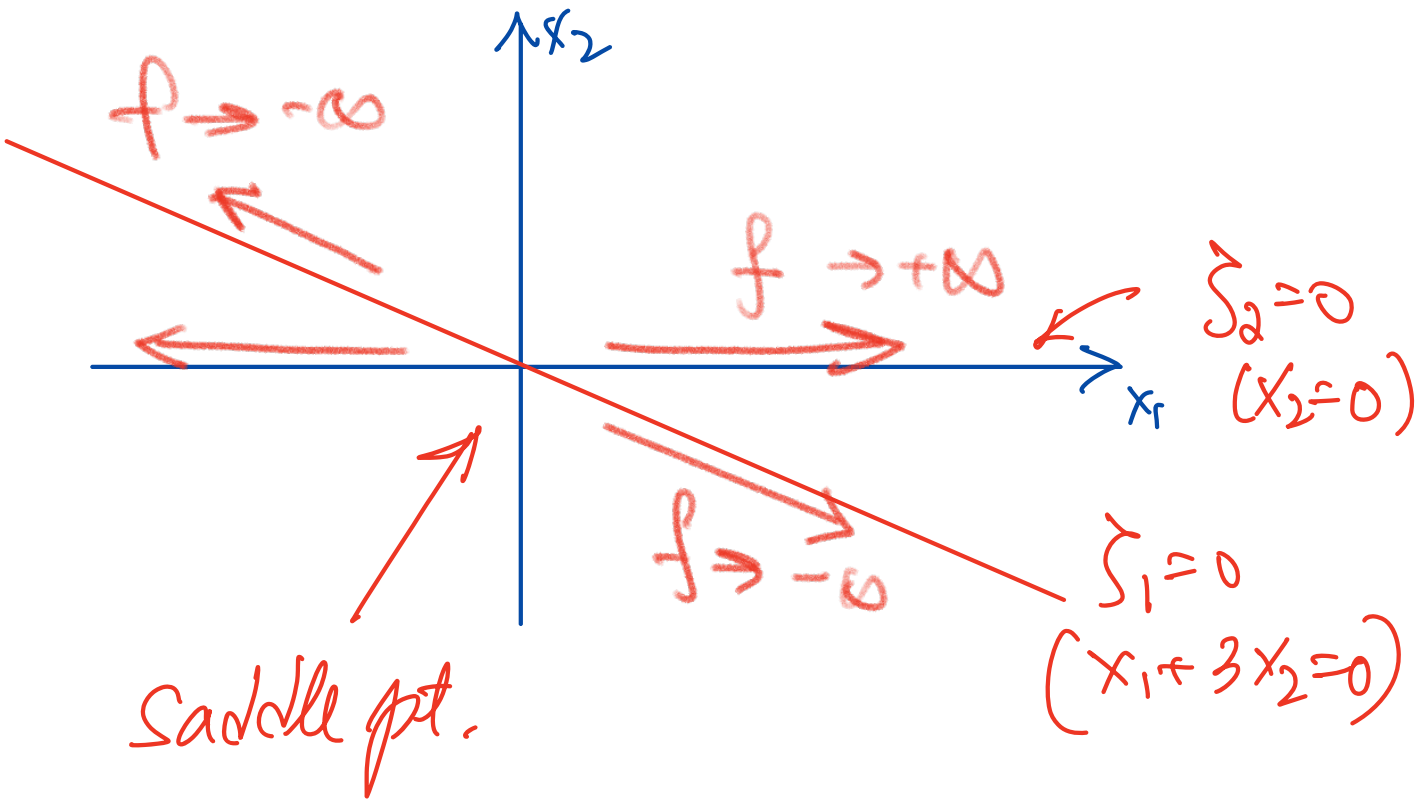
$$= (x_1 + 3x_2)^2 - (\sqrt{2}x_2)^2 + 3x_1 + 5x_2 + 6$$

$\zeta_1 = x_1 + 3x_2$ $\quad \zeta_2 = \sqrt{2}x_2$

↗ quadratic terms dominate linear terms

$f \to +\infty$ as $\zeta_1 \to \infty$

$f \to -\infty$ as $\zeta_2 \to \infty$



6.10 Consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f(\boldsymbol{x}) = \boldsymbol{x}^\top \begin{bmatrix} 2 & 5 \\ -1 & 1 \end{bmatrix} \boldsymbol{x} + \boldsymbol{x}^\top \begin{bmatrix} 3 \\ 4 \end{bmatrix} + 7.$$

a. Find the directional derivative of f at $[0, 1]^\top$ in the direction $[1, 0]^\top$.

