

At this point, all we have established is that the pairs $(\pm \mathbf{x}^*, \lambda^*)$ satisfy the Lagrange condition. We now show that the points $\pm \mathbf{x}^*$ are, in fact, strict local maximizers. We do this for the point $\mathbf{x}^*$. A similar procedure applies to $-\mathbf{x}^*$. We first compute the Hessian matrix of the Lagrangian function. We have

$$L(\mathbf{x}^*, \lambda^*) = 2\mathbf{Q} - 2\lambda^* \mathbf{P} = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}.$$

The tangent space $T(\mathbf{x}^*)$ to $\{\mathbf{x} : 1 - \mathbf{x}^\top \mathbf{P} \mathbf{x} = 0\}$ is

$$\begin{aligned} T(\mathbf{x}^*) &= \{\mathbf{y} \in \mathbb{R}^2 : \mathbf{x}^{*\top} \mathbf{P} \mathbf{y} = 0\} \\ &= \{\mathbf{y} : [\sqrt{2}, 0] \mathbf{y} = 0\} \\ &= \{\mathbf{y} : \mathbf{y} = [0, a]^\top, a \in \mathbb{R}\}. \end{aligned}$$

Note that for each $\mathbf{y} \in T(\mathbf{x}^*)$, $\mathbf{y} \neq \mathbf{0}$,

$$\mathbf{y}^\top L(\mathbf{x}^*, \lambda^*) \mathbf{y} = [0, a] \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ a \end{bmatrix} = -2a^2 < 0.$$

Hence, $L(\mathbf{x}^*, \lambda^*) < 0$ on $T(\mathbf{x}^*)$, and thus $\mathbf{x}^* = [1/\sqrt{2}, 0]^\top$ is a strict local maximizer. The same is true for the point $-\mathbf{x}^*$. Note that

$$\frac{\mathbf{x}^{*\top} \mathbf{Q} \mathbf{x}^*}{\mathbf{x}^{*\top} \mathbf{P} \mathbf{x}^*} = 2,$$

which, as expected, is the value of the maximal eigenvalue of $\mathbf{P}^{-1} \mathbf{Q}$. Finally, we point out that any scalar multiple $t\mathbf{x}^*$ of $\mathbf{x}^*$, $t \neq 0$, is a solution to the original problem of maximizing $\mathbf{x}^\top \mathbf{Q} \mathbf{x} / \mathbf{x}^\top \mathbf{P} \mathbf{x}$. ■

20.6 Minimizing Quadratics Subject to Linear Constraints

Consider the problem

$$\begin{aligned} &\text{minimize} && \frac{1}{2} \mathbf{x}^\top \mathbf{Q} \mathbf{x} \\ &\text{subject to} && \mathbf{A} \mathbf{x} = \mathbf{b}, \end{aligned}$$

where $\mathbf{Q} > 0$, $\mathbf{A} \in \mathbb{R}^{m \times n}$, $m < n$, $\text{rank } \mathbf{A} = m$. This problem is a special case of what is called a *quadratic programming problem* (the general form of a quadratic programming problem includes the constraint $\mathbf{x} \geq \mathbf{0}$). Note that the constraint set contains an infinite number of points (see Section 2.3). We now show, using Lagrange's theorem, that there is a unique solution to the optimization problem above. Following that, we provide an example illustrating the application of this solution to an optimal control problem.

To solve the problem, we first form the Lagrangian function

$$l(\mathbf{x}, \boldsymbol{\lambda}) = \frac{1}{2} \mathbf{x}^\top \mathbf{Q} \mathbf{x} + \boldsymbol{\lambda}^\top (\mathbf{b} - \mathbf{A} \mathbf{x}).$$

The Lagrange condition yields

$$D_x l(\mathbf{x}^*, \boldsymbol{\lambda}^*) = \mathbf{x}^{*\top} \mathbf{Q} - \boldsymbol{\lambda}^{*\top} \mathbf{A} = \mathbf{0}^\top.$$

