MA 421 Fall 2025 (Aaron N. K. Yip)
Homework 6, due on Thursday, Oct 9th, 11:59pm, in Gradescope

1. [C], p.116, #7.5[1]. Of course, try to solve this *without* multiplying out $E_1E_2E_3E_4$.

2. (This question explores the so-called "Big-M" method.)
Consider the following problem:

$$\begin{aligned} \max \quad & -2x_1 - x_2 \\ \text{s.t.} \quad & 3x_1 + 4x_2 = 12, \\ & -x_1 + 2x_2 \leq 2, \\ & x_1 + 4x_2 \geq 6, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Note that the origin is not feasible (and it is not immediately clear if the problem has any feasible solution at all). Instead of going through the usual Phases I and II, the Big-M method considers the following one single problem:

$$\begin{aligned} \max \quad & -2x_1 - x_2 - My_1 - My_2 \\ \text{s.t.} \quad & 3x_1 + 4x_2 + y_1 = 12, \\ & -x_1 + 2x_2 + w_1 = 2, \\ & x_1 + 4x_2 - w_2 + y_2 = 6, \\ & x_1, x_2, w_1, w_2, y_1, y_2 \geq 0, \end{aligned}$$

where M is some "very large number". Note that w_1, w_2 are slack variables and y_1, y_2 are auxiliary variables. The idea behind this method is that the original problem is feasible if and only if $y_1 = y_2 = 0$ is feasible and that happens if and only if for (any) large enough M, the new objective function is not "$-\infty$". With this in mind, proceed to solve this problem by first choosing y_1, w_1, y_2 as basic variables so as to have an initial feasible dictionary[2].

3. Consider the following *multi-objective* minimization problem[3]:

$$\begin{aligned} \min \quad & \{\zeta_1(X),\ \zeta_2(X),\ \zeta_3(X), \ldots\} \\ \text{s.t.} \quad & AX \geq b. \end{aligned}$$

[1] Note that in [C], y is a row vector.

[2] Either you perform simplex symbolically by hand or substitute M by some "very large" number such as 1000. Of course, how "large" is sufficient depends on the actual problem (and numbers).

[3] The minimization problem is just for convenience.

A feasible point X_* is called a *Pareto (efficient) solution* if there is *no* other feasible point X such that

$$\zeta_i(X) \le \zeta_i(X_*), \ \text{ for } i = 1, 2, \ldots$$

and for at least one i,

$$\zeta_i(X) < \zeta_i(X_*).$$

The above is also equivalent to the following two (and maybe other) characterizations:

(a) X_* is a Pareto solution if there is no other feasible point X that would decrease some objectives without increase at least one other objective.

(b) X_* is a Pareto solution if for any other feasible point X, suppose one objective decreases, then at least one other objective must increase.

Find (all) the Pareto solution(s) for the following problems.

(a) $\min\{\zeta_1(x) = x, \ \zeta_2(x) = -x\}$
s.t. $1 \le x \le 2$. (This is a one-dimensional problem!)

(b) $\min\{\zeta_1(x_1, x_2) = x_1 + 2x_2, \ \zeta_2(x_1, x_2) = 2x_1 + x_2\}$
s.t. $1 \le x_1 \le 2, \ 1 \le x_2 \le 2$.

(c) $\min\{\zeta_1(x_1, x_2) = x_1 + x_2, \ \zeta_2(x_1, x_2) = -x_1 + x_2\}$
s.t. $1 \le x_1 \le 2, \ 1 \le x_2 \le 2$.

(d) $\min\{\zeta_1(x_1, x_2) = x_1 + x_2, \ \zeta_2(x_1, x_2) = -x_1 + x_2, \ \zeta_3(x_1, x_2) = -x_2\}$
s.t. $1 \le x_1 \le 2, \ 1 \le x_2 \le 2$.

Note: instead of going over the general theory of multi-objective problems, simply solve the above problems by graphical methods, a.k.a. try to "visualize" the objective functions and "move around inside" the feasible sets.