b. Find all points that satisfy the first-order necessary condition for f. Does f have a minimizer? If it does, then find all minimizer(s); otherwise, explain why it does not.

$$f(x_1, x_2) = 2x_1^2 + 4x_1x_2 + x_2^2 + 3x_1 + 4x_2 + 7$$

$$\nabla f = (4x_1 + 4x_2 + 3,\ 4x_1 + 2x_2 + 4)^T$$

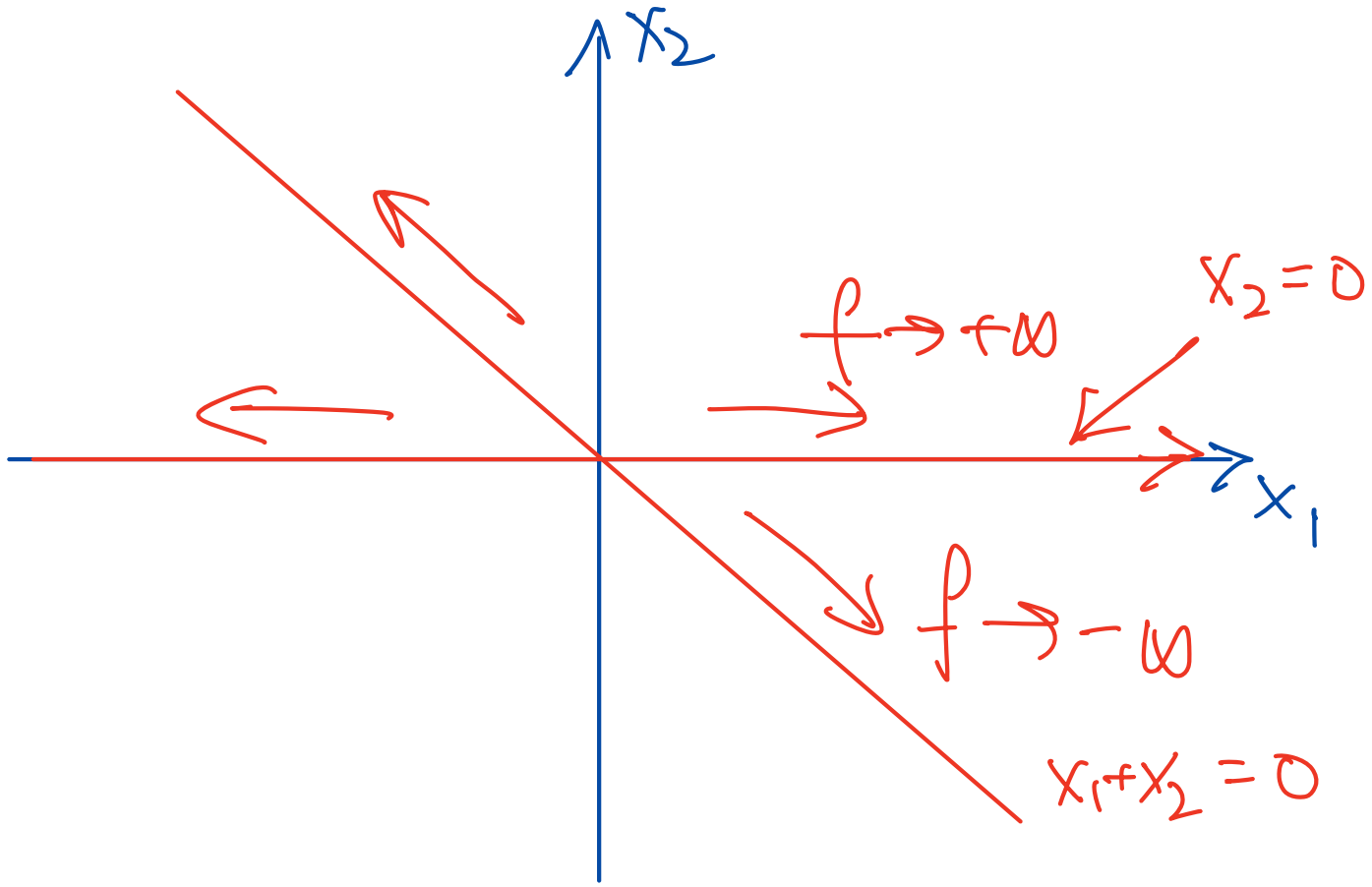
$$\nabla f = 0 \implies \begin{matrix} 4x_1 + 4x_2 + 3 = 0 \\ 4x_1 + 2x_2 + 4 = 0 \end{matrix} \implies \begin{pmatrix} x_1 = -\frac{5}{4} \\ x_2 = \frac{1}{2} \end{pmatrix}$$