Rewriting, we get

$$\mathbf{x}^* = \mathbf{Q}^{-1} \mathbf{A}^\top \boldsymbol{\lambda}^*.$$

Premultiplying both sides of the above by $\mathbf{A}$ gives

$$\mathbf{A} \mathbf{x}^* = \mathbf{A} \mathbf{Q}^{-1} \mathbf{A}^\top \boldsymbol{\lambda}^*.$$

Using the fact that $\mathbf{A} \mathbf{x}^* = \mathbf{b}$, and noting that $\mathbf{A} \mathbf{Q}^{-1} \mathbf{A}^\top$ is invertible because $\mathbf{Q} > 0$ and $\text{rank } \mathbf{A} = m$, we can solve for $\boldsymbol{\lambda}^*$ to obtain

$$\boldsymbol{\lambda}^* = (\mathbf{A} \mathbf{Q}^{-1} \mathbf{A}^\top)^{-1} \mathbf{b}.$$

Therefore, we obtain

$$\mathbf{x}^* = \mathbf{Q}^{-1} \mathbf{A}^\top (\mathbf{A} \mathbf{Q}^{-1} \mathbf{A}^\top)^{-1} \mathbf{b}.$$

The point $\mathbf{x}^*$ is the only candidate for a minimizer. To establish that $\mathbf{x}^*$ is indeed a minimizer, we verify that $\mathbf{x}^*$ satisfies the second-order sufficient conditions. For this, we first find the Hessian matrix of the Lagrangian function at $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$. We have

$$L(\mathbf{x}^*, \boldsymbol{\lambda}^*) = \mathbf{Q},$$

which is positive definite. Thus, the point $\mathbf{x}^*$ is a strict local minimizer. We will see in Chapter 22 that $\mathbf{x}^*$ is, in fact, a global minimizer.

The special case where $\mathbf{Q} = \mathbf{I}_n$, the $n \times n$ identity matrix, reduces to the problem considered in Section 12.3. Specifically, the problem in Section 12.3 is to minimize the norm $\|\mathbf{x}\|$ subject to $\mathbf{A} \mathbf{x} = \mathbf{b}$. The objective function here is $f(\mathbf{x}) = \|\mathbf{x}\|$, which is not differentiable at $\mathbf{x} = \mathbf{0}$. This precludes the use of Lagrange's theorem because the theorem requires differentiability of the objective function. We can overcome this difficulty by considering an equivalent optimization problem:

$$\begin{aligned} & \text{minimize} && \frac{1}{2} \|\mathbf{x}\|^2 \\ & \text{subject to} && \mathbf{A} \mathbf{x} = \mathbf{b}. \end{aligned}$$

The objective function $\|\mathbf{x}\|^2/2$ has the same minimizer as the previous objective function $\|\mathbf{x}\|$. Indeed, if $\mathbf{x}^*$ is such that for all $\mathbf{x} \in \mathbb{R}^n$ satisfying $\mathbf{A} \mathbf{x} = \mathbf{b}$, $\|\mathbf{x}^*\| \leq \|\mathbf{x}\|$, then $\|\mathbf{x}^*\|^2/2 \leq \|\mathbf{x}\|^2/2$. The same is true for the converse. Because the problem of minimizing $\|\mathbf{x}\|^2/2$ subject to $\mathbf{A} \mathbf{x} = \mathbf{b}$ is simply the

problem considered above with $Q = I_n$, we easily deduce the solution to be $\mathbf{x}^* = \mathbf{A}^\top(\mathbf{A}\mathbf{A}^\top)^{-1}\mathbf{b}$, which agrees with the solution in Section 12.3.

