**MA 421 Fall 2025 (Aaron N. K. Yip)**
**Homework 3, due on Thursday, Sept 18th, in class**

1. [V] (*Linear Programming, Foundations and Extensions*, **5th edition**)

   p.55: 4.1, 4.2, 4.3. (For each problem, provide graphic illustration.)

2. (In class, we have define a set to be convex.) Now, for a *function* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, it is called *convex* if

$$f(\lambda X + (1-\lambda)Y) \le \lambda f(Y) + (1-\lambda)f(Y), \quad \text{for all } X, Y \in \mathbb{R}^n \text{ and } 0 \le \lambda \le 1.$$

   It is called *strictly convex* if in the above inequality, equality holds *only* at $\lambda = 0, 1$. (Typical examples of (one-dimensional) convex functions include $f(x) = x^2$, $e^x$. Note that these examples satisfy $f''(x) > 0$.)

   (a) Prove that a linear function $f(X) = AX + b$ is convex, but not strictly convex.

   (b) Consider the minimization problem of a convex function $f$: $\min_{X\in\mathbb{R}^n} f(X)$. Prove that if there are two points $X_*$ and $Y_*$ that achieve the minimum value of $f$, then the *whole line segment* connecting $X$ and $Y$ also achieve the minimum value. (In a sense, if a convex function has two minimum points, then it has infinitely many minimum points.)

   (c) Consider the minimization problem of a *strictly* convex function $f$: $\min_{X\in\mathbb{R}^n} f(X)$. Prove that if there is a minimum point $X_*$, then it must be unique, i.e. there cannot be any other minimum point.

3. Consider the following LP problem:

$$\begin{array}{lll} \max & 4x_1 + 2x_2 - x_3 + 2x_4 & \\ \text{s.t.} & x_1 + x_2 + 3x_3 + 4x_4 & = 8, \\ & x_1 + x_2 + x_3 + x_4 & = 4, \\ & x_1, x_2, x_3, x_4 \ge 0. & \end{array}$$

   In the following, you will solve this problem using three methods.

   M1. Since there are two equalities for four variables, in principle, you can solve for two of them in terms of two other (free) variables. Choose two variables to solve so that you can have an initial feasible dictionary. Now the original problem becomes an LP with only two variables. Solve this reduced LP problem. (The main question is which variables to choose in the first place. It is not immediately clear when the number of variables is big. But for small problems, you can certainly do it by guessing.)

M2. Convert the above LP into the standard form $AX \leq b$ and find an initial feasible dictionary by solving an auxiliary problem (by introducing the variable $x_0$ as in [V, p.18]). Then solve the overall LP problem.

M3. Use the auxiliary variable approach (as discussed in class and also outlined in the note in Week 3: LP in general form) to find an initial feasible dictionary. Then solve the overall LP problem.