$$f(x_1, x_2) = 2\left[x_1^2 + 2x_1x_2 + \frac{1}{2}x_2^2\right] + 3x_1 + 4x_2 + 7$$

$$= 2\left[x_1^2 + 2x_1x_2 + x_2^2 + \frac{1}{2}x_2^2 - x_1^2\right] + 3x_1 + 4x_2 + 7$$

$$= 2\left[(x_1 + x_2)^2 - \left(\frac{1}{\sqrt{2}}x_2\right)^2\right] + 3x_1 + 4x_2 + 7$$

Dominates linear terms



saddle pt.

12.19 Solve the problem

$$\begin{aligned} &\text{minimize} && \|\boldsymbol{x}-\boldsymbol{x}_0\|^2 \\ &\text{subject to} && \begin{bmatrix}1 & 1 & 1\end{bmatrix}\boldsymbol{x}=1, \end{aligned}$$

where $\boldsymbol{x}_0=[0,-3,0]^\top$.

$$f(x) = x_1^2+(x_2+3)^2+x_3^2$$

$$g(x) = (x_1+x_2+x_3-1)$$

$$\nabla f+\lambda\nabla g=0, \quad g=0$$

$$\begin{aligned} 2x_1 &= \lambda \\ 2(x_2+3) &= \lambda \\ 2x_3 &= \lambda \end{aligned} \longrightarrow x_1+x_2+x_3=1$$

$$\frac{\lambda}{2}+\frac{\lambda}{2}-3+\frac{\lambda}{2}=1$$

$$\lambda=\frac{9}{3}$$

Hence $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4/3 \\ -5/3 \\ 4/3 \end{pmatrix}$

20.2 Find local extremizers for the following optimization problems:

a.

$$\text{Minimize} \quad x_1^2 + 2x_1x_2 + 3x_2^2 + 4x_1 + 5x_2 + 6x_3$$

$$\text{subject to} \quad x_1 + 2x_2 = 3$$
$$4x_1 + 5x_3 = 6.$$

b.

$$\text{Maximize} \quad 4x_1 + x_2^2$$

$$\text{subject to} \quad x_1^2 + x_2^2 = 9.$$

c.

$$\text{Maximize} \quad x_1x_2$$

$$\text{subject to} \quad x_1^2 + 4x_2^2 = 1.$$

(a) $x_1^2 + 2x_1x_2 + 3x_2^2 + 4x_1 + 5x_2 + 6x_3$
$+ \lambda_1(x_1 + 2x_2 - 3)$
$+ \lambda_2(4x_1 + 5x_3 - 6)$

$\partial x_1, \partial x_2 \Rightarrow$ $2x_1 + 2x_2 + 4 + \lambda_1 + 4\lambda_2 = 0$
$2x_1 + 6x_2 + 5 + 2\lambda_1 + 5\lambda_2 = 0$

$\Leftrightarrow$ $\begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}^{-1} \left(-\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \end{bmatrix} \right)$

$\partial\lambda_1, \partial\lambda_2 \Rightarrow \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}\left(\begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}^{-1}\left(-\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}-\begin{bmatrix} 4 \\ 5 \end{bmatrix}\right)\right)=\begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$$=-\begin{bmatrix} 3 \\ 6 \end{bmatrix}-\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}^{-1}\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

(Note: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}=\dfrac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad-bc}$)

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}\frac{\begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix}}{8}\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$$=-\begin{bmatrix} 3 \\ 6 \end{bmatrix}-\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}\frac{\begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix}}{8}\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$=-\begin{bmatrix} 3 \\ 6 \end{bmatrix}-\frac{1}{8}\begin{bmatrix} 2 & 2 \\ 14 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$= -\begin{bmatrix}3\\6\end{bmatrix} - \frac{1}{4}\begin{bmatrix}9\\33\end{bmatrix} = -\frac{1}{4}\begin{bmatrix}21\\57\end{bmatrix}$$

$$\frac{1}{8}\begin{bmatrix}2 & 2\\14 & 2\end{bmatrix}\begin{bmatrix}1 & 4\\2 & 5\end{bmatrix}\begin{bmatrix}\lambda_1\\\lambda_2\end{bmatrix} = -\frac{1}{4}\begin{bmatrix}21\\57\end{bmatrix}$$