Example 20.10 Consider the discrete-time linear system model

$$x_k = ax_{k-1} + bu_k, \quad k \geq 1,$$

with initial condition x_0 given. We can think of $\{x_k\}$ as a discrete-time signal that is controlled by an external input signal $\{u_k\}$. In the control literature, x_k is called the *state* at time k . For a given x_0 , our goal is to choose the control signal $\{u_k\}$ so that the state remains “small” over a time interval $[1, N]$, but at the same time the control signal is “not too large.” To express the desire to keep the state $\{x_k\}$ small, we choose the control sequence to minimize

$$\frac{1}{2} \sum_{i=1}^N x_i^2.$$

On the other hand, maintaining a control signal that is not too large, we minimize

$$\frac{1}{2} \sum_{i=1}^N u_i^2.$$

The two objectives above are conflicting in the sense that they cannot, in general, be achieved simultaneously—minimizing the first may result in a large control effort, while minimizing the second may result in large states. This is clearly a problem that requires compromise. One way to approach the problem is to minimize a weighted sum of the two functions above. Specifically, we can formulate the problem as

$$\text{minimize} \quad \frac{1}{2} \sum_{i=1}^N (qx_i^2 + ru_i^2)$$

$$\text{subject to} \quad x_k = ax_{k-1} + bu_k, \quad k = 1, \dots, N, \quad x_0 \text{ given},$$

where the parameters q and r reflect the relative importance of keeping the state small versus keeping the control effort not too large. This problem is an instance of the *linear quadratic regulator (LQR) problem* (see, e.g., [15], [20], [85], [86], or [99]). Combining the two conflicting objectives of keeping the state small while keeping the control effort small is an instance of the *weighted sum* approach (see Section 24.4).

To solve the problem above, we can rewrite it as a quadratic programming problem. Define

$$\begin{aligned}
 \mathbf{Q} &= \begin{bmatrix} q\mathbf{I}_N & \mathbf{O} \\ \mathbf{O} & r\mathbf{I}_N \end{bmatrix}, \\
 \mathbf{A} &= \begin{bmatrix} 1 & \cdots & 0 & -b & \cdots & 0 \\ -a & 1 & \vdots & & -b & \vdots \\ & \ddots & \ddots & \vdots & & \ddots \\ 0 & & -a & 1 & 0 & \cdots & -b \end{bmatrix}, \\
 \mathbf{b} &= \begin{bmatrix} ax_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{z} = [x_1, \dots, x_N, u_1, \dots, u_N]^\top.
 \end{aligned}$$

With these definitions, the problem reduces to the previously considered quadratic programming problem,

$$\begin{aligned}
 &\text{minimize} \quad \frac{1}{2} \mathbf{z}^\top \mathbf{Q} \mathbf{z} \\
 &\text{subject to} \quad \mathbf{A} \mathbf{z} = \mathbf{b},
 \end{aligned}$$

where $\mathbf{Q}$ is $2N \times 2N$, $\mathbf{A}$ is $N \times 2N$, and $\mathbf{b} \in \mathbb{R}^N$. The solution is

$$\mathbf{z}^* = \mathbf{Q}^{-1} \mathbf{A}^\top (\mathbf{A} \mathbf{Q}^{-1} \mathbf{A}^\top)^{-1} \mathbf{b}.$$

The first N components of $\mathbf{z}^*$ represent the optimal state signal in the interval $[1, N]$, whereas the second N components represent the optimal control signal.

In practice, computation of the matrix inverses in the formula for $\mathbf{z}^*$ above may be too costly. There are other ways to tackle the problem by exploiting its special structure. This is the study of *optimal control* (see, e.g., [15], [20], [85], [86], or [99]). ■

The following example illustrates an application of the above discussion.

Example 20.11 Credit-Card Holder Dilemma. Suppose that we currently have a credit-card debt of \$10,000. Credit-card debts are subject to a monthly interest rate of 2%, and the account balance is increased by the interest amount every month. Each month we have the option of reducing the account balance by contributing a payment to the account. Over the next 10 months, we plan to contribute a payment every month in such a way as to minimize the overall debt level while minimizing the hardship of making monthly payments.

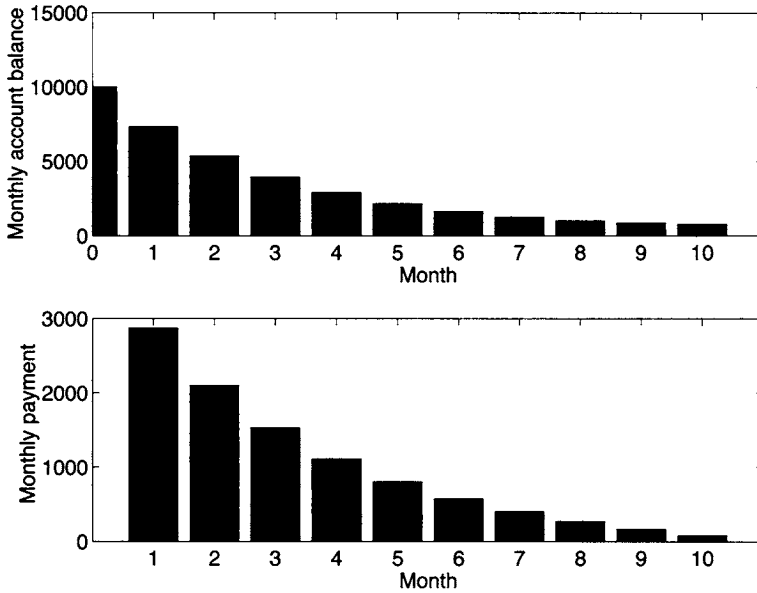


Figure 20.15 Plots for Example 20.11 with $q = 1$ and $r = 10$.

We solve our problem using the LQR framework described in Example 20.10. Let the current time be 0, x_k the account balance at the end of month k , and u_k our payment in month k . We have

$$x_k = 1.02x_{k-1} - u_k, \quad k = 1, \dots, 10;$$

that is, the account balance in a given month is equal to the account balance in the previous month plus the monthly interest on that balance minus our payment that month. Our optimization problem is then

$$\text{minimize} \quad \frac{1}{2} \sum_{i=1}^{10} (qx_i^2 + ru_i^2)$$

$$\text{subject to} \quad x_k = 1.02x_{k-1} - u_k, \quad k = 1, \dots, 10, \quad x_0 = 10,000,$$

which is an instance of the LQR problem. The parameters q and r reflect our priority in trading off between debt reduction and hardship in making payments. The more anxious we are to reduce our debt, the larger the value of q relative to r . On the other hand, the more reluctant we are to make payments, the larger the value of r relative to q .

The solution to the problem above is given by the formula derived in Example 20.10. In Figure 20.15 we plot the monthly account balances and payments over the next 10 months using $q = 1$ and $r = 10$. We can see here that our debt has been reduced to less than \$1000 after 10 months, but with a first

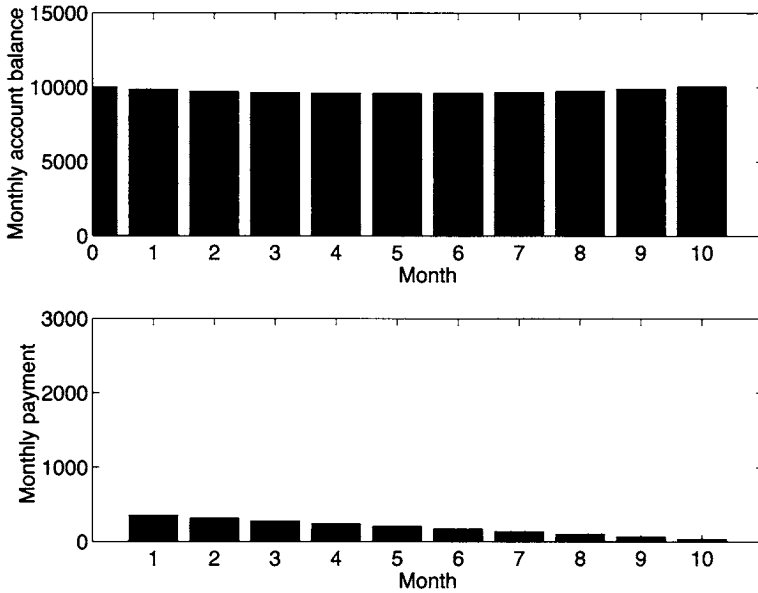


Figure 20.16 Plots for Example 20.11 with $q = 1$ and $r = 300$.