$$\begin{bmatrix}1 & 1\\7 & 1\end{bmatrix}\begin{bmatrix}1 & 4\\2 & 5\end{bmatrix}\begin{bmatrix}\lambda_1\\\lambda_2\end{bmatrix} = -3\begin{bmatrix}7\\19\end{bmatrix}$$

$$\begin{bmatrix}3 & 9\\9 & 33\end{bmatrix}\begin{bmatrix}\lambda_1\\\lambda_2\end{bmatrix} = -3\begin{bmatrix}7\\19\end{bmatrix}$$

$$\begin{bmatrix}1 & 3\\3 & 11\end{bmatrix}\begin{bmatrix}\lambda_1\\\lambda_2\end{bmatrix} = -\begin{bmatrix}7\\19\end{bmatrix}$$

$$\begin{bmatrix}\lambda_1\\\lambda_2\end{bmatrix} = -\frac{\begin{bmatrix}11 & -3\\-3 & 1\end{bmatrix}}{2}\begin{bmatrix}7\\19\end{bmatrix} = -\begin{bmatrix}10\\-1\end{bmatrix}$$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}2 & 2\\2 & 6\end{bmatrix}^{-1}\left(-\begin{bmatrix}1 & 4\\2 & 5\end{bmatrix}\begin{bmatrix}\lambda_1\\\lambda_2\end{bmatrix} - \begin{bmatrix}4\\5\end{bmatrix}\right)$$

$$= \frac{1}{8}\begin{bmatrix}6 & -2\\-2 & 2\end{bmatrix}\left(\begin{bmatrix}1 & 4\\2 & 5\end{bmatrix}\begin{bmatrix}10\\-1\end{bmatrix} - \begin{bmatrix}4\\5\end{bmatrix}\right) = \begin{pmatrix}-1\\2\end{pmatrix}$$

20.3 Find minimizers and maximizers of the function

$$f(\boldsymbol{x}) = (\boldsymbol{a}^\top \boldsymbol{x})(\boldsymbol{b}^\top \boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^3,$$

subject to

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_2 + x_3 &= 0, \end{aligned}$$

where

$$\boldsymbol{a} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

$f(x) = x_2(x_1 + x_3)$

$x_2(x_1+x_3) + \lambda_1(x_1+x_2) + \lambda_2(x_2+x_3)$

$\partial x_1, \partial x_2, \partial x_3 \Rightarrow$

$$\begin{aligned} x_2 + \lambda_1 &= 0 \\ x_1 + x_3 + \lambda_1 + \lambda_2 &= 0 \\ x_2 + \lambda_2 &= 0 \end{aligned}$$

$$\left(\Longleftrightarrow \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{\text{not invertible}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \right)$$

Solve directly: $x_1 = -x_2$, $x_3 = -x_2$

$f(x) = x_2(x_1+x_3) = x_2(-2x_2) = -2x_2^2$

$\max f(x) = 0$ at $x_2 = 0$

$\min f(x) = -\infty$

20.5 Consider the problem

$$\begin{aligned} \text{minimize} \quad & \|\boldsymbol{x} - \boldsymbol{x}_0\|^2 \\ \text{subject to} \quad & \|\boldsymbol{x}\|^2 = 9, \end{aligned}$$

where $\boldsymbol{x}_0 = [1, \sqrt{3}]^\top$.

a. Find all points satisfying the Lagrange condition for the problem.

$$(x_1-1)^2+(x_2-\sqrt{3})^2+\lambda(x_1^2+x_2^2-9)$$

$$\begin{cases} 2(x_1-1)+2\lambda x_1=0 \\ 2(x_2-\sqrt{3})+2\lambda x_2=0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} (1+\lambda)x_1=1 \\ (1+\lambda)x_2=\sqrt{3} \end{cases} \quad \begin{cases} x_1=\dfrac{1}{1+\lambda} \\ x_2=\dfrac{\sqrt{3}}{1+\lambda} \end{cases} \qquad \lambda+1\neq 0$$

$$\Rightarrow \quad \frac{1}{(1+\lambda)^2}+\frac{3}{(1+\lambda)^2}=9$$

$$\Rightarrow \quad (1+\lambda)^2=\frac{4}{9} \quad \Rightarrow \quad 1+\lambda=\pm\frac{2}{3}$$

$$\left(x_1=\frac{3}{2},\ x_2=\frac{3\sqrt{3}}{2}\right)$$

$$\text{or} \quad \left(x_1=-\frac{3}{2},\ x_2=-\frac{3\sqrt{3}}{2}\right)$$