payment close to \$3000. If we feel that a payment of \$3000 is too high, then we can try to reduce this amount by increasing the value of r relative to q . However, going too far along these lines can lead to trouble. Indeed, if we use $q = 1$ and $r = 300$ (see Figure 20.16), although the monthly payments do not exceed \$400, the account balance is never reduced by much below \$10,000. In this case, the interest on the account balance eats up a significant portion of our monthly payments. In fact, our debt after 10 months will be higher than \$10,000. ■

For a treatment of optimization problems with quadratic objective functions, subject to linear or quadratic constraints, arising in communication and signal processing, see [105] and [106].

EXERCISES

20.1 Consider the following constraints on $\mathbb{R}^2$:

$$h(x_1, x_2) = (x_1 - 2)^2 = 0 \quad \text{and} \quad g(x_1, x_2) = (x_2 + 1)^3 \leq 0.$$

Find the set of feasible points. Are the feasible points regular? Justify your answer.

20.2 Find local extremizers for the following optimization problems:

a. Minimize $x_1^2 + 2x_1x_2 + 3x_2^2 + 4x_1 + 5x_2 + 6x_3$

subject to $x_1 + 2x_2 = 3$

$4x_1 + 5x_3 = 6.$

b. Maximize $4x_1 + x_2^2$

subject to $x_1^2 + x_2^2 = 9.$

c. Maximize x_1x_2

subject to $x_1^2 + 4x_2^2 = 1.$

20.3 Find minimizers and maximizers of the function

$$f(\mathbf{x}) = (\mathbf{a}^\top \mathbf{x})(\mathbf{b}^\top \mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3,$$

subject to

$$x_1 + x_2 = 0$$

$$x_2 + x_3 = 0,$$

where

$$\mathbf{a} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

20.4 Consider the problem

$$\text{minimize } f(\mathbf{x})$$

$$\text{subject to } h(\mathbf{x}) = 0,$$

where $f: \mathbb{R}^2 \rightarrow \mathbb{R}$, $h: \mathbb{R}^2 \rightarrow \mathbb{R}$, and $\nabla f(\mathbf{x}) = [x_1, x_1 + 4]^\top$. Suppose that $\mathbf{x}^*$ is an optimal solution and $\nabla h(\mathbf{x}^*) = [1, 4]^\top$. Find $\nabla f(\mathbf{x}^*)$.

20.5 Consider the problem

$$\text{minimize } \|\mathbf{x} - \mathbf{x}_0\|^2$$

$$\text{subject to } \|\mathbf{x}\|^2 = 9,$$

where $\mathbf{x}_0 = [1, \sqrt{3}]^\top$.

a. Find all points satisfying the Lagrange condition for the problem.

- b. Using second-order conditions, determine whether or not each of the points in part a is a local minimizer.

20.6 We wish to construct a closed box with minimum surface area that encloses a volume of V cubic feet, where $V > 0$.

- Let a , b , and c denote the dimensions of the box with minimum surface area (with volume V). Derive the Lagrange condition that must be satisfied by a , b , and c .
- What does it mean for a point $\mathbf{x}^*$ to be a *regular* point in this problem? Is the point $\mathbf{x}^* = [a, b, c]^\top$ a regular point?
- Find a , b , and c .
- Does the point $\mathbf{x}^* = [a, b, c]^\top$ found in part c satisfy the second-order sufficient condition?

20.7 Find local extremizers of

- $f(x_1, x_2, x_3) = x_1^2 + 3x_2^2 + x_3$ subject to $x_1^2 + x_2^2 + x_3^2 = 16$.
- $f(x_1, x_2) = x_1^2 + x_2^2$ subject to $3x_1^2 + 4x_1x_2 + 6x_2^2 = 140$.

20.8 Consider the problem

$$\begin{aligned} &\text{minimize} && 2x_1 + 3x_2 - 4, && x_1, x_2 \in \mathbb{R} \\ &\text{subject to} && x_1x_2 = 6. \end{aligned}$$

- Use Lagrange's theorem to find all possible local minimizers and maximizers.
- Use the second-order sufficient conditions to specify which points are strict local minimizers and which are strict local maximizers.
- Are the points in part b global minimizers or maximizers? Explain.

20.9 Find all maximizers of the function

$$f(x_1, x_2) = \frac{18x_1^2 - 8x_1x_2 + 12x_2^2}{2x_1^2 + 2x_2^2}.$$

20.10 Find all solutions to the problem

$$\begin{aligned} &\text{maximize} && \mathbf{x}^\top \begin{bmatrix} 3 & 4 \\ 0 & 3 \end{bmatrix} \mathbf{x} \\ &\text{subject to} && \|\mathbf{x}\|^2 = 1. \end{aligned}$$

20.11 Consider a matrix $\mathbf{A}$ with the property that $\mathbf{A}^\top \mathbf{A}$ has eigenvalues ranging from 1 to 20 (i.e., the smallest eigenvalue is 1 and the largest is 20). Let $\mathbf{x}$ be a vector such that $\|\mathbf{x}\| = 1$, and let $\mathbf{y} = \mathbf{A}\mathbf{x}$. Use Lagrange multiplier methods to find the range of values that $\|\mathbf{y}\|$ can take.

Hint: What is the largest value that $\|\mathbf{y}\|$ can take? What is the smallest value that $\|\mathbf{y}\|$ can take?

20.12 Consider a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$. Define the *induced 2-norm* of $\mathbf{A}$, denoted $\|\mathbf{A}\|_2$, to be the number

$$\|\mathbf{A}\|_2 = \max\{\|\mathbf{A}\mathbf{x}\| : \mathbf{x} \in \mathbb{R}^n, \|\mathbf{x}\| = 1\},$$

where the norm $\|\cdot\|$ on the right-hand side above is the usual Euclidean norm.

Suppose that the eigenvalues of $\mathbf{A}^\top \mathbf{A}$ are $\lambda_1, \dots, \lambda_n$ (ordered from largest to smallest). Use Lagrange's theorem to express $\|\mathbf{A}\|_2$ in terms of the eigenvalues above (cf. Theorem 3.8).

20.13 Let $\mathbf{P} = \mathbf{P}^\top$ be a positive definite matrix. Show that any point $\mathbf{x}$ satisfying $1 - \mathbf{x}^\top \mathbf{P}\mathbf{x} = 0$ is a regular point.

20.14 Consider the problem

$$\begin{aligned} &\text{maximize} && ax_1 + bx_2, && x_1, x_2 \in \mathbb{R} \\ &\text{subject to} && x_1^2 + x_2^2 = 2, \end{aligned}$$

where $a, b \in \mathbb{R}$. Show that if $[1, 1]^\top$ is a solution to the problem, then $a = b$.

20.15 Consider the problem

$$\begin{aligned} &\text{minimize} && x_1x_2 - 2x_1, && x_1, x_2 \in \mathbb{R} \\ &\text{subject to} && x_1^2 - x_2^2 = 0. \end{aligned}$$

- Apply Lagrange's theorem directly to the problem to show that if a solution exists, it must be either $[1, 1]^\top$ or $[-1, 1]^\top$.
- Use the second-order necessary conditions to show that $[-1, 1]^\top$ cannot possibly be the solution.
- Use the second-order sufficient conditions to show that $[1, 1]^\top$ is a strict local minimizer.

20.16 Let $\mathbf{A} \in \mathbb{R}^{m \times n}$, $m \leq n$, $\text{rank } \mathbf{A} = m$, and $\mathbf{x}_0 \in \mathbb{R}^n$. Let $\mathbf{x}^*$ be the point on the nullspace of $\mathbf{A}$ that is closest to $\mathbf{x}_0$ (in the sense of Euclidean norm).

- Show that $\mathbf{x}^*$ is orthogonal to $\mathbf{x}^* - \mathbf{x}_0$.

- b. Find a formula for $\mathbf{x}^*$ in terms of $\mathbf{A}$ and $\mathbf{x}_0$.

20.17 Consider the problem

$$\begin{aligned} & \text{minimize} && \frac{1}{2} \|\mathbf{Ax} - \mathbf{b}\|^2 \\ & \text{subject to} && \mathbf{Cx} = \mathbf{d}, \end{aligned}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $m > n$, $\mathbf{C} \in \mathbb{R}^{p \times n}$, $p < n$, and both $\mathbf{A}$ and $\mathbf{C}$ are of full rank. We wish to find an expression for the solution (in terms of $\mathbf{A}$, $\mathbf{b}$, $\mathbf{C}$, and $\mathbf{d}$).

- Apply Lagrange's theorem to solve this problem.
- As an alternative, rewrite the given optimization problem in the form of a quadratic programming problem and apply the formula in Section 20.6 to obtain the solution.

20.18 Consider the problem of minimizing a general quadratic function subject to a linear constraint:

$$\begin{aligned} & \text{minimize} && \frac{1}{2} \mathbf{x}^\top \mathbf{Qx} - \mathbf{c}^\top \mathbf{x} + d \\ & \text{subject to} && \mathbf{Ax} = \mathbf{b}, \end{aligned}$$

where $\mathbf{Q} = \mathbf{Q}^\top > 0$, $\mathbf{A} \in \mathbb{R}^{m \times n}$, $m < n$, $\text{rank } \mathbf{A} = m$, and d is a constant. Derive a closed-form solution to the problem.

20.19 Let $\mathbf{L}$ be an $n \times n$ real symmetric matrix, and let $\mathcal{M}$ be a subspace of $\mathbb{R}^n$ with dimension $m < n$. Let $\{\mathbf{b}_1, \dots, \mathbf{b}_m\} \subset \mathbb{R}^n$ be a basis for $\mathcal{M}$, and let $\mathbf{B}$ be the $n \times m$ matrix with $\mathbf{b}_i$ as the i th column. Let $\mathbf{L}_{\mathcal{M}}$ be the $m \times m$ matrix defined by $\mathbf{L}_{\mathcal{M}} = \mathbf{B}^\top \mathbf{L} \mathbf{B}$. Show that $\mathbf{L}$ is positive semidefinite (definite) on $\mathcal{M}$ if and only if $\mathbf{L}_{\mathcal{M}}$ is positive semidefinite (definite).

Note: This result is useful for checking that the Hessian of the Lagrangian function at a point is positive definite on the tangent space at that point.

20.20 Consider the sequence $\{x_k\}$, $x_k \in \mathbb{R}$, generated by the recursion

$$x_{k+1} = ax_k + bu_k, \quad k \geq 0 \quad (a, b \in \mathbb{R}, a, b \neq 0),$$

where $u_0, u_1, u_2, \dots$ is a sequence of "control inputs," and the initial condition $x_0 \neq 0$ is given. The recursion above is also called a *discrete-time linear system*. We wish to find values of control inputs u_0 and u_1 such that $x_2 = 0$, and the average input energy $(u_0^2 + u_1^2)/2$ is minimized. Denote the optimal inputs by u_0^* and u_1^* .

- a. Find expressions for u_0^* and u_1^* in terms of a , b , and x_0 .
- b. Use the second-order sufficient conditions to show that the point $\mathbf{u}^* = [u_0^*, u_1^*]^\top$ in part a is a strict local minimizer.

20.21 Consider the discrete-time linear system $x_k = 2x_{k-1} + u_k$, $k \geq 1$, with $x_0 = 1$. Find the values of the control inputs u_1 and u_2 to minimize

$$x_2^2 + \frac{1}{2}u_1^2 + \frac{1}{3}u_2^2.$$

20.22 Consider the discrete-time linear system $x_{k+1} = x_k + 2u_k$, $0 \leq k \leq 2$, with $x_0 = 3$. Use the Lagrange multiplier approach to calculate the optimal control sequence $\{u_0, u_1, u_2\}$ that transfers the initial state x_0 to $x_3 = 9$ while minimizing

$$\frac{1}{2} \sum_{k=0}^2 u_k^2